ON THE ELASTIC STABILITY OF A RECTANGULAR PLATE, WHEN SUBJECTED TO A VARIABLE EDGE THRUST

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1. *Introduction and summary.* The stability of a rectangular plate, subjected to constant thrust over opposite pairs of edges, has been treated with some degree of completeness for various boundary conditions*. The more general problem, in which the thrusts are no longer constant, has not yet received any treatment apart from the approximate method developed by E. Schwerin†, which would appear to be capable of only limited extension. The object of this paper is accordingly the detailed consideration of a simple case when the thrust is no longer constant.

Although only one particularly simple case of linear thrust has been considered here, the method used would appear to be easily extensible to the case of general linear thrust. The importance of the problem lies in its ability to demonstrate the effect on stability of local variation in an external force system.

The writer wishes to take this opportunity to express his thanks to Mr W. R. Dean, Trinity College, Cambridge, at whose suggestion the problem was attacked, and for his valuable criticism and advice.

2. Referred to ordinary rectangular axes ξ, η, we are going to consider the stability of a rectangular plate $\xi = \pm a$, $\eta = \pm b$, subject to a thrust $K\xi$ over the two edges $\eta = \pm b$.

Using Love's notation‡ for the stress resultants, the differential equations and boundary conditions which they have to satisfy give rise to a possible solution

$$T_1 = S_1 = 0; \quad T_2 = -K\xi. \tag{1}$$

Moreover, since we can find a displacement system which produces the above stress resultants, this shows that they are correct for our particular problem.

* See paper by G. I. Taylor in *Zeitschrift f. angew. Math. u. Mechanik*, 13 (1933), 147.

† E. Schwerin, *Zeitschrift f. angew. Math. u. Mechanik*, 3, 422.

‡ A. E. H. Love, *Mathematical Theory of Elasticity* (4th ed.), ch. 22.

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Employing the method of neutral equilibrium*, the equation of neutral equilibrium for a rectangular plate subject to external forces at its edges only is

$$D\nabla_1^4\omega - T_1\frac{\partial^2\omega}{\partial\xi^2} - 2S_1\frac{\partial^2\omega}{\partial\xi\,\partial\eta} - T_2\frac{\partial^2\omega}{\partial\eta^2} = 0, \tag{2}$$

where ω is the normal displacement of a point in the middle plane of the plate, and D is the flexural rigidity. Substituting for T_1, S_1 and T_2, from (1), we get

$$D\nabla_1^4\omega + K\xi\frac{\partial^2\omega}{\partial\eta^2} = 0. \tag{3}$$

3. Case I. *All edges freely supported.* In addition to (3) we now have the boundary conditions

$$\left.\begin{aligned} &\omega = 0, \\ &\frac{\partial^2\omega}{\partial\xi^2} + \sigma\frac{\partial^2\omega}{\partial\eta^2} = 0, \\ &\text{for } \xi = \pm a; \end{aligned}\right\} \tag{4}$$

$$\left.\begin{aligned} &\omega = 0, \\ &\frac{\partial^2\omega}{\partial\eta^2} + \sigma\frac{\partial^2\omega}{\partial\xi^2} = 0, \\ &\text{for } \eta = \pm b. \end{aligned}\right\} \tag{5}$$

Now change the coordinate system by making the substitution

$$\xi = \frac{2ax}{\pi} - a; \quad \eta = \frac{2by}{\pi} - b.$$

Then (3), (4) and (5) take the form

$$D\left[\frac{\partial^4\omega}{\partial x^4} + 2\left(\frac{a}{b}\right)^2\frac{\partial^4\omega}{\partial x^2\,\partial y^2} + \left(\frac{a}{b}\right)^4\frac{\partial^4\omega}{\partial y^4}\right] + K\left(\frac{2a}{\pi}\right)^3\left(\frac{a}{b}\right)^2\frac{\partial^2\omega}{\partial y^2}\left(x - \frac{\pi}{2}\right) = 0,$$

$$\left.\begin{aligned} &\omega = 0, \\ &b^2\frac{\partial^2\omega}{\partial x^2} + \sigma a^2\frac{\partial^2\omega}{\partial y^2} = 0, \\ &\text{for } x = 0, \pi; \end{aligned}\right\} \qquad \left.\begin{aligned} &\omega = 0, \\ &a^2\frac{\partial^2\omega}{\partial y^2} + \sigma b^2\frac{\partial^2\omega}{\partial x^2} = 0, \\ &\text{for } y = 0, \pi. \end{aligned}\right\}$$

Putting $\dfrac{K}{D}\dfrac{a^5}{b^2}\left(\dfrac{2}{\pi}\right)^3 = \lambda$, and $\dfrac{a}{b} = \rho$, we get the above equations in non-dimensional form,

$$\frac{\partial^4\omega}{\partial x^4} + 2\rho^2\frac{\partial^4\omega}{\partial x^2\,\partial y^2} + \rho^4\frac{\partial^4\omega}{\partial y^4} + \lambda\frac{\partial^2\omega}{\partial y^2}\left(x - \frac{\pi}{2}\right) = 0, \tag{6}$$

$$\left.\begin{aligned} &\omega = 0, \\ &\frac{\partial^2\omega}{\partial x^2} + \rho^2\sigma\frac{\partial^2\omega}{\partial y^2} = 0, \\ &\text{for } x = 0, \pi; \end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned} &\omega = 0, \\ &\rho^2\frac{\partial^2\omega}{\partial y^2} + \sigma\frac{\partial^2\omega}{\partial x^2} = 0, \\ &\text{for } y = 0, \pi. \end{aligned}\right\} \tag{8}$$

We now endeavour to find out for what values of λ a non-zero solution of (6) exists which satisfies the boundary conditions (7) and (8).

* See, for example, R. V. Southwell, *Phil. Trans. Roy. Soc.* A, 213, 187.

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From purely physical considerations it is natural to make the assumption that the solution, when it exists, can be put in the form

$$\omega = \omega_x \sin my,$$

where ω_x is a function of x only, which vanishes at $x = 0, \pi$, and m is some positive integer. The equation for ω_x is, from (6),

$$\frac{d^4\omega_x}{dx^4} - 2\rho^2 m^2 \frac{d^2\omega_x}{dx^2} + \rho^4 m^4 \omega_x = \lambda m^2 \left(x - \frac{\pi}{2}\right) \omega_x. \tag{9}$$

Again on physical grounds, ω_x and $\frac{d\omega_x}{dx}$ will both be capable of expansion in Fourier series. We put $\omega_x = \sum_{n=1}^{\infty} A_n \sin nx$,

and $$\frac{d\omega_x}{dx} = \sum_{n=0}^{\infty} B_n \cos nx.$$

Then, as a result of general theory, the cosine series can be integrated term by term, and we deduce that B_0 is zero, and that $\Sigma |A_r|$ is convergent. In order to obtain a solution of (9) in the form required, it is convenient to expand the right-hand side of (9) as a sine series in $(0, \pi)$. This process consists fundamentally in multiplying the series $\Sigma A_n \sin nx$ by the cosine series for $x - \pi/2$. Since each of these series is absolutely convergent this process is legitimate.

Now $$x - \frac{\pi}{2} = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} \quad (0 \leqslant x \leqslant \pi),$$

and we put

$$-\frac{\pi}{2}\left(x - \frac{\pi}{2}\right) \sum_{n=1}^{\infty} A_n \sin nx \equiv \sum_{r=1}^{\infty} c_r \sin rx. \tag{10}$$

By equating to zero the coefficients of the sines, we determine each of the c's in terms of an infinite series of A's. From (10), (9) now takes the form

$$\frac{d^4\omega_x}{dx^4} - 2\rho^2 m^2 \frac{d^2\omega_x}{dx^2} + \rho^4 m^4 \omega_x = -\lambda m^2 \frac{2}{\pi} \sum_{r=1}^{\infty} c_r \sin rx. \tag{11}$$

From this equation it is now necessary to find a solution for ω_x which satisfies all the boundary conditions and which can be expressed as a sine series in $(0, \pi)$. A particular integral of the above equation is

$$\omega_x = -\lambda m^2 \frac{2}{\pi} \sum_{r=1}^{\infty} \frac{c_r \sin rx}{(r^2 + \rho^2 m^2)^2}, \tag{12}$$

and we show that it is the solution which we want. Since the series $\Sigma |A_r|$ is convergent, we deduce that the c's are all bounded, and accordingly we may differentiate (12) twice term by term, and we get $\frac{d^2\omega_x}{dx^2}$ expressed as a convergent sine series. It follows therefore that the solution (12) is such that ω_x and $\frac{d^2\omega_x}{dx^2}$ each

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vanish for $x=0$, π and is hence the required solution. But this solution must be identical with

$$\omega_x = \sum_{n=1}^{\infty} A_n \sin nx;$$

hence we deduce the system of equations

$$A_r(r^2+\rho^2m^2)^2 + \lambda m^2 \frac{2}{\pi} c_r = 0, \tag{13}$$

where r is any positive integer.

On substituting for the c's in terms of the A's, (13) takes the form of an infinite system of equations, linear in the A's. In general the only solution of these equations is that in which all the A's vanish; but if the infinite determinant formed from the coefficients of the A's vanishes, this is no longer true. There will then exist a solution in which one of the A's can be regarded as arbitrary—say A_1—and the ratios of the other A's to A_1 are finite but not all zero. By equating to zero the above determinant, we get the required relation connecting ρm and the corresponding force which is just large enough to cause instability.

Put $$\mu = \lambda m^2 \frac{2}{\pi};$$

the determinantal equation is

$$\begin{vmatrix} \frac{(1^2+\rho^2m^2)^2}{\mu} & \frac{1}{1^2}-\frac{1}{3^2} & 0 & \frac{1}{3^2}-\frac{1}{5^2} \\ \frac{1}{1^2}-\frac{1}{3^2} & \frac{(2^2+\rho^2m^2)^2}{\mu} & \frac{1}{1^2}-\frac{1}{5^2} & 0 \\ 0 & \frac{1}{1^2}-\frac{1}{5^2} & \frac{(3^2+\rho^2m^2)^2}{\mu} & \frac{1}{1^2}-\frac{1}{7^2} \\ \frac{1}{3^2}-\frac{1}{5^2} & 0 & \frac{1}{1^2}-\frac{1}{7^2} & \frac{(4^2+\rho^2m^2)^2}{\mu} \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0. \tag{14}$$

The above, however, is only formal and has no practical significance unless the determinant is convergent. This we now consider.

Multiply each element of the determinant by μ—which is finite and not zero—and divide the rth row and the rth column by r^2. The sum of the non-diagonal elements in this new form of the determinant is less than

$$\mu \left\{ \sum_{r=1}^{\infty} \frac{1}{r^2} \right\}^2,$$

and therefore, since all the elements in the determinant are positive, the sum of the non-diagonal elements converges absolutely. Since, in addition, the product

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of the diagonal elements converges absolutely, it follows that the determinant is convergent.*

We now consider the determinantal equation (14). For any given value of ρm, it is an equation in μ having an infinite number of roots, but it is of course only with the smallest of these roots that we are actually concerned. We endeavour to find this value of μ as a function of ρm, and then having done this we find for what value of ρm it is a minimum. Since, however, μ cannot be obtained directly as a function of ρm, we approximate to it by considering the values of μ obtained from the determinantal equations in which the left-hand side contains successively the 4, 9, 16, ... elements at the top left-hand corner of (14). We consider the accuracy of the results obtained in this way after we have actually found them.

4. *Numerical evaluation.* Now K is always related to μ by

$$K=\frac{D}{a^3}\left(\frac{\pi}{2}\right)^4\frac{\mu}{\rho^2m^2},$$

so that we require μ in terms of ρm.

As a *first* approximation we get

$$\mu=\frac{9}{2}\left(1+\frac{\rho^2m^2}{1^2}\right)\left(1+\frac{\rho^2m^2}{2^2}\right),$$

from which we deduce that K has a minimum value of $61{\cdot}5\,D/a^3$ when $\rho m=\sqrt{2}$.

As a *second* approximation we get

$$\mu=\frac{\left(1+\frac{\rho^2m^2}{1^2}\right)\left(1+\frac{\rho^2m^2}{2^2}\right)\left(1+\frac{\rho^2m^2}{3^2}\right)}{\left[\frac{2^2}{3^25^4}\left(1+\frac{\rho^2m^2}{1^2}\right)^2+\frac{2^2}{1^43^4}\left(1+\frac{\rho^2m^2}{3^2}\right)\right]^{\frac{1}{2}}}.$$

The *third* approximation is more complicated; μ is the smallest root of the biquadratic equation

$$A-B\mu^2+C\mu^4=0,$$

where

$$A=\left[\left(1+\frac{\rho^2m^2}{1^2}\right)\left(1+\frac{\rho^2m^2}{2^2}\right)\left(1+\frac{\rho^2m^2}{3^2}\right)\left(1+\frac{\rho^2m^2}{4^2}\right)\right]^2,$$

$$B=\left[\left(1+\frac{\rho^2m^2}{1^2}\right)^2\left(1+\frac{\rho^2m^2}{2^2}\right)^2\frac{1}{3^44^4}\left(\frac{1}{1^2}-\frac{1}{7^2}\right)^2+\left(1+\frac{\rho^2m^2}{1^2}\right)^2\left(1+\frac{\rho^2m^2}{4^2}\right)\frac{1}{2^43^4}\left(\frac{1}{1^2}-\frac{1}{5^2}\right)^2\right.$$
$$\left.+\left(1+\frac{\rho^2m^2}{3^2}\right)^2\left(1+\frac{\rho^2m^2}{4^2}\right)^2\frac{1}{1^42^4}\left(\frac{1}{1^2}-\frac{1}{3^2}\right)^2+\left(1+\frac{\rho^2m^2}{2^2}\right)^2\left(1+\frac{\rho^2m^2}{3^2}\right)\frac{1}{1^44^4}\left(\frac{1}{3^2}-\frac{1}{5^2}\right)^2\right],$$

and

$$C=\frac{1}{1^42^43^44^4}\left[\left(\frac{1}{1^2}-\frac{1}{3^2}\right)^2\left(\frac{1}{1^2}-\frac{1}{7^2}\right)^2+\left(\frac{1}{1^2}-\frac{1}{5^2}\right)^2\left(\frac{1}{3^2}-\frac{1}{5^2}\right)^2\right.$$
$$\left.-2\left(\frac{1}{1^2}-\frac{1}{3^2}\right)\left(\frac{1}{1^2}-\frac{1}{5^2}\right)\left(\frac{1}{1^2}-\frac{1}{7^2}\right)\left(\frac{1}{3^3}-\frac{1}{5^2}\right)\right].$$

* Whittaker and Watson, *Modern Analysis* (4th ed.), ch. 2, § 2·8.

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We now evaluate Ka^3/D correct to the nearest integer, for various values of ρm; the results are given in Table I.

TABLE I. *Values of* Ka^3/D

ρm	First approximation	Second approximation	Third approximation
0·5	146	144	144
1·0	68	67	67
1·5	62	59	59
2·0	68	63	63
2·5	81	72	72

In order to find for what value of ρm Ka^3/D is a minimum, we make further calculations which are shown in Table II.

TABLE II. *Values* of* Ka^3/D

ρm	Second approximation	ρm	Third approximation
1·40	59·147	1·465	58·934
1·44	59·057	1·470	58·928
1·46	59·024	1·475	58·926
1·48	59·008	1·480	58·924
1·50	59·021	1·485	58·918
1·52	59·041	1·490	58·922
1·60	59·230	1·495	58·924

* The calculations in this Table were effected by using five figure logarithm tables. The last figure is accordingly unreliable.

From Table I we see that the second and third approximations for the value of Ka^3/D agree to the nearest integer for the range of ρm which has been considered. Since, however, the third approximation is an appreciably closer approximation than the second, we deduce that the values of Ka^3/D for the third approximation are accurate to at least three figures for values of ρm lying within the range considered.

From Table II we deduce that to three figures the accurate value of ρm for which the smallest force causing instability is a minimum is 1·48, and that the corresponding value of Ka^3/D is 58·9. The importance of this last result lies in the fact that, if K is less than $58{\cdot}9\,D/a^3$, then the plate is stable whatever the value of ρm.

Suppose now that ρ is given, and that we want to find the maximum value of K consistent with stability. For this purpose we must construct a graph connecting ρm and the critical value of K. The graph given below is an approximation to this, being accurate enough to demonstrate the method.

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Since ρ is given, ρm may take any one of the values ρ, 2ρ, 3ρ, ..., and corresponding to each of these values of ρm there exists a unique value of K, deducible from the graph below. It is with the least of these values that we are concerned, since for the given ρ it is the maximum value of K consistent with stability. As a particular case take $\rho = 0{\cdot}5$. It is clear from the figure that the value of m which gives rise to the smallest value of K is in this case 3; and that this value of K is approximately $59\,D/a^3$. As K passes this critical value, the plate will tend to buckle into three waves, the displacement for x constant being proportional to $\sin 3y$, where y goes from 0 to π. For ρ equal to one, the critical value of m is two; while for ρ greater than one and a half, the critical value of m is always one.

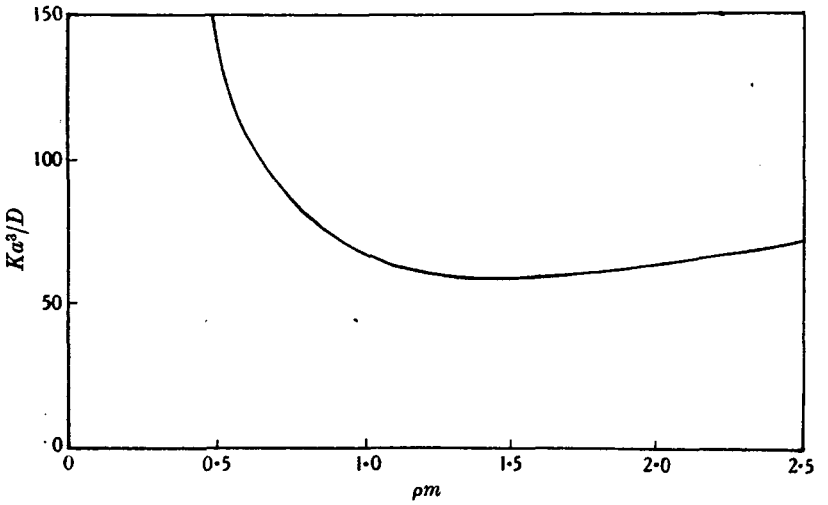


Fig. 1.

Given ρ, i.e. given the ratio of the sides of the rectangle, the critical thrust can be found from the relation, shown in Fig. 1, between K and ρm. Since m must be integral, ρm can take any one of the values ρ, 2ρ, ... Corresponding to each of these values there is a value of K, and the smallest of these values of K determines the critical thrust; the corresponding value of m shows the type of distortion that will take place when the critical thrust is just exceeded.

We now suppose that the critical thrust consistent with stability has been slightly exceeded, and endeavour to find an approximate form for the section made with the surface by y constant.

Since the results obtained from the third approximation are accurate to three figures, the ratios of the A's found by making the same assumptions are sufficiently accurate to give us the general form of the section. From (13) we get three independent equations connecting A_1, A_2, A_3, A_4; whence we have

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$$A_1 \qquad\qquad -A_2$$

$$\begin{vmatrix} \frac{(2^2+\rho^2m^2)^2}{\mu} & \left(\frac{1}{1^2}-\frac{1}{5^2}\right) & 0 \\ \left(\frac{1}{1^2}-\frac{1}{5^2}\right) & \frac{(3^2+\rho^2m^2)^2}{\mu} & \left(\frac{1}{1^2}-\frac{1}{7^2}\right) \\ 0 & \left(\frac{1}{1^2}-\frac{1}{7^2}\right) & \frac{(4^2+\rho^2m^2)^2}{\mu} \end{vmatrix} = \begin{vmatrix} \left(\frac{1}{1^2}-\frac{1}{3^2}\right) & \left(\frac{1}{1^2}-\frac{1}{5^2}\right) & 0 \\ 0 & \frac{(3^2+\rho^2m^2)^2}{\mu} & \left(\frac{1}{1^2}-\frac{1}{7^2}\right) \\ \left(\frac{1}{3^2}-\frac{1}{5^2}\right) & \left(\frac{1}{1^2}-\frac{1}{7^2}\right) & \frac{(4^2+\rho^2m^2)^2}{\mu} \end{vmatrix}$$

$$= \qquad A_3 \qquad\qquad -A_4$$

$$\begin{vmatrix} \left(\frac{1}{1^2}-\frac{1}{3^2}\right) & \frac{(2^2+\rho^2m^2)^2}{\mu} & 0 \\ 0 & \left(\frac{1}{1^2}-\frac{1}{5^2}\right) & \left(\frac{1}{1^2}-\frac{1}{7^2}\right) \\ \left(\frac{1}{3^2}-\frac{1}{5^2}\right) & 0 & \frac{(4^2+\rho^2m^2)^2}{\mu} \end{vmatrix} = \begin{vmatrix} \left(\frac{1}{1^2}-\frac{1}{3^2}\right) & \frac{(2^2+\rho^2m^2)^2}{\mu} & \left(\frac{1}{1^2}-\frac{1}{5^2}\right) \\ 0 & \left(\frac{1}{1^2}-\frac{1}{5^2}\right) & \frac{(3^2+\rho^2m^2)^2}{\mu} \\ \left(\frac{1}{3^2}-\frac{1}{5^2}\right) & 0 & \left(\frac{1}{1^2}-\frac{1}{7^2}\right) \end{vmatrix}$$

Since these equations are dependent on ρm and μ, we take the particular case of $\rho m = 1{\cdot}48$; whereupon we have

$$A_1 = \frac{-A_2}{0{\cdot}540} = \frac{A_3}{0{\cdot}089} = \frac{-A_4}{0{\cdot}0103},$$

so that the displacement is proportional to

$$\sin x - 0{\cdot}540 \sin 2x + 0{\cdot}089 \sin 3x - 0{\cdot}010 \sin 4x.$$

We now evaluate this expression for various values of x, and we get the results shown in Table III correct to two places of decimals.

TABLE III

x	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°	180°
Displacement	0·04	0·11	0·23	0·41	0·64	0·91	1·16	1·33	1·31	1·06	0·59	0·0

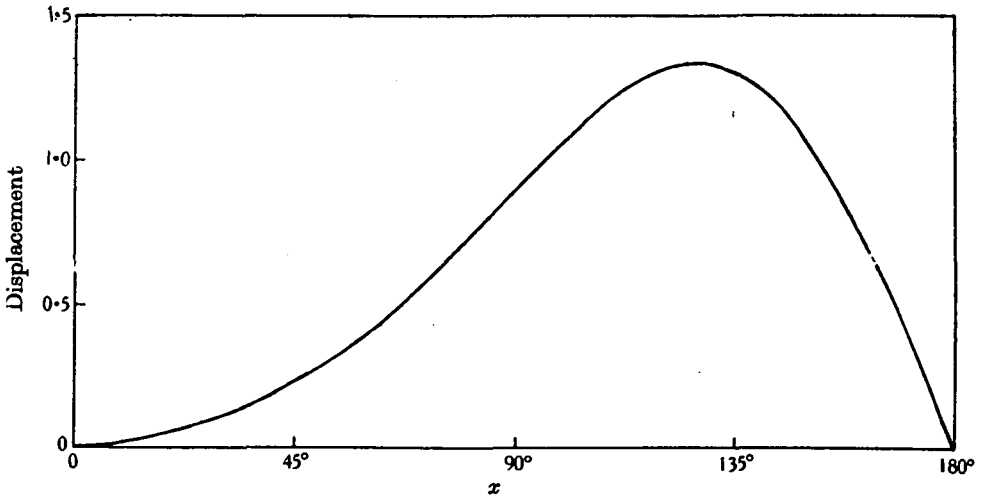


Fig. 2.

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5. CASE II. *The two edges $\eta = \pm b$ simply supported, the other two edges clamped.* Of the three fundamental equations (6), (7) and (8) in Case I, (6) and (8) are unaltered, but (7) now takes the form

$$\left.\begin{aligned} \omega &= 0, \\ \frac{\partial \omega}{\partial x} &= 0, \end{aligned}\right\} \text{ for } x = 0,\ \pi. \tag{15}$$

As in Case I, we assume that

$$\omega = \omega_x \sin my,$$

where m is some positive integer, and that ω_x can be expanded as a Fourier series of sines of the form

$$\omega_x = \sum_{n=1}^{\infty} A_n \sin nx.$$

Moreover, by reasoning similar to that in Case I, we deduce that ω_x satisfies the differential equation (11), when the c's and A's are related as in that case. The boundary conditions (8) and (15) are satisfied provided that ω_x and $\frac{d\omega_x}{dx}$ each vanish for $x = 0, \pi$. As in Case I it is now necessary to find a solution of (11) for ω_x, which satisfies the above boundary conditions; and which can be expressed as a sine series in $(0, \pi)$.

We deduce from (11) the particular integral (12), but since this does not satisfy all the boundary conditions, it is necessary to write down the general solution of (11), determine the arbitrary constants contained in it so as to satisfy all the boundary conditions, and then express the result as a sine series. The general solution of (11) is

$$\begin{aligned} \omega_x = a \sinh \rho m x + d \cosh \rho m x + x\,[b \cosh \rho m x + c \sinh \rho m x] \\ - \lambda m^2 \frac{2}{\pi} \sum_{r=1}^{\infty} \frac{c_r \sin rx}{(r^2 + \rho^2 m^2)^2}. \end{aligned} \tag{16}$$

Now, since $\Sigma\,|A_r|$ is convergent, the c's are all bounded, and we may therefore differentiate the sine series for ω_x term by term. Accordingly we have four equations from which to determine a, b, c, and d. These are

$$d = 0,$$

$$a \sinh \rho m\pi + b\pi \cosh \rho m\pi + c\pi \sinh \rho m\pi = 0, \tag{17}$$

$$a\rho m + b - \mu \sum_{r=1}^{\infty} \frac{r c_r}{(r^2 + \rho^2 m^2)^2} = 0, \tag{18}$$

and

$$\begin{aligned} b\,[\cosh \rho m\pi + \pi \rho m \sinh \rho m\pi] + c\,[\sinh \rho m\pi + \pi \rho m \cosh \rho m\pi] \\ + a\rho m \cosh \rho m\pi - \mu \sum_{r=1}^{\infty} \frac{r c_r \cos r\pi}{(r^2 + \rho^2 m^2)^2} = 0, \end{aligned} \tag{19}$$

where

$$\mu = \lambda m^2 \frac{2}{\pi}.$$

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Solving the above equations for b and c, we get

$$b = \frac{\mu\pi\rho m}{D}\sinh\rho m\pi \sum_{r=1}^{\infty}\frac{rc_r\cos r\pi}{(r^2+\rho^2m^2)^2} + \frac{\mu}{D}\sinh^2\rho m\pi \sum_{r=1}^{\infty}\frac{rc_r}{(r^2+\rho^2m^2)^2},$$

$$c = -[\pi\rho m\cosh\rho m\pi - \sinh\rho m\pi]\frac{\mu}{D}\sum_{r=1}^{\infty}\frac{rc_r\cos r\pi}{(r^2+\rho^2m^2)^2}$$

$$+[\sinh\rho m\pi\cosh\rho m\pi - \rho m\pi]\frac{\mu}{D}\sum_{r=1}^{\infty}\frac{rc_r}{(r^2+\rho^2m^2)^2},$$

where $$D = \sinh^2\rho m\pi - \rho^2m^2\pi^2.$$

Simplifying further we get

$$b = \frac{\mu\sinh\rho m\pi}{\sinh\rho m\pi+\rho m\pi}\sum_{r=1,3,\ldots}^{\infty}\frac{rc_r}{(r^2+\rho^2m^2)^2} + \frac{\mu\sinh\rho m\pi}{\sinh\rho m\pi-\rho m\pi}\sum_{r=2,4,\ldots}^{\infty}\frac{rc_r}{(r^2+\rho^2m^2)^2}, \tag{20}$$

$$c = -\frac{\mu(1+\cosh\rho m\pi)}{\sinh\rho m\pi+\rho m\pi}\sum_{r=1,3,\ldots}^{\infty}\frac{rc_r}{(r^2+\rho^2m^2)^2} + \frac{\mu(1-\cosh\rho m\pi)}{\sinh\rho m\pi-\rho m\pi}\sum_{r=2,4,\ldots}^{\infty}\frac{rc_r}{(r^2+\rho^2m^2)^2}. \tag{21}$$

We now express ω_x as a sine series. The particular integral is already so expressed, so that it remains to express in a sine series the complementary function

$$a\sinh\rho mx + x[b\cosh\rho mx + c\sinh\rho mx].$$

Writing it as $\Sigma B_r\sin rx$, we get

$$\frac{\pi}{2}B_r = \int_0^{\pi}\sin rx\,[a\sinh\rho mx + x(b\cosh\rho mx + c\sinh\rho mx)]\,dx. \tag{22}$$

Denoting the coefficients of a, b, and c on the right-hand side of (22) by I_1, I_2, and I_3, we have

$$I_1 = -\frac{r\cos r\pi\sinh\rho m\pi}{(r^2+\rho^2m^2)},$$

$$I_2 = -\frac{r\cos r\pi}{(r^2+\rho^2m^2)^2}[\pi(r^2+\rho^2m^2)\cosh\rho m\pi - 2\rho m\sinh\rho m\pi],$$

$$I_3 = -\frac{r}{(r^2+\rho^2m^2)^2}[\pi(r^2+\rho^2m^2)\cos r\pi\sinh\rho m\pi - 2\rho m(\cos r\pi\cosh\rho m\pi - 1)].$$

From (17), (22), and the above expressions for I_1, I_2, and I_3, we deduce

$$\frac{\pi}{2}B_r = \frac{2\rho mr}{(r^2+\rho^2m^2)^2}[b\cos r\pi\sinh\rho m\pi + c(\cos r\pi\cosh\rho m\pi - 1)].$$

Hence from (16) the solution for ω_x which satisfies all the boundary conditions is

$$\omega_x = \sum_{r=1}^{\infty}\frac{\sin rx}{(r^2+\rho^2m^2)^2}\left[\frac{4\rho mr}{\pi}\{b\cos r\pi\sinh\rho m\pi + c(\cos r\pi\cosh\rho m\pi - 1)\} - \mu c_r\right],$$

where b and c are given by (20) and (21).

But this solution is identical with

$$\omega_x = \sum_{n=1}^{\infty}A_n\sin nx.$$

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Therefore we have the following infinite system of equations, linear in the A's,

$$A_r(r^2+\rho^2m^2)^2+\mu c_r-\frac{4\rho mr}{\pi}\left[b\cos r\pi\sinh\rho m\pi+c\left(\cosh\rho m\pi\cos r\pi-1\right)\right]=0. \tag{23}$$

From (20) and (21) we deduce that

$$b\sinh\rho m\pi+c\cosh\rho m\pi$$
$$=-\mu\frac{\cosh\rho m\pi+1}{\sinh\rho m\pi+\rho m\pi}\sum_{r=1,3,\ldots}^{\infty}\frac{rc_r}{(r^2+\rho^2m^2)^2}$$
$$+\mu\frac{\cosh\rho m\pi-1}{\sinh\rho m\pi-\rho m\pi}\sum_{r=2,4,\ldots}^{\infty}\frac{rc_r}{(r^2+\rho^2m^2)^2}.$$

On changing the summation suffix in the above from r to s in order to avoid confusion, we deduce that

$$\left.\begin{aligned} &b\cos r\pi\sinh\rho m\pi+c\left(\cosh\rho m\pi\cos r\pi-1\right)\\ &=2\mu\frac{\cosh\rho m\pi-1}{\sinh\rho m\pi-\rho m\pi}\sum_{s=2,4,\ldots}^{\infty}\frac{sc_s}{(s^2+\rho^2m^2)^2}\quad(r\text{ even})\\ &=2\mu\frac{\cosh\rho m\pi+1}{\sinh\rho m\pi+\rho m\pi}\sum_{s=1,3,\ldots}^{\infty}\frac{sc_s}{(s^2+\rho^2m^2)^2}\quad(r\text{ odd}).\end{aligned}\right\} \tag{24}$$

Hence using (24), (23) is

$$A_r(r^2+\rho^2m^2)^2+\mu c_r-\frac{8\rho mr\mu}{\pi}\frac{\cosh\rho m\pi-1}{\sinh\rho m\pi-\rho m\pi}\sum_{s=2,4,\ldots}^{\infty}\frac{sc_s}{(s^2+\rho^2m^2)^2}=0\quad(r\text{ even}),$$

and (25)

$$A_r(r^2+\rho^2m^2)^2+\mu c_r-\frac{8\rho mr\mu}{\pi}\frac{\cosh\rho m\pi+1}{\sinh\rho m\pi+\rho m\pi}\sum_{s=1,3,\ldots}^{\infty}\frac{sc_s}{(s^2+\rho^2m^2)^2}=0\quad(r\text{ odd}). \tag{26}$$

The equations (25) and (26) correspond to the equation (13) in Case I, and the same remarks apply for obtaining a non-zero solution.

Now introduce u_r, defined by the two identities

$$\sum_{s=2,4,\ldots}^{\infty}u_sA_s\equiv\frac{8\rho m}{\pi}\frac{\cosh\rho m\pi+1}{\sinh\rho m\pi+\rho m\pi}\sum_{s=1,3,\ldots}^{\infty}\frac{sc_s}{(s^2+\rho^2m^2)^2}$$

and

$$\sum_{s=1,3,\ldots}^{\infty}u_sA_s\equiv\frac{8\rho m}{\pi}\frac{\cosh\rho m\pi-1}{\sinh\rho m\pi-\rho m\pi}\sum_{s=2,4,\ldots}^{\infty}\frac{sc_s}{(s^2+\rho^2m^2)^2};$$

then equations (25) and (26) take the form

$$A_r(r^2+\rho^2m^2)^2+\mu c_r-r\mu\sum_{s=1,3,\ldots}^{\infty}u_sA_s=0\quad(r\text{ even}), \tag{27}$$

and

$$A_r(r^2+\rho^2m^2)^2+\mu c_r-r\mu\sum_{s=2,4,\ldots}^{\infty}u_sA_s=0\quad(r\text{ odd}). \tag{28}$$

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Eliminating the A's from (27) and (28), we get the infinite determinantal equation

$$\begin{vmatrix} \frac{(1^2+\rho^2m^2)^2}{\mu} & \left(\frac{1}{1^2}-\frac{1}{3^2}\right)-u_2 & 0 & \left(\frac{1}{3^2}-\frac{1}{5^2}\right)-u_4 \\ \left(\frac{1}{1^2}-\frac{1}{3^2}\right)-2u_1 & \frac{(2^2+\rho^2m^2)^2}{\mu} & \left(\frac{1}{1^2}-\frac{1}{5^2}\right)-2u_3 & 0 \\ 0 & \left(\frac{1}{1^2}-\frac{1}{5^2}\right)-3u_2 & \frac{(3^2+\rho^2m^2)^2}{\mu} & \left(\frac{1}{1^2}-\frac{1}{7^2}\right)-3u_4 \\ \left(\frac{1}{1^2}-\frac{1}{5^2}\right)-4u_1 & 0 & \left(\frac{1}{1^2}-\frac{1}{7^2}\right)-4u_3 & \frac{(4^2+\rho^2m^2)^2}{\mu} \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \quad (29)$$

As in Case I, however, this equation is only formal and has no practical significance unless it is convergent.

In order to consider this, it is first necessary to rearrange the determinant. Multiply each element by μ, and divide the rth row by r^4, where r has every value from unity upwards. Then from the definition of the u's it can be easily shown that the u's are positive, and that Σu_r is convergent—say to a sum C, a function of ρm. The sum of the moduli of the non-diagonal elements in the rearranged determinant is less than

$$\mu\left\{\sum_{r=1}^{\infty}\frac{1}{r^4}\right\}\left\{2\sum_{r=1}^{\infty}\frac{1}{r^2}\right\}+\mu C\left\{\sum_{r=1}^{\infty}\frac{1}{r^3}\right\}$$

and is therefore convergent. Moreover, since the product of the diagonal elements converges absolutely, it follows that the determinant is convergent.

6. *Numerical evaluation.* Table IV gives the values of the first four u's evaluated for various values of ρm, while Table V gives the corresponding values of Ka^3/D obtained from the second and third approximations.

TABLE IV. *Values of the u's*

ρm	u_1	u_2	u_3	u_4
0·5	0·2620	0·6962	0·2795	0·1014
1·0	0·2316	0·5494	0·2914	0·1196
1·5	0·1904	0·3858	0·2665	0·1324
2·0	0·1484	0·2658	0·2090	0·1319
2·5	0·1124	0·1889	0·1701	0·1224

TABLE V. *Values of Ka^3/D*

ρm	Second approximation	Third approximation
0·5	519·4	514·1
1·0	166·3	165·0
1·5	110·2	109·8
2·0	98·52	98·70
2·5	100·4	100·8

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From Table V we make the same deductions about accuracy as in Case I, except for $\rho m = 0{\cdot}5$. In this case we can state that the third approximation is accurate to within 0·5 %.

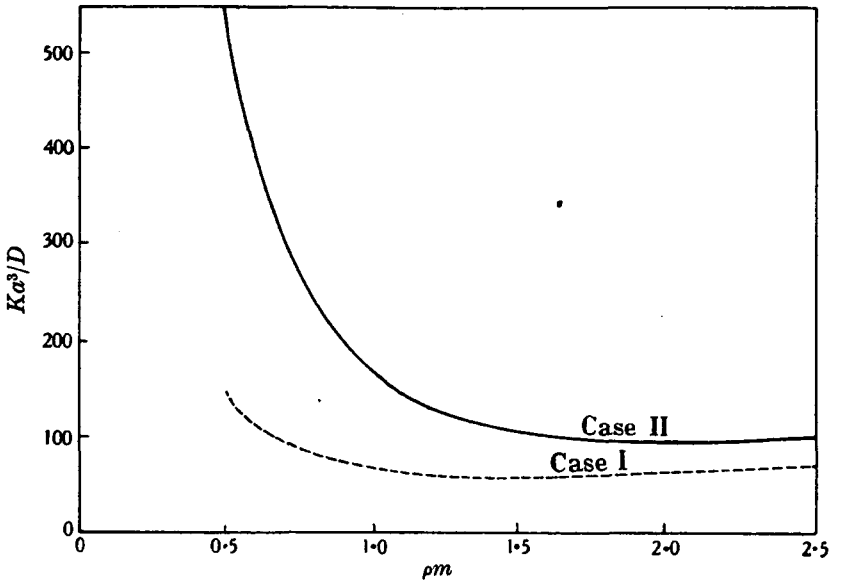


Fig. 3.

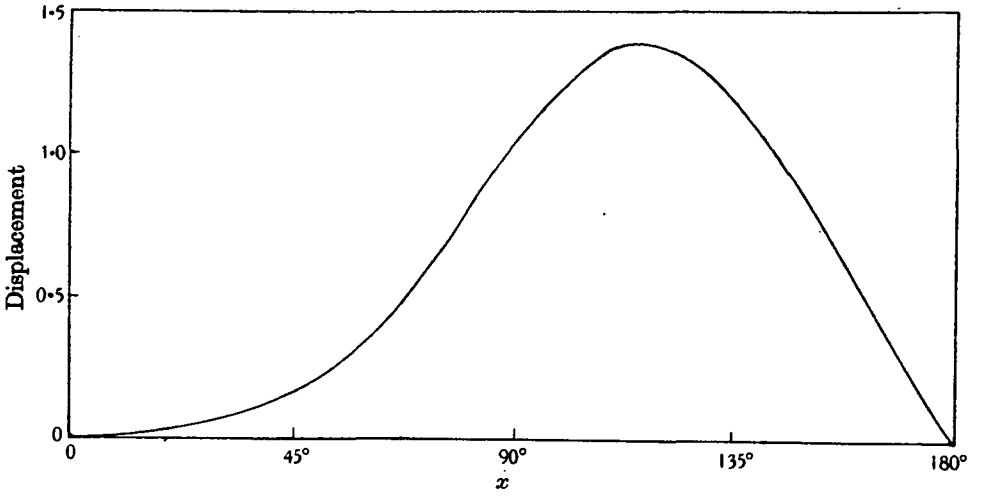


Fig. 4.

We now suppose that the plate has suffered a distortion owing to the critical thrust for $\rho m = 1{\cdot}5$ having been slightly exceeded, and we proceed to find its form for the section by the plane x constant. Using a similar method to that of Case I, we deduce that the displacement is roughly proportional to

$$\sin x - 0{\cdot}508 \sin 2x - 0{\cdot}048 \sin 3x + 0{\cdot}079 \sin 4x.$$

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TABLE VI

x	15°	30°	45°	60°	75°	90°	105°	120°	135°	150°	165°
Displacement	0·04	0·08	0·17	0·36	0·68	1·05	1·32	1·37	1·18	0·82	0·41

It should be noted that the above is only approximate and not sufficiently accurate to be differentiated without considerable error. We deduce, however, that the actual form is as shown in Fig. 4.

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