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On operators which depend on some parameter

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Abstract. The purpose of this paper is to study the mapping properties of certain operators which depend on a variable parameter. We prove a general theorem which is applied to some special cases. Among results obtained are new inequalities involving the Poisson operator and its conjugate.

1. *Introduction.* Given a measure space (S, σ) we denote by $L^p(S)$ the space of complex valued function f such that

$$\|f\|_{L^p(S)} = \left(\int_S |f(t)|^p \, d\sigma\right)^{1/p} < \infty.$$

(When there is no danger of ambiguity we shall denote $\|f\|_{L^p(S)}$ by $\|f\|_p$.) The numbers p and p' will be connected by $1/p + 1/p' = 1$. The n-dimensional Euclidean space will be denoted by E_n, and we shall write L^p for $L^p(E_n)$.

A large class of operators on L^p are either of convolution type: that is defined by

$$T(f)(x) = (f * g)(x) = \int_{E_n} g(x-t) f(t) \, dt$$

for some measurable function g, or are defined by kernels which are 'homogeneous' in some sense. It is well known that the inequality of Young is adequate for dealing with some of those defined by convolution. Also, for dealing with some integrals with homogeneous kernels we have the inequality of Schur, and a recent extension of this inequality given in (4). The purpose of this paper is to describe a method for studying certain integrals which cannot be treated by the above inequalities. To do this we exploit their dependence on some variable parameter.

It is usual to study operators of this form by introducing the norm $\|f(x,h)\|_{pq}$ defined by

$$\|f(x,h)\|_{pq} = \left(\int_{E_m} \|f(x,h)\|_p^q \, dh/|h|^m\right)^{1/q},$$

where f is a complex-valued measurable function on $E_n \times E_m$. For example, see Hörmander (3) and also Taibleson (7). Although results in this paper will at some stage involve norms of this form, we shall be primarily interested in the boundedness of operators on L^p spaces. In general we shall, by integrating with respect to the parameter on which the operator depends, be able to obtain an operator to which the above inequalities can be applied. We shall apply the general result to certain special

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cases among which are extensions of the Poisson operator and its conjugate. Some cases of these results will generalize some well-known inequalities.

2. *The main theorem.* The general theorem introduced in this section will depend mainly on the continuous form of Minkowski's inequality. We shall consider two measure spaces (S_1, σ_1) and (S_2, σ_2).

LEMMA 1. *Let $1 \leqslant p \leqslant \infty$ and let ψ be a complex-valued $\sigma_2 \times \sigma_1$-measurable function on $S_2 \times S_1$. Then*

$$\left(\int_{S_2}\left|\int_{S_1}\psi(x,t)\,d\sigma_1\right|^p d\sigma_2\right)^{1/p} \leqslant \int_{S_1}\left(\int_{S_2}|\psi(x,t)|^p\,d\sigma_2\right)^{1/p} d\sigma_1.$$

For a proof of this result see, for example, Theorem 202 of (2).

THEOREM 1. *Let I be a measurable subset of some measure space, let $\{\chi_\lambda, \lambda \in I\}$ be a class of complex-valued measurable functions on $S_2 \times S_1$, and let*

$$\left(\int_I |\chi_\lambda(x,t)|^s\,d\lambda\right)^{1/s} \leqslant \rho_s(x,t)$$

for almost all $(x,t) \in S_2 \times S_1$, where $1 \leqslant s < \infty$. Let $\{T_\lambda, \lambda \in I\}$ be a class of operators defined by

$$T_\lambda(f)(x) = \int_{S_1} f(t)\,\chi_\lambda(x,t)\,d\sigma_1, \quad (x \in S_2),$$

and let $0 \leqslant r/s \leqslant 1$. Then

$$\left(\int_I \|T_\lambda(f)\|_r^s\,d\lambda\right)^{1/s} \leqslant \left\|\int_{S_1}|f(t)|\,\rho_s(x,t)\,d\sigma_1\right\|_r.$$

Proof. It follows from Lemma 1 that

$$\begin{aligned}\left(\int_I |T_\lambda(f)(x)|^s\,d\lambda\right)^{1/s} &= \left(\int_I\left|\int_{S_1} f(t)\,\chi_\lambda(x,t)\,d\sigma_1\right|^s d\lambda\right)^{1/s}\\ &\leqslant \int_{S_1}\left(\int_I |f(t)|^s|\chi_\lambda(x,t)|^s\,d\lambda\right)^{1/s} d\sigma_1\\ &\leqslant \int_{S_1}|f(t)|\rho_s(x,t)\,d\sigma_1,\end{aligned}$$

so that $$\left\{\int_{S_2}\left(\int_I |T_\lambda(f)(x)|^s\,d\lambda\right)^{r/s} d\sigma_2\right\}^{1/r} \leqslant \left\|\int_{S_1}|f(t)|\,\rho_s(x,t)\,d\sigma_1\right\|_r.$$

Now if $0 \leqslant r/s \leqslant 1$, we have, by Lemma 1,

$$\begin{aligned}\int_{S_2}\left(\int_I |T_\lambda(f)(x)|^s\,d\lambda\right)^{r/s} d\sigma_2 &= \int_{S_2}\left(\int_I T_\lambda(f)(x)|^{r(s/r)}\,d\lambda\right)^{r/s} d\sigma_2\\ &\geqslant \left(\int_I\left|\int_{S_2}|T_\lambda(f)(x)|^r\,d\sigma_2\right|^{s/r} d\lambda\right)^{r/s}\\ &= \left(\int_I \|T_\lambda(f)\|_r^s\,d\lambda\right)^{r/s}.\end{aligned}$$

This gives the required result.

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We shall be mainly interested in cases of Theorem 1 for which $I = (0, \infty)$, and where $\rho_s(x,t)$ defines a bounded operator. In the examples given here we shall also use the fact that some modification of the kernel $\chi_\lambda(x,t)$ is a non-increasing function of the parameter λ.

COROLLARY 1·1. *Let $\{\chi_\lambda, \lambda \in (0,\infty)\}$ be a class of complex-valued measurable functions on $S_2 \times S_1$ such that*

(i) $\left(\int_0^\infty |\chi_\lambda(x,t)|^s d\lambda\right)^{1/s} \leqslant \rho_s(x,t)$ *for almost all* $(x,t) \in S_2 \times S_1$, *where* $1 \leqslant s < \infty$, *and* ρ_s *is measurable on* $S_2 \times S_1$;

(ii) *for some number* $\tau < 1/s$ *and for almost all* $(x,t) \in S_2 \times S_1$, $\lambda^\tau \chi_\lambda(x,t)$ *is a non-increasing function of* λ;

(iii) *there is a constant* $A_{p,r,s}$ *such that*

$$\left\| \int_{S_1} |f(t)| \rho_s(x,t)\, d\sigma_1 \right\|_r \leqslant A_{p,r,s} \|f\|_p \quad (f \in L^p(S_1)).$$

Let $\{T_\lambda, \lambda \in (0,\infty)\}$ be a class of operators defined by

$$T_\lambda(f)(x) = \int_{S_1} f(t)\, \chi_\lambda(x,t)\, d\sigma_1, \quad (x \in S_2),$$

and let $0 \leqslant r/s \leqslant 1$. *Then*

$$\|T_\lambda(f)\|_r \leqslant \lambda^{-(1/s)} B_{p,r,s} \|f\|_p, \quad (f \in L^p(S_1)),$$

where $\lambda > 0$ *and* $B_{p,r,s} = A_{p,r,s}(1-\tau s)^{1/s}$.

Proof. We consider the operator T_λ^+ defined by

$$T_\lambda^+(f)(x) = \int_{S_1} |\chi_\lambda(x,t)|\, |f(t)|\, dt.$$

It is obvious that $|T_\lambda(f)(x)| \leqslant T_\lambda^+(f)(x)$.

Now it follows from Theorem 1 that, for $f \in L^p(S_1)$,

$$\left(\int_0^\infty \|T_\mu^+(f)\|_r^s\, d\mu\right)^{1/s} \leqslant A_{p,r,s} \|f\|_p.$$

Let $\lambda > 0$ be fixed. Then, since $\mu^\tau T_\mu^+(f)$ is a non-increasing function of μ, we have

$$\begin{aligned} \|\lambda^\tau T_\lambda^+(f)\|_r \left(\int_0^\lambda \mu^{-\tau s} d\mu\right)^{1/s} &\leqslant \left(\int_0^\lambda \|T_\mu^+(f)\|_r^s\, d\mu\right)^{1/s} \\ &\leqslant A_{p,r,s} \|f\|_p, \end{aligned}$$

and the required result follows.

3. *Extensions of the Poisson operators and its conjugate.* We shall first consider certain extensions of the Poisson operator and its conjugate in one-dimension, and then treat the n-dimensional case of the Poisson operator. We define the operators $P_{a,s}^{(\nu,\beta)}$ and $Q_{a,s}^{(\nu,\beta)}$ by

$$P_{a,s}^{(\nu,\beta)}(f)(x) = \pi^{-1} |x|^{\beta-\nu-(1/s)} \int_{-\infty}^{\infty} \frac{a|t|^\nu}{a^2 + (t-x)^2} f(t)\, dt,$$

$$Q_{a,s}^{(\nu,\beta)}(f)(x) = \pi^{-1} |x|^{\beta-\nu-(1/s)} \int_{-\infty}^{\infty} \frac{(t-x)|t|^\nu}{a^2 + (t-x)^2} f(t)\, dt.$$

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We denote $P_{a,s}^{(0,0)}$ by $P_{a,s}$, and $Q_{a,s}^{(0,0)}$ by $Q_{a,s}$, so that the Poisson operator P_a and its conjugate Q_a correspond respectively to $P_{a,\infty}$ and $Q_{a,\infty}$. We note that

$$P_{a,s}(f)(x) = |x|^{-(1/s)} P_a(f)(x),$$

and similarly for $Q_{a,s}$.

In the proof of results concerning these operators we shall require results involving the fractional integral $K_{\alpha,\beta}^{(\nu)}$ defined in the one-dimensional case by

$$K_{\alpha,\beta}^{(\nu)}(f)(x) = |x|^{\beta-\nu-\alpha} \int_{-\infty}^{\infty} f(t)\,|t|^{\nu}\,|t-x|^{\alpha-1}\,dt.$$

LEMMA 2. *Let*

$$0 \leqslant \beta < 1/p, \quad 1/q = (1/p)-\beta, \quad p > 1,$$

$$(1/p)-\nu-\alpha > 0, \quad \alpha > \beta, \quad \nu+1-(1/p) > 0.$$

Then there is a constant $k_{p,\alpha,\beta,\nu}$ such that

$$\|K_{\alpha,\beta}^{(\nu)}(f)\|_q \leqslant k_{p,\alpha,\beta,\nu}\|f\|_p.$$

A proof of this lemma is given in (4), where the result is obtained as a special case of an extension of the inequality of Schur. It is shown there that

$$k_{p,\alpha,\beta,\nu} = \left(\int_{-\infty}^{\infty} |t|^{(\beta+\nu-(1/p))/(1-\beta)}|t-1|^{(\alpha-1)/(1-\beta)}dt\right)^{(1-\beta)}$$

THEOREM 2. *Suppose that*

$$0 \leqslant \beta < 1/p, \quad 1/q = (1/p)-\beta \geqslant (1/s), \quad \beta < 1/(2p)$$

$$(1/p)-\nu > (1/s) > \beta, \quad \nu > (1/p)-1, \quad a > 0.$$

Then there are constants $A_{p,s,\beta,\nu}$ and $A'_{p,s,\beta,\nu}$ such that

(i) $\|Q_{(\nu,\beta)}^{a,s}(f)\|_q \leqslant a^{-(1/s)} A_{p,s,\beta,\nu}\|f\|_p$,

(ii) $\|P_{a,s}^{(\nu,\beta)}(f)\|_q \leqslant a^{-(1/s)} A'_{p,s,\beta,\nu}\|f\|_p$.

Proof. In the first case we set

$$\chi_a(x,t) = \pi^{-1}|x|^{\beta-\nu-(1/s)}|t|^{\nu}(t-x)\{a^2+(t-x)^2\}^{-1}.$$

Then

$$\left(\int_0^{\infty} |\chi_a(x,t)|^s\,da\right)^{1/s} = A_s|x|^{\beta-\nu-(1/s)}|t|^{\nu}|t-x|^{(1/s)-1} = \rho_s(x,t),$$

where

$$A_s = \pi^{-1}\left(\int_0^{\infty}(a^2+1)^{-s}\,da\right)^{1/s}.$$

Since $|\chi_a(x,t)|$ is a non-increasing function of a, the result (i) follows from Corollary 1·1 and Lemma 2 with $\alpha = (1/s)$.

The result (ii) follows similarly. We set

$$\chi_a(x,t) = \pi^{-1}|x|^{\beta-\nu-(1/s)}|t|^{\nu}\,a\{a^2+(t-x)^2\}^{-1},$$

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and observe that

$$\left(\int_0^\infty |\chi_a(x,t)|^s da\right)^{1/s} = A_s'|x|^{\beta-\nu-(1/s)}|t|^\nu|t-x|^{(1/s)-1}$$
$$= \rho_s(x,t),$$

where
$$A_s' = \pi^{-1}\left(\int_0^\infty a^s(a^2+1)^{-s}da\right)^{1/s}.$$

In this case $a^{-1}|\chi_a(x,t)|$ is a non-increasing function of a.

COROLLARY 2·1. *Let $1 < p < s < \infty$. Then there are constants $A_{p,s}$ and $A'_{p,s}$ such that*

(i) $\||x|^{-(1/s)}Q_a(f)(x)\|_p \leqslant a^{-(1/s)}A_{p,s}\|f\|_p, \quad (a>0),$

(ii) $\||x|^{-(1/s)}P_a(f)(x)\|_p \leqslant a^{-(1/s)}A'_{p,s}\|f\|_p, \quad (a>0).$

The well-known result

$$\|P_a(f)\|_p \leqslant \|f\|_p$$

follows easily from Young's inequality. Also, the result

$$\|Q_a(f)\|_p \leqslant A_{p,\infty}\|f\|_p$$

follows from Riesz's theorem on conjugate functions; see Chapter IV of (6).

Remarks. (i) We observe that the restriction $p < s$ of Corollary 2·1 is best possible since there is a function $f_1(t) \in L^p$ such that

$$|x|^{-(1/s)}P_a(f_1)(x) \notin L^p \quad \text{for} \quad 0 < s \leqslant p.$$

In fact if we choose $f_1(t) = b(b^2+t^2)^{-1}$, $b > 0$, then it is well known that

$$P_a(f_1)(x) = (a+b)\{(a+b)^2+x^2\}^{-1},$$

and it is easy to verify that

$$\||x|^{-(1/s)}P_a(f_1)(x)\|_p$$

is not finite if $0 < s \leqslant p$.

(ii) On putting $\beta = 0$, $\nu = -1/s$ and p' for p in Theorem 2, we deduce that if $\max(p,p') < s < \infty$, then

$$\|Q_a\{|t|^{-(1/s)}f(t)\}\|_p \leqslant A^*_{p,s}a^{-(1/s)}\|f\|_p.$$

Hence by applying Fubini's theorem to the integral

$$\int_{-\infty}^\infty g(t)\,Q_{a,s}(f)(t)\,dt \quad (f \in L^p, g \in L^{p'}),$$

we verify that the adjoint of $Q_{a,s}$ is the operator $Q^*_{a,s}$ defined by

$$Q^*_{a,s}(f)(x) = -Q_a\{|t|^{-(1/s)}f(t)\}(x).$$

We prove similarly that

$$P^*_{a,s}(f)(x) = P_a\{|t|^{-(1/s)}f(t)\}(x).$$

The n-dimensional case. We shall now obtain an n-dimensional analogue of Theorem 2. In fact, we shall obtain results involving mappings between Euclidean spaces of different dimensions. This will depend on a similar result for the operator $K^{(\nu)}_{\alpha,\beta}$.

In each of the results that follow, we shall regard the Euclidean space of lower dimension as a subspace of the one of higher dimension, so that if $1 \leqslant s \leqslant t$ E_s will be regarded as a subspace of E_t, and $x \in E_s$ will imply that $x = (x_1, \ldots x_s, 0, 0) \in E_t$.

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The n-dimensional case of $K^{(\nu)}_{\alpha,\beta}$ is given by

$$K^{(\nu)}_{\alpha,\beta}(f)(x) = |x|^{(\beta-\nu-\alpha)n}\int_{E_n}|t|^{n\nu}|t-x|^{(\alpha-1)n}f(t)\,dt \quad (x\in E_m).$$

LEMMA 3. *Let*

$$m\geqslant 1,\quad n\geqslant 1,\quad p>1,\quad 1/q=(n/m)\{(1/p)-\beta\}>0,\quad 1/p\geqslant 1/q,$$

$$1-(1/p)+\nu>0,\quad \alpha>\beta>1-(m/n),\quad 0<\alpha<1,\quad 1/p>\alpha+\nu.$$

Then there is a constant $k_{m,n,p,\alpha,\beta,\nu}\equiv k$ *such that*

$$\|K^{(\nu)}_{\alpha,\beta}(f)\|_q\leqslant k\|f\|_p.$$

Proof. This result has been proved by Cotlar and Ortiz (1). The special case in which $\alpha-1+(1/np')+(1/m)\{(1/p)-\beta\}>0$ can be deduced from an n-dimensional analogue of the extension of Schur's inequality. The case $m=n$ was proved by Stein and Weiss (5).

We define the n-dimensional case of the operator $P^{(\nu,\beta)}_{a,s}$ by

$$P^{(\nu,\beta)}_{a,s}(f)(x) = c_n^{-1}|x|^{n(\beta-\nu)-(1/s)}\int_{E_n}\frac{a|t|^{\nu n}}{(a^2+|t-x|^2)^{\frac{1}{2}(n+1)}}f(t)\,dt,\ (x\in E_m),$$

where $$c_n=\pi^{\frac{1}{2}(n+1)}/\Gamma(\tfrac{1}{2}(n+1)).$$

Now if $$\chi_a(x,t)=c_n^{-1}|x|^{n(\beta-\nu)-(1/s)}|t|^{\nu n}a(a^2+|t-x|^2)^{-\frac{1}{2}(n+1)},$$

we have $$\left(\int_0^\infty|\chi_a(x,t)|^s\,da\right)^{1/s}=A_{a,n,s}|x|^{n(\beta-\nu)-(1/s)}|t|^{\nu n}|t-x|^{(1/s)-n},$$

where $$A_{a,n,s}=c_n^{-1}\left(\int_0^\infty a^s(a^2+1)^{-\frac{1}{2}(n+1)s}\,da\right)^{1/s}.$$

Hence on applying Corollary 1·1 and Lemma 3 with $n\alpha=1/s$, and observing that $a^{-1}|\chi_a(x,t)|$ is a non-increasing function of a, we obtain

THEOREM 3. *Let* $a>0$, $m\geqslant 1$, $n\geqslant 1$, $p>1$, $1/q=(n/m)\{(1/p)-\beta\}\geqslant(1/s)>0$, $1/p\geqslant 1/q, \nu+1-(1/p)>0$, $1/p>1/(ns)+\nu$, $1/(ns)\geqslant\beta>1-(m/n)$. *Then there is a constant* $B_{m,n,p,s,\beta,\nu}=B$ *such that*

$$\|P^{(\nu,\beta)}_{a,s}(f)\|_q\leqslant Ba^{-(1/s)}\|f\|_p.$$

Note. The conditions of the theorem imply that $n\leqslant m$.

4. *A remark on harmonic functions.* We define the space $E_{n+1,+}$ by

$$E_{n+1,+}=\{(x,y):x\in E_n, y\in E_1, y>0\}.$$

It has been proved in (6) (see (7) for other references) that a harmonic function $f(x,y)$ on $E_{n+1,+}$ satisfying

$$\|f(x,y)\|_p\leqslant M,\quad \|f(x,y_1)-f(x,y_2)\|_p\to 0\quad\text{as}\quad y_1\to 0,\quad y_2\to 0,$$

can be represented as the Poisson transform of a function $f\in L^p$. Using this result and the special case $m=n$, $\nu=\beta=0$ of Theorem 3, we obtain

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THEOREM 4. *Let $1 < p < \infty$ and let $f(x, y)$ be a harmonic function on $E_{n+1,+}$ satisfying*

(i) $\|f(x,y)\|_p \leqslant M$, (ii) $\|f(x,y_1)-f(x,y_2)\|_p \to 0$ as $y_1 \to 0$, $y_2 \to 0$.

Then for $p < s < \infty$, $y > 0$, there is a constant $A_{p,s}$ such that

$$\| |x|^{-(1/s)} f(x,y)\|_p \leqslant y^{-(1/s)} A_{p,s} M.$$

5. *The Weierstrass kernel.* As a further example we consider integrals involving the Weierstrass kernel. We define

$$W_{a,s}^{(\nu,\beta)}(f)(x) = (\pi a)^{-\frac{1}{2}} |x|^{\beta-\nu-(2/s)} \int_{-\infty}^{\infty} |t|^{\nu} e^{-(x-t)^2/a} f(t)\, dt \quad (a > 0).$$

If
$$\chi_a(x,t) = \pi^{-\frac{1}{2}} |x|^{\beta-\nu-(2/s)} |t|^{\nu} a^{-\frac{1}{2}} e^{-(x-t)^2/a},$$

we have
$$\left(\int_0^{\infty} |\chi_a(x,t)|^s da\right)^{1/s} = \pi^{-\frac{1}{2}} |x|^{\beta-\nu-(2/s)} |t|^{\nu} \left(\int_0^{\infty} a^{-\frac{1}{2}s} e^{-(x-t)^2 s/a} da\right)^{1/s}$$
$$= |x|^{\beta-\nu-(2/s)} |t|^{\nu} |x-t|^{(2/s)-1} A_s$$
$$= \rho_s(x,t),$$

where the constant A_s is finite if $\frac{1}{2}s > 1$.

Since, by Lemma 2, $\rho_s(x,t)$ defines a bounded operator from L^p to L^q

$$(1/q = (1/p)-\beta) \quad \text{if} \quad 0 \leqslant \beta < 1/p, \quad p > 1, \quad (1/p)-\nu-(2/s) > 0,$$
$$(2/s)-\beta > 0, \quad \nu+1-(1/p) > 0,$$

it follows from Theorem 1 that, under these conditions,

$$\left(\int_0^{\infty} \| W_{a,s}^{(\nu,\beta)}(f)\|_q^s da\right)^{1/s} \leqslant A_s k_{p,s,\beta,\nu} \|f\|_p, \quad (0 < 1/s \leqslant 1/q).$$

We observe, however, that Corollary 1·1 is not directly applicable in this case since condition (ii) is not satisfied. On putting $1/a$ for a in the integral on the left-hand side of the above inequality, we obtain

$$\left(\int_0^{\infty} a^{-2} \| W_{1/a,s}^{(\nu,\beta)}(f)\|_q^s da\right)^{1/s} \leqslant A_s k_{p,s,\beta,\nu} \|f\|_p.$$

Now the kernel of $a^{-2/s} W_{1/a,s}^{(\nu,\beta)}$ is

$$a^{-2/s} \chi_{1/a}(x,t) = \pi^{-\frac{1}{2}} |x|^{\beta-\nu-(2/s)} |t|^{\nu} a^{\frac{1}{2}-(2/s)} e^{-(x-t)^2 a},$$

so that if $\tau + \frac{1}{2} - (2/s) \leqslant 0$, then $a^{\tau-(2/s)} \chi_{1/a}(x,t)$ defines a non-increasing function of a. If we also have $\tau < 1/s$, then it follows by proceeding as in the proof of Corollary 1·1 that

$$a^{-(2/s)} \| W_{1/a,s}^{(\nu,\beta)}(f)\|_q \leqslant a^{-(1/s)} B \|f\|_p.$$

Now we can always choose τ to be less than $\min\{(2/s)-\frac{1}{2}, (1/s)\}$. Hence on putting $1/a$ for a in the last inequality we obtain:

THEOREM 5. *Let $0 \leqslant \beta < 1/p$, $1/q = (1/p)-\beta \geqslant (1/s), p > 1$, $(1/p)-\nu-(2/s) > 0$, $(2/s)-\beta > 0, \nu+1-(1/p) > 0, a > 0$. Then there is a constant $B_{p,s,\beta,\nu} \equiv B$ such that*

$$\| W_{a,s}^{(\nu,\beta)}(f)\|_q \leqslant a^{-(1/s)} B \|f\|_p.$$

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Let W_a denote the Weierstrass operator; i.e.

$$W_a(f)(x) = (\pi a)^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t)\, e^{-(x-t)^2/a}\, dt.$$

Then we have

COROLLARY 5·1. *Let* $1 < p < (\frac{1}{2}s) < \infty$, $a > 0$. *Then there are constants* $A_{p,s}$ *and* $A^*_{p,s}$ *such that*

(i) $\| |x|^{-(2/s)} W_a(f)(x) \|_p \leqslant a^{-(1/s)} A_{p,s} \|f\|_p$,

(ii) $\| W_a\{|t|^{-(2/s)} f(t)\} \|_{p'} \leqslant a^{-(1/s)} A^*_{p,s} \|f\|_p$, $\quad (s \geqslant p')$.

Proof. The result (i) is the special case $\beta = \nu = 0$ of Theorem 5; (ii) follows by putting $\beta = 0$, $\nu = -2/s$, and p' for p.

The n-dimensional case of the operator $W^{(\nu,\beta)}_{a,s}$ is defined by

$$W^{(\nu,\beta)}_{a,s}(f)(x) = (\pi a)^{-\frac{1}{2}n} \int_{E_n} f(t)\, e^{-(x-t)^2/a}\, dt \quad (x \in E_m).$$

On proceeding as in the proof of the one-dimensional case and using Lemma 3, we obtain

THEOREM 6. *Let* $m \geqslant 1$, $n \geqslant 1$, $p > 1$, $1/q = (n/m)\{(1/p) - \beta\} \geqslant (1/s) > 0$, $1/p \geqslant 1/q$, $\nu + 1 - (1/p) > 0$, $1/p > 2/(ns) + \nu$, $2/(ns) \geqslant \beta > 1 - (m/n)$, $a > 0$. *Then there is a constant* $B_{m,n,p,s,\beta,\nu} \equiv B$ *such that*

$$\| W^{(\nu,\beta)}_{a,s}(f) \|_q \leqslant a^{-(1/s)} B \|f\|_p.$$

Note. The conditions of the theorem imply that $n \leqslant m + 1$.

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