Reference

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100.34 Tilings from a square template with asymmetric edges

Question: How many distinct shapes that tile the plane can be formed on a square template with identical asymmetric edges?

Shapes formed on a square template with identical asymmetric edges can tessellate only if they have two edges that are 'ins' and two edges that are 'outs'. Further, through rotations or flips, any shape with two 'in' edges and two 'out' edges can be arranged so that an 'out' will be in a chosen fixed position. With one edge fixed, there are three ways to position the other 'out' edge and for each position it can have two orientations. Once the second 'out' edge is fixed, there are two slots left for 'in' edges. Each of them can take two orientations. Thus there are $3 \times 2 \times 2 \times 2 = 24$ shapes that need to be considered. These 24 shapes are shown as puzzle pieces in Figure 1, with sides also distinguished by letters. To answer the question we need to determine whether each shape tessellates and to eliminate duplicates.

One way to determine which of the 24 shapes tessellate is, by trial and error, to fit copies of each shape together, which was the author's initial approach. A more elegant alternative is to analyse each shape to see if it satisfies conditions of a type known to tessellate. All tiles with shapeable edges that tile isohedrally, that is, edges fit together in only one way throughout a periodic tiling, belong to one of 28 types, named after mathematician Heinrich Heesch (1906-1996). Eleven of them are quadrilateral types but only five play a role in the following discussion, those that are limited to translations, glide reflections, and corner rotations of 90°. (Corner rotations of 60° and 120° do not form squares and centre-point rotations have symmetry so are not used as edges for this question.) A TTTT type translates or moves the top edge to the bottom and the left edge to the right edge. A TGTG type translates one pair of opposite edges while the other pair are flipped versions of each other. In a G1G2G1G2 type both opposite pairs contain edges that are flipped versions of one another, while a G1G1G2G2 type flips one edge and rotates it to form an adjacent edge and does the same for the other adjacent pair. Finally, in a C4C4C4C4 type one edge is rotated 90° to form an adjacent edge and the other two edges are formed in the same way. If a shape can be shown to fit one of these types, it must tessellate.

To determine if a shape fits one of these Heesch types, there are six isometries that relate edges.

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A vertical glide-reflection interchanges top and bottom edges: $a \leftrightarrow b$, $c \leftrightarrow d$.

A horizontal glide-reflection interchanges left-right edges: $e \leftrightarrow f$, $g \leftrightarrow h$.

A 90° clockwise rotation transforms edges as follows: $a \rightarrow g, b \rightarrow h$, $c \rightarrow e, d \rightarrow f, e \rightarrow b, f \rightarrow a, g \rightarrow d, h \rightarrow c$.

A 90° counterclockwise rotation transforms edges as follows: $a \rightarrow f$, $b \rightarrow e, c \rightarrow h, d \rightarrow g, e \rightarrow c, f \rightarrow d, g \rightarrow a, h \rightarrow b$.

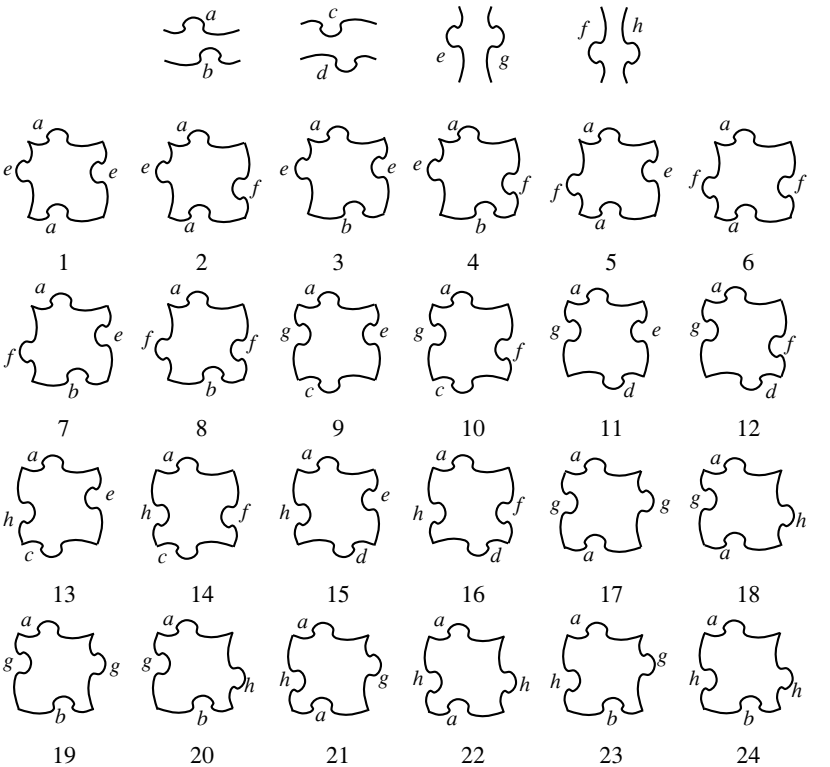


FIGURE 1

A diagonal glide reflection from bottom to right edge or from left edge to top interchanges adjacent sides: $a \leftrightarrow h, b \leftrightarrow g, c \leftrightarrow f, d \leftrightarrow e$.

A diagonal glide reflection from bottom to left edge or from right edge to top interchanges adjacent sides: $a \leftrightarrow e, b \leftrightarrow f, c \leftrightarrow g, d \leftrightarrow h$.

From these we can identify a Heesch type, though some tiles fit more than one type. Those that fit no Heesch type (NH) will be examined later.

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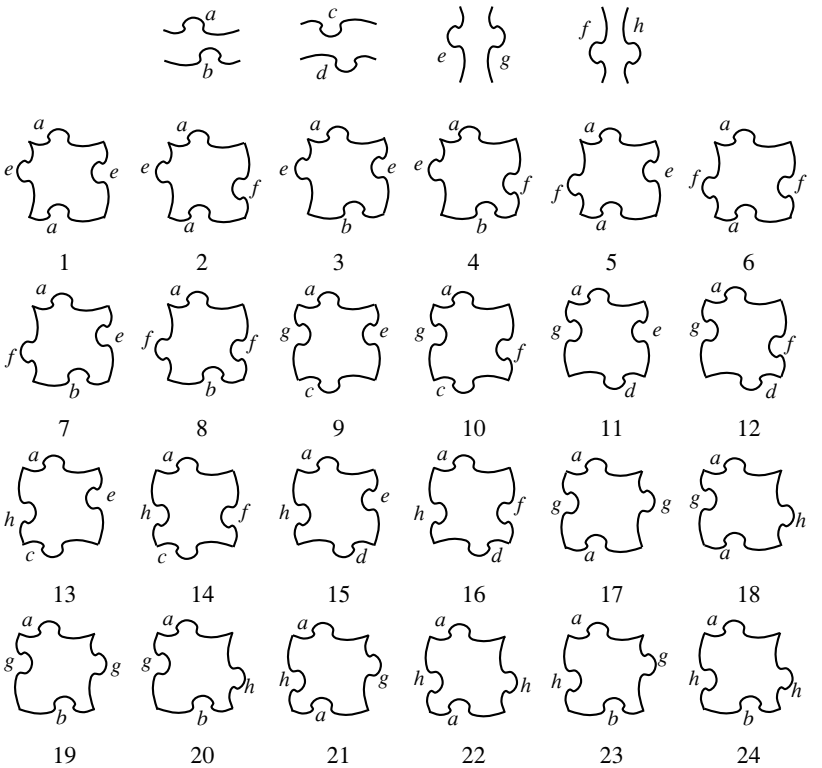


FIGURE 1

Row 1:	1: TTTT	2: TGTG	3: TGTG
	4: G1G2G1G2	5: TGTG	6: TTTT
Row 2:	7: G1G2G1G2	8: TGTG	9: C4C4C4C4
	10: NH	11: NH	12: C4C4C4C4
Row3:	13: NH	14: C4C4C4C4	15: G1G1G2G2
	16: NH	17: TTTT	18: TGTG
Row 4:	19: TGTG	20: C4C4C4C4	21: TGTG
	22: TTTT	23: G1G2G1G2	24: TGTG.

To check for duplicates, each shape is assigned a signature by listing the letters of its sides in clockwise order:

Row 1:	*aeae*	*afae*	*aebe*	*afbe*	*aeaf*	*afaf*
Row 2:	*aebf*	*afbf*	*aecg*	*afcg*	*aedg*	*afdg*
Row 3:	*aech*	*afch*	*aedh*	*afdh*	*agag*	*ahag*
Row 4:	*agbg*	*ahbg*	*agah*	*ahah*	*agbh*	*ahbh.*

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The square has eight symmetry transformations that will map the square back onto itself. They are the identity (I); clockwise rotations of 90° (R1), 180° (R2) and 270° (R3); and flips over a horizontal line (H), a vertical line (V), the diagonal from the bottom right to the top left corner (DL) and the diagonal from the bottom left to the top right corner (DR).

The listing above has all possible sequences with an a on the top edge. If any of the signatures when transformed returns a signature with an a on the top, it must be itself or a duplicate of another shape. To check for duplicates, we need to examine only signatures for which a transformation will result in an a on the top.

The eight transformations alter edges as follows:

I: sends each edge to itself

R1: $a \to g, b \to h, c \to e, d \to f, e \to b, f \to a, g \to d, h \to c$

R2: $a \to b, b \to c, c \to d, d \to a, e \to h, f \to g, g \to f, h \to e$

R3: $a \to f, b \to e, c \to h, d \to g, e \to c, f \to d, g \to a, h \to b$

H: $a \to c, b \to d, c \to a, d \to b, e \to f, f \to e, g \to h, h \to g$

V: $a \to b, b \to a, c \to d, d \to c, e \to g, f \to h, g \to e, h \to f$

DL: $a \to e, b \to f, c \to g, d \to h, e \to a, f \to b, g \to c, h \to d$

DR: $a \to h, b \to g, c \to f, d \to e, e \to d, f \to c, g \to b, h \to a.$

We can ignore the I transformation.

The R1 transformation returns an a on top when an f is on the left edge. There are four in the first two rows, tiles 5, 6, 7 and 8 with sequences $aeaf$, $afaf$, $aebf$, $afbf$. Applying the R1 transformation, they become $agbg$, $agag$, $agbh$ and $agah$. These sequences appear in the last two rows. Duplicate pairs are (5,19), (6,17), (7,23) and (8,21). Four tiles can be eliminated.

The R2 transformation will return an a on top when there is a d on the bottom, or tiles 11, 12, 15 and 16 in the second and third rows with sequences $aedg$, $afdg$, $aedh$, $afdh$. Transformed, they become $afdh$, $afdg$, $aedh$ and $aedg$. Duplicate pairs are (11,16), (12,12), (15,15) and (16,11). Two tiles are mapped to themselves, and two tiles are mapped to each other. One tile can be eliminated.

The R3 transformation is a counterclockwise rotation of 90°. It will return an a on the top when a g is on the right side, or the four sequences $agag$, $agbg$, $agah$, $agbh$, tiles 17, 19, 21 and 23 in rows three and four. They are mapped to $afaf$, $aeaf$, $afbf$ and $aebf$. These tiles have already been paired and eliminated in the R1 transformation.

The H transformation flips bottom and top and changes the orientation on right and left, but leaves right side right and left side left. A sequence with a c on the bottom will become a sequence with an a on the top when flipped this way. The tiles with a c on the bottom are 9, 10, 13 and 14 in rows two and three, sequences $aecg$, $afcg$, $aech$, $afch$. The transformation rule yields $cfah$, $ceah$, $cfag$ and $ceag$. However, transformations that flip the tile change the order of sides from clockwise to counterclockwise.

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Correcting direction and putting the *a* on top yields *afch*, *aech*, *afcg* and *aecg*. The duplicates are (9,14), (10,13), (13,10) and (14,9). Two pairs are mapped to each other so two more tiles are eliminated.

No tiles return an *a* with the V transformation.

The DL transformation will put an *a* on top when the left side is an *e*, sequences 1, 2, 3 and 4 in the first row with signatures *aeae*, *afae*, *aebe*, *afbe*. After transforming and correcting the ordering, they become *aeae*, *aebe*, *afae* and *afbe*. The duplicate pairings are (1,1), (2,3), (3,2) and (4,4). Two tiles are mapped to themselves and two to each other. Another tile is eliminated.

Finally, the DR transformation will put an *a* on the top when the right edge is an *h*. The four tiles affected are tiles 18, 20, 22 and 24 in rows three and four, sequences *ahag*, *ahbg*, *ahah* and *ahbh*. Transforming and correcting the order yields *ahbh*, *ahbg*, *ahah* and *ahag*. Duplicate pairings are (18,24), (20,20), (22,22) and (24,18). Again two tiles are mapped to each other and two to themselves. One more tile is eliminated.

Applying the transformations reveals that nine of the 24 tiles are duplicates. A list of unique tile shapes is 1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 18, 20 and 22. Thirteen tessellate isohedrally but two (10 and 11) do not. However, the two will tile anisohedrally with translation blocks of four, as in Figure 2.

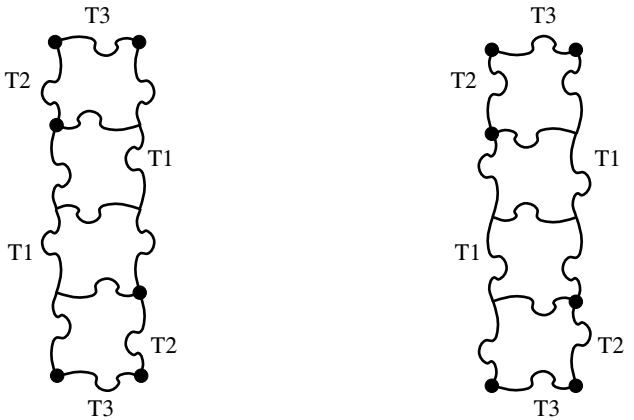


FIGURE 2

May be the easiest anisohedral tiles to form. Start with a tile that fits isohedral class IH61, a C4C4C4C4 tiling with all sides shaped the same with asymmetric edges (shown in Figure 3 on the left). Rotate the tile 90° repeatedly about one of its vertices to form a block of four (as in Figure 3). If the edge between any two tiles in the block is flipped over the midpoint of its vertices, the two new tiles (stuck together as shown in Figure 3) will be the two anisohedral tiles discussed above.

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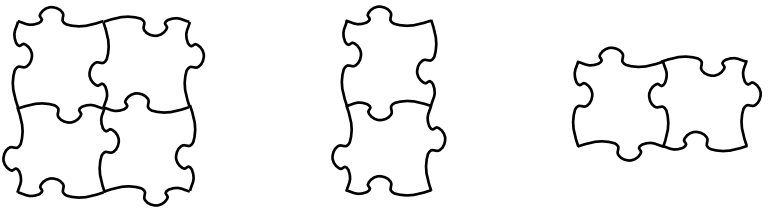


FIGURE 3

Hence, exactly 15 distinct tiles that tessellate can be formed on a square template with identical asymmetric edges. Thirteen are isohedral and two are anisohedral.

Acknowledgement: I am heavily indebted to Doris Schattschneider for extensive comments on an earlier draft of this Note.

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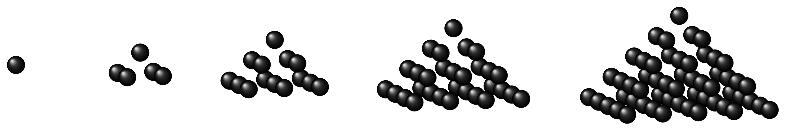
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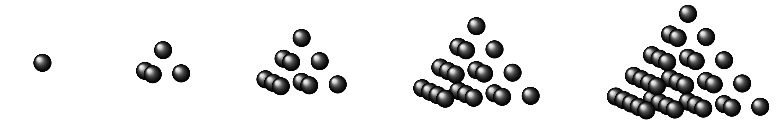
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100.35 Proof without words: the average of square pyramidal and triangular is tetrahedral

Let P_n represent the number of dots in a square pyramid with n levels and T_n represent the number of dots in a triangular pyramid with n levels both illustrated below.



$P_1 = 1$ $P_2 = 5$ $P_3 = 14$ $P_4 = 30$ $P_5 = 55$



$T_1 = 1$ $T_2 = 4$ $T_3 = 10$ $T_4 = 20$ $T_5 = 35$

We call these the square pyramidal and tetrahedral numbers, respectively. Also, let $t_n = 1 + 2 + \ldots + n$ represent the nth ordinary triangular numbers.

Theorem: The nth tetrahedral number is the average of the nth square pyramidal number and the nth triangular number, i.e. $T_n = \dfrac{P_n + t_n}{2}$.

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