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A PREDICATIVE VARIANT OF HYLAND'S EFFECTIVE TOPOS

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Abstract. Here, we present a category **pEff** which can be considered a predicative variant of Hyland's Effective Topos **Eff** for the following reasons. First, its construction is carried in Feferman's predicative theory of non-iterative fixpoints $\widehat{ID}_1$. Second, **pEff** is a list-arithmetic locally cartesian closed pretopos with a full subcategory $\mathbf{pEff}_{set}$ of small objects having the same categorical structure which is preserved by the embedding in **pEff**; furthermore subobjects in $\mathbf{pEff}_{set}$ are classified by a non-small object in **pEff**. Third **pEff** happens to coincide with the exact completion of the lex category defined as a predicative rendering in $\widehat{ID}_1$ of the subcategory of **Eff** of recursive functions and it validates the Formal Church's thesis. Hence **pEff** turns out to be itself a predicative rendering of a full subcategory of **Eff**.

§1. Introduction. As reported in [24] Hyland's paper "The Effective Topos" [7] gave input to a whole new strand of research about realizability and its applications to logic, mathematics and computer science. Hyland applied the tripos-to-topos construction in [8] producing one of the first examples of elementary topos, denoted with **Eff**, that is not a Grothendieck topos. A characteristics of **Eff** which attracted a lot of interest in logic is the fact that it provides a realizability interpretation of high-order logic extending Kleene realizability for intuitionistic arithmetic and hence it validates the Formal Church's thesis (see [23]).

A predicative study of **Eff** had been already developed in the context of algebraic set theory by Van den Berg and Moerdijk (see [22]) by taking Aczel's CZF in [1] as the set theory to be realized in their categorical structures.

The aim of this paper is to produce a strictly predicative version of **Eff**, called **pEff**, which can be formalized in Feferman's predicative theory of non-iterative fixpoints $\widehat{ID}_1$ (see [5]) whose proof-theoretic strength is much lower than that of CZF. Our ultimate goal is to use **pEff** as the effective universe where to validate proofs done in the Minimalist Foundation in order to extract their computational contents.

The Minimalist Foundation (**MF**) is a predicative foundation for constructive mathematics ideated in joint work of the first author with Sambin in [19] and completed to a two-level system in [9]. **MF** is weaker than CZF (in terms of proof-theoretic strength) because it can be interpreted in Martin-Löf's type theory with one universe [20] as shown in [9], and directly also in $\widehat{ID}_1$ (see [12] and [13]).

MF was called *minimalist* in [19] because it was intended to constitute a common core among the most relevant constructive and classical foundations. It consists of two levels with an interpretation of one into the other: an *intensional level* suitable

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as a base for a proof-assistant and for the extraction of computational contents from proofs, an *extensional level* formulated in a language close to that of ordinary mathematics, and an interpretation of the extensional level in the intensional one by means of a quotient completion (see [9]). Both the intensional level and the extensional level of **MF** consist of dependent type systems based on versions of Martin-Löf's type theory.

A key difference between **MF** and Martin-Löf's type theories is that in **MF** propositions are defined in such a way that choice principles, including the axiom of unique choice, are no longer necessarily valid.

Moreover, there is an analogy between the two-level formal system of **MF** and the tripos-to-topos construction of a realizability topos: the role of the tripos is taken by the intensional level of **MF**, the role of the realizability topos construction is taken by the quotient model used in [9] to interpret the extensional level of **MF** in its intensional one, and the internal language of a generic elementary topos corresponds to the extensional level of **MF**. The key difference between the tripos-to-topos construction and the construction of **MF** is that the quotient completion employed in **MF** does not yield to an exact category. This quotient completion had been studied categorically in joint work of the first author with Rosolini in [17], [16] and [18] under the name of "elementary quotient completion of a Lawvere's elementary doctrine". Such a completion turned out to be a *generalization of the well-known notion of exact completion on a lex category*.

Here we build a predicative variant of **Eff**, called **pEff**, by applying the elementary quotient completion in [17] to a Lawvere's hyperdoctrine $\overline{\mathbf{Prop}^r}$. The base category $\mathcal{C}_r$ of $\overline{\mathbf{Prop}^r}$ is a predicative rendering of the subcategory of **Eff** of recursive functions within $\widehat{ID_1}$ and the logical hyperdoctrine structure extends Kleene realizability interpretation of connectives and supports an interpretation (whilst with a pure combinatory non-categorical interpretation of λ-abstraction) of the intensional level of **MF** extended with the Formal Church's thesis and the full axiom of choice as in [6]. Then, from results in [9], it follows that **pEff** validates the extensional level of **MF**.

pEff can be seen as a predicative variant of **Eff** for the following reasons. First, the construction of **pEff** is formalizable in the predicative theory $\widehat{ID_1}$. Second, **pEff** is a list-arithmetic locally cartesian closed pretopos with a full subcategory $\mathbf{pEff}_{set}$ of small objects having the same categorical structure which is preserved by the embedding in **pEff**; furthermore subobjects in $\mathbf{pEff}_{set}$ are classified by a non-small object in **pEff**. Third, the elementary quotient completion construction used to build **pEff** coincides with the exact completion on the lex base category $\mathcal{C}_r$ of the doctrine $\overline{\mathbf{Prop}^r}$. Finally, **pEff** turns out to be a predicative rendering of a full subcategory of **Eff** which validates the Formal Church's thesis.

§2. Categorical preliminaries. We just recall some categorical definitions we will use later.

DEFINITION 2.1. A *first-order hyperdoctrine* is a functor $\mathbf{P} : \mathcal{C}^{op} \to \mathbf{preHeyt}$ from a finite product category $\mathcal{C}$ to the category of Heyting prealgebras (see, e.g., [25]) such that, for every f in $\mathcal{C}$, $\mathbf{P}_f$ has left and right adjoints $\exists_f$ and $\forall_f$ satisfying the

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Beck–Chevalley condition (see [25]). For every object I of $\mathbb{C}$, we use $\sqsubseteq_I$ to denote the preorder relation in $\mathbf{P}(I)$.

We say that $\mathbf{P}$ has *weak comprehensions* if for every object A of $\mathcal{C}$ and for every $p \in \mathbf{P}(A)$, there exists an arrow $\mathsf{cmp} : \mathsf{Cmp}(p) \to A$ in $\mathcal{C}$ such that $\top \sqsubseteq_{\mathsf{Cmp}(p)} \mathbf{P}_{\mathsf{cmp}}(p)$ and for every arrow $f : B \to A$ such that $\top \sqsubseteq_B \mathbf{P}_f(p)$, there exists an arrow $f' : B \to \mathsf{Cmp}(p)$ such that $\mathsf{cmp} \circ f' = f$.

Definition 2.2. Let **Pord** be the category of preorders and **Pos** be the category of partially ordered sets. If $\mathbf{C} : \mathcal{C}^{op} \to \mathbf{Cat}$ is a functor, then its *preorder reflection* $\mathbf{P}[\mathbf{C}] : \mathcal{C}^{op} \to \mathbf{Pord}$ is the functor whose fibres are the preorder reflections of the fibres of $\mathbf{C}$ (and for which $\mathbf{P}[\mathbf{C}](f)$ is determined by $\mathbf{C}(f)$ for every arrow f of $\mathcal{C}$), while its *posetal reflection* $\mathbf{PR}[\mathbf{C}] : \mathcal{C}^{op} \to \mathbf{Pos}$ is the functor whose fibres are the posetal reflections of those of $\mathbf{C}$ (and for which $\mathbf{PR}[\mathbf{C}](f)$ is determined by $\mathbf{C}(f)$ for every arrow f of $\mathcal{C}$). The posetal reflection produces a functor also when applied to a pseudofunctor $\mathbf{C}$.

Definition 2.3. If $\mathcal{C}$ is a finitely complete category, *the doctrine of its weak subobjects* $\mathbf{wSub}_{\mathcal{C}}$ is the posetal reflection of the slice pseudofunctor.

§3. Feferman's weak theory of inductive definitions $\widehat{ID_1}$. We give here a brief presentation of Feferman's theory $\widehat{ID_1}$ ([5]). Let us consider the language of second-order arithmetic given by countably many individual and set variables, a constant 0, a unary successor functional symbol *succ*, an n-ary functional symbol for every n-ary (definition of a) primitive recursive function, the equality predicate $=$ between individuals, the membership predicate ε between individuals and sets, connectives $\wedge, \vee, \to, \neg$ and individual and set quantifiers. The occurrence of the set variable X in $t \varepsilon X$ is *positive*, while an occurrence of X in a nonatomic formula φ with no set quantifiers is *positive* (*negative* resp.) if one of the following holds: φ is $\psi \wedge \rho$ or $\psi \vee \rho$ and the occurrence is positive (neg.) in ψ or in ρ; φ is $\psi \to \rho$ and the occurrence is positive (neg.) in ρ or negative (pos.) in ψ; φ is $\neg\psi$ and the occurrence is negative (pos.) in ψ; φ is $\exists x\, \psi$ or $\forall x\, \psi$ and the occurrence is positive (neg.) in ψ. A second-order formula φ is *admissible* if it does not contain set quantifiers and it has at most one free individual variable x and one free set variable X of which all occurrences are positive.

$\widehat{ID_1}$ is a first-order classical theory with equality containing 0, *succ*, the functional symbols for the (definitions of) primitive recursive functions and a unary predicate symbol P_φ for every admissible second-order formula $\varphi(x, X)$. As usual, $\perp$ is the formula $0 = succ(0)$.

The axioms of $\widehat{ID_1}$ include those of Peano arithmetic (including defining equations for primitive recursive functions) plus the *induction principle* for every formula of $\widehat{ID_1}$ and a *fixpoint schema*: $P_\varphi(x) \leftrightarrow \varphi[P_\varphi/X]$ for every admissible second-order formula φ, where $\varphi[P_\varphi/X]$ is the result of substituting in φ all the subformulas of the form $t \varepsilon X$ with $P_\varphi(t)$.

3.1. Notation of recursive functions. In $\widehat{ID_1}$ one can encode Kleene's application $\{x\}(y)$ via Kleene's predicate $T(x, y, z)$ and primitive recursive function $U(z)$: for every formula $P(z)$, $P(\{x\}(y))$ means $\exists z\,(T(x, y, z) \wedge P(U(z)))$. We define $\{t\}()$ as t and $\{t\}(s_1, ..., s_{n+1})$ as $\{\{t\}(s_1, ..., s_n)\}(s_{n+1})$. In $\widehat{ID_1}$ one can also encode pairs of

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natural numbers with natural numbers via a bijective primitive recursive function *pair* with primitive recursive projections p_1, p_2 represented by numerals $\mathbf{p}_1, \mathbf{p}_2, \mathbf{pair}$ such that $pair(x,y) = \{\mathbf{pair}\}(x,y)$ and $p_i(x) = \{\mathbf{p}_i\}(x)$ ($i = 1,2$) in $\widehat{ID_1}$. Moreover we encode finite lists of natural numbers with natural numbers in a surjective way such that 0 encodes the empty list, the concatenation function is primitive recursive and it is represented by a numeral **cnc**, the length function *lh* is primitive recursive, the component function $(x,j) \mapsto (x)_j$ which sends each natural number x, seen as a list of natural numbers, to its j-component if $j < lh(x)$ or to 0 otherwise, is primitive recursive, and there is a numeral **listrec** representing the list recursor.

The successor function can be represented by a numeral **succ** and the natural numbers recursor by a numeral **rec**. In $\widehat{ID_1}$ one can also define λ-astraction $\Lambda x.t$ of terms built with Kleene brackets, variables and numerals as in any partial combinatory algebra.

§4. The Kleene effective p-tripos.

4.1. The category $\mathcal{C}_r$ of realized collections. Here we define the category $\mathcal{C}_r$ of realized collections which is a rendering of the subcategory of recursive functions of **Eff** (see [21]) in $\widehat{ID_1}$. Then, we will describe its categorical structure. In order to prove that $\mathcal{C}_r$ is weakly locally cartesian closed in a straightforward way, we will introduce an indexed category of realized collections over it.

DEFINITION 4.1. A *realized collection* (or simply a *class*) A of $\widehat{ID_1}$ is a formal expression $\{x | \varphi_{\mathsf{A}}(x)\}$ where $\varphi_{\mathsf{A}}(x)$ is a formula of $\widehat{ID_1}$ with at most x as free variable. We write $x \varepsilon \mathsf{A}$ as an abbreviation for $\varphi_{\mathsf{A}}(x)$. Classes with provably equivalent membership relations in $\widehat{ID_1}$ are identified. An *operation* between classes of $\widehat{ID_1}$ from A to B is an equivalence class $[\mathbf{n}]_{\approx_{\mathsf{A},\mathsf{B}}}$ of numerals with $x \varepsilon \mathsf{A} \vdash \{\mathbf{n}\}(x) \varepsilon \mathsf{B}$ in $\widehat{ID_1}$. For such $\mathbf{n}$ and $\mathbf{m}$ we define $\mathbf{n} \approx_{\mathsf{A},\mathsf{B}} \mathbf{m}$ as $x \varepsilon \mathsf{A} \vdash \{\mathbf{n}\}(x) = \{\mathbf{m}\}(x)$ in $\widehat{ID_1}$.

If $[\mathbf{n}]_{\approx_{\mathsf{A},\mathsf{B}}} : \mathsf{A} \to \mathsf{B}$ and $[\mathbf{m}]_{\approx_{\mathsf{B},\mathsf{C}}} : \mathsf{B} \to \mathsf{C}$ are operations between classes of $\widehat{ID_1}$, then their *composition* is defined by $[\mathbf{m}]_{\approx_{\mathsf{B},\mathsf{C}}} \circ [\mathbf{n}]_{\approx_{\mathsf{A},\mathsf{B}}} := [\Lambda x.\{\mathbf{m}\}(\{\mathbf{n}\}(x))]_{\approx_{\mathsf{A},\mathsf{C}}}$.

The *identity* operation for the class A of $\widehat{ID_1}$ is defined as $\mathsf{id}_{\mathsf{A}} := [\Lambda x.x]_{\approx_{\mathsf{A},\mathsf{A}}}$.

DEFINITION 4.2. We denote with $\mathcal{C}_r$ the category whose objects are realized collections of $\widehat{ID_1}$ and arrows are operations in $\widehat{ID_1}$ between them with their composition and identity operations.

We will omit subscripts of $\approx$ when they will be clear from the context.

THEOREM 4.3. *$\mathcal{C}_r$ is a finitely complete category with disjoint stable finite coproducts, parameterized list objects and weak exponentials* (*see* [10] *and* [4]).

PROOF. In $\mathcal{C}_r$ the object $1 := \{x | x = 0\}$ is a terminal object; $\mathsf{A} \times \mathsf{B} := \{x | p_1(x) \varepsilon \mathsf{A} \wedge p_2(x) \varepsilon \mathsf{B}\}$ is a binary product for A and B together with the projections defined by $[\mathbf{p}_1]$ and $[\mathbf{p}_2]$.

If $\mathsf{f} = [\mathbf{n}]_{\approx}$ and $\mathsf{g} = [\mathbf{m}]_{\approx}$ are arrows in $\mathcal{C}_r$ from A to B, then an equalizer for them is $\mathsf{Eq}(\mathsf{f},\mathsf{g}) := \{x | x \varepsilon \mathsf{A} \wedge \{\mathbf{n}\}(x) = \{\mathbf{m}\}(x)\}$ together with the arrow $[\Lambda x.x]_{\approx} : \mathsf{Eq}(\mathsf{f},\mathsf{g}) \to \mathsf{A}$; $0 := \{x | \bot\}$ is an initial object; $\mathsf{A} + \mathsf{B} := \{x | (p_1(x) = 0 \wedge p_2(x) \varepsilon \mathsf{A}) \vee (p_1(x) = 1 \wedge p_2(x) \varepsilon \mathsf{B})\}$ gives a binary coproduct for A and B together with the injections $\mathsf{j}_1 := [\Lambda x.\{\mathbf{pair}\}(0,x)]_{\approx}$ from A to $\mathsf{A} + \mathsf{B}$ and $\mathsf{j}_2 := [\Lambda x.\{\mathbf{pair}\}(1,x)]_{\approx}$

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from B to $\mathsf{A}+\mathsf{B}$; the object $\mathsf{List}(\mathsf{A})$ defined as $\{x|\forall j(\,j<lh(x)\rightarrow(x)_j\,\varepsilon\,\mathsf{A})\}$ is a parameterized list object for A together with the empty list arrow $\varepsilon:=[\Lambda x.0]_{\approx}:1\rightarrow\mathsf{List}(\mathsf{A})$ and the append arrow cons defined as $[\Lambda x.\{\mathbf{cnc}\}(\{\mathbf{p}_1\}(x),\{\mathbf{p}_2\}(x))]_{\approx}$; the object $\mathsf{A}\Rightarrow\mathsf{B}:=\{x|\forall u\,(u\,\varepsilon\,\mathsf{A}\rightarrow\{x\}(u)\,\varepsilon\,\mathsf{B})\}$ defines a weak exponential for A and B together with the evaluation $\mathsf{ev}:=[\Lambda x.\{\{\mathbf{p}_1\}(x)\}(\{\mathbf{p}_2\}(x))]_{\approx}$ from $(\mathsf{A}\Rightarrow\mathsf{B})\times\mathsf{A}$ to B. After having noticed that an arrow $\mathsf{j}:=[\mathbf{j}]_{\approx}:\mathsf{A}\rightarrow\mathsf{B}$ in $\mathcal{C}_r$ is mono if and only if $x\,\varepsilon\,\mathsf{A}\wedge y\,\varepsilon\,\mathsf{A}\wedge\{\mathbf{j}\}(x)=\{\mathbf{j}\}(y)\vdash x=y$ in $\widehat{ID_1}$, one can easily verify that finite coproducts are stable and disjoint. ⊣

REMARK 4.4. A parameterized natural numbers object can be defined using the list object $\mathsf{List}(1)$. However one can directly define a parameterized natural numbers object N as $\{x|\,x=x\}$ together with the arrows $[\Lambda x.0]_{\approx}:1\rightarrow\mathsf{N}$ and $[\mathbf{succ}]_{\approx}:\mathsf{N}\rightarrow\mathsf{N}$. This presentation of natural numbers object will help to avoid encodings when showing the validity of the Formal Church's thesis in the doctrine.

DEFINITION 4.5. Suppose A is an object of $\mathcal{C}_r$. A *family of realized collections* on A is a formal expression $\{x'|\varphi_{\mathsf{C}}(x,x')\}$ where $\varphi_{\mathsf{C}}(x,x')$ is a formula of $\widehat{ID_1}$ with at most x and x' as free variables (we write $x'\,\varepsilon\,\mathsf{C}(x)$ as an abbreviation for $\varphi_{\mathsf{C}}(x,x')$) for which $x'\,\varepsilon\,\mathsf{C}(x)\vdash x\,\varepsilon\,\mathsf{A}$ in $\widehat{ID_1}$. Families of realized collections on A with provably (in $\widehat{ID_1}$) equivalent membership relations are identified.

An *operation* from a family of realized collections $\mathsf{C}(x)$ on A to another $\mathsf{D}(x)$ is given by an equivalence class $[\mathbf{n}]_{\approx_{\mathsf{C}(x),\mathsf{D}(x)}}$ of numerals such that in $\widehat{ID_1}$ we have $x'\,\varepsilon\,\mathsf{C}(x)\vdash\{\mathbf{n}\}(x,x')\,\varepsilon\,\mathsf{D}(x)$ with respect to the equivalence relation defined as follows: $\mathbf{n}\approx_{\mathsf{C}(x),\mathsf{D}(x)}\mathbf{m}$ if and only if $x'\,\varepsilon\,\mathsf{C}(x)\vdash_{\widehat{ID_1}}\{\mathbf{n}\}(x,x')=\{\mathbf{m}\}(x,x')$.

If $\mathsf{f}=[\mathbf{n}]_{\approx_{\mathsf{C}(x),\mathsf{D}(x)}}:\mathsf{C}(x)\rightarrow\mathsf{D}(x)$ and $\mathsf{g}=[\mathbf{m}]_{\approx_{\mathsf{D}(x),\mathsf{E}(x)}}:\mathsf{D}(x)\rightarrow\mathsf{E}(x)$ are operations between families of realized collection on A, then their *composition* is defined as $\mathsf{g}\circ\mathsf{f}:=[\Lambda x.\Lambda x'.\{\mathbf{m}\}(x,\{\mathbf{n}\}(x,x'))]_{\approx_{\mathsf{C}(x),\mathsf{E}(x)}}:\mathsf{C}(x)\rightarrow\mathsf{E}(x)$.[1]

If $\mathsf{C}(x)$ is a family of realized collections on A, then its *identity* operation is defined by $\mathsf{id}_{\mathsf{C}(x)}:=[\Lambda x.\Lambda x'.x']_{\approx_{\mathsf{C}(x),\mathsf{C}(x)}}:\mathsf{C}(x)\rightarrow\mathsf{C}(x)$.

The proof of the following theorem is an immediate verification.

THEOREM 4.6. *For every object* A *of* $\mathcal{C}_r$, *families of realized collections on* A *and operations between them together with their composition and identity operations define a category. We denote this category with* $\mathbf{Col}^r(\mathsf{A})$.

If $\mathsf{f}:=[\mathbf{n}]_{\approx}:\mathsf{A}\rightarrow\mathsf{B}$, *then the following assignments give rise to a functor* $\mathbf{Col}^r_{\mathsf{f}}$ *from* $\mathbf{Col}^r(\mathsf{B})$ *to* $\mathbf{Col}^r(\mathsf{A})$:

1. *for every object* $\mathsf{C}(x)$ *of* $\mathbf{Col}^r(\mathsf{B})$, $\mathbf{Col}^r_{\mathsf{f}}(\mathsf{C}(x)):=\{x'|\,x\,\varepsilon\,\mathsf{A}\wedge x'\,\varepsilon\,\mathsf{C}(\{\mathbf{n}\}(x))\}$;
2. *for every arrow* $\mathsf{g}:=[\mathbf{k}]_{\approx}:\mathsf{C}(x)\rightarrow\mathsf{D}(x)$ *of* $\mathbf{Col}^r(\mathsf{B})$,

$$\mathbf{Col}^r_{\mathsf{f}}(\mathsf{g}):=[\Lambda x.\Lambda x'.\{\mathbf{k}\}(\{\mathbf{n}\}(x),x')]_{\approx}:\mathbf{Col}^r_{\mathsf{f}}(\mathsf{C}(x))\rightarrow\mathbf{Col}^r_{\mathsf{f}}(\mathsf{D}(x)).$$

Moreover, the assignments $\mathsf{A}\mapsto\mathbf{Col}^r(\mathsf{A})$ *and* $\mathsf{f}\mapsto\mathbf{Col}^r_{\mathsf{f}}$ *define an indexed category* $\mathbf{Col}^r:\mathcal{C}_r^{op}\rightarrow\mathbf{Cat}$.

The functors $\mathbf{Col}^r_{\mathsf{f}}$ are also called *substitution functors*. The proof of the following theorem is similar to that of Theorem 4.3.

[1] We will omit the subscripts of $\approx$ when they will be clear from the context.

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THEOREM 4.7. *For every object* A *of* $\mathcal{C}_r$, $\mathbf{Col}^r(\mathsf{A})$ *is a finitely complete category with parameterized list objects, finite coproducts and weak exponentials. Moreover for every arrow* $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ *in* $\mathcal{C}_r$, *the functor* $\mathbf{Col}^r_{\mathsf{f}}$ *preserves this structure.*

Moreover substitution functors have left adjoints:

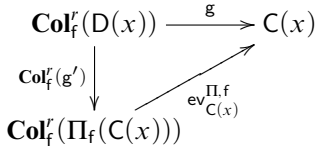
THEOREM 4.8. *For every* $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ *in* $\mathcal{C}_r$, *the functor* $\mathbf{Col}^r_{\mathsf{f}}$ *has a left adjoint*

$$\Sigma_{\mathsf{f}} : \mathbf{Col}^r(\mathsf{A}) \to \mathbf{Col}^r(\mathsf{B}).$$

PROOF. If $\mathsf{f} = [\mathbf{n}]_{\approx_{\mathsf{A},\mathsf{B}}} : \mathsf{A} \to \mathsf{B}$ is an arrow in $\mathcal{C}_r$, then a left adjoint Σ_{f} to $\mathbf{Col}^r_{\mathsf{f}}$ is defined by the following conditions: if $\mathsf{C}(x)$ is an object of $\mathbf{Col}^r(\mathsf{A})$, then $\Sigma_{\mathsf{f}}(\mathsf{C}(x)) := \{x' | x = \{\mathbf{n}\}(p_1(x')) \wedge p_2(x') \varepsilon\, \mathsf{C}(p_1(x'))\}$; if $\mathsf{g} := [\mathbf{m}]_{\approx}$ is an arrow from $\mathsf{C}(x)$ to $\mathsf{D}(x)$ in $\mathbf{Col}^r(\mathsf{A})$, then $\Sigma_{\mathsf{f}}(\mathsf{g}) : \Sigma_{\mathsf{f}}(\mathsf{C}(x)) \to \Sigma_{\mathsf{f}}(\mathsf{D}(x))$ is defined as $[\Lambda x.\Lambda x'.\{\mathbf{pair}\}(\{\mathbf{p}_1\}(x'), \{\mathbf{m}\}(\{\mathbf{p}_1\}(x'), \{\mathbf{p}_2\}(x')))]_{\approx}$. ⊣

Now we prove that substitution functors have weak versions of right adjoints.

THEOREM 4.9. *For every arrow* $\mathsf{f} := [\mathbf{n}]_{\approx} : \mathsf{A} \to \mathsf{B}$ *in* $\mathcal{C}_r$ *and every object* $\mathsf{C}(x)$ *in* $\mathbf{Col}^r(\mathsf{A})$, *there exists an object* $\Pi_{\mathsf{f}}(\mathsf{C}(x))$ *in* $\mathbf{Col}^r(\mathsf{B})$ *and an arrow* $\mathsf{ev}^{\Pi,\mathsf{f}}_{\mathsf{C}(x)} : \mathbf{Col}^r_{\mathsf{f}}(\Pi_{\mathsf{f}}(\mathsf{C}(x))) \to \mathsf{C}(x)$ *in* $\mathbf{Col}^r(\mathsf{B})$ *such that for every* $\mathsf{D}(x) \in \mathbf{Col}^r(\mathsf{B})$ *and every arrow* $\mathsf{g} : \mathbf{Col}^r_f(\mathsf{D}(x)) \to \mathsf{C}(x)$ *in* $\mathbf{Col}^r(\mathsf{A})$, *there exists an arrow* $\mathsf{g}' : \mathsf{D}(x) \to \Pi_{\mathsf{f}}(\mathsf{C}(x))$ *in* $\mathbf{Col}^r(\mathsf{B})$ *such that the following diagram commutes in* $\mathbf{Col}^r(\mathsf{A})$:

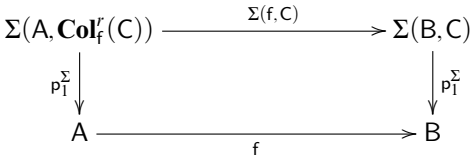


PROOF. Take $\Pi_{\mathsf{f}}(\mathsf{C}(x)) := \{x' | x \varepsilon\, \mathsf{B} \wedge \forall y\, (x = \{\mathbf{n}\}(y) \wedge y \varepsilon\, \mathsf{A} \to \{x'\}(y) \varepsilon\, \mathsf{C}(y))\}$ and $\mathsf{ev}^{\Pi,\mathsf{f}}_{\mathsf{C}(x)} := [\Lambda x.\Lambda x'.\{x'\}(x)]_{\approx}$. ⊣

DEFINITION 4.10. If A is an object of $\mathcal{C}_r$ and $\mathsf{B}(x)$ is an object of $\mathbf{Col}^r(\mathsf{A})$, then $\Sigma(\mathsf{A}, \mathsf{B}(x))$ is the object of $\mathcal{C}_r$ defined as $\{x | \, p_1(x) \varepsilon\, \mathsf{A} \wedge p_2(x) \varepsilon\, \mathsf{B}(p_1(x))\}$.

The proof of the following lemma consists of an easy verification.

LEMMA 4.11. *For every* $\mathsf{f} := [\mathbf{n}]_{\approx} : \mathsf{A} \to \mathsf{B}$ *in* $\mathcal{C}_r$ *and every* C *in* $\mathbf{Col}^r(\mathsf{B})$, *if* $\Sigma(\mathsf{f}, \mathsf{C})$ *is* $[\Lambda x.\{\mathbf{pair}\}(\{\mathbf{n}\}(\{\mathbf{p}_1\}(x)), \{\mathbf{p}_2\}(x))]_{\approx}$, *the following diagram is a pullback where* p^{Σ}_1 *means* $[\mathbf{p}_1]_{\approx}$



THEOREM 4.12. *For every* A *in* $\mathcal{C}_r$, $\mathcal{C}_r/\mathsf{A}$ *and* $\mathbf{Col}^r(\mathsf{A})$ *are equivalent*.

PROOF. Suppose A is an object of $\mathcal{C}_r$. It is sufficient to consider the functor $\mathbf{J}_{\mathsf{A}} : \mathcal{C}_r/\mathsf{A} \to \mathbf{Col}^r(\mathsf{A})$ defined by the assignments

$$[\mathbf{b}]_{\approx} : \mathsf{B} \to \mathsf{A} \mapsto \{x' | \, x' \varepsilon\, \mathsf{B} \wedge x = \{\mathbf{b}\}(x')\}, \qquad [\mathbf{n}]_{\approx} \mapsto [\Lambda x.\Lambda x'.\{\mathbf{n}\}(x')]_{\approx}$$

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and the functor $\mathbf{I}_{\mathsf{A}} : \mathbf{Col}^r(\mathsf{A}) \to \mathcal{C}_r/\mathsf{A}$ defined by the assignments (see Definition 4.10)

$$\mathsf{B}(x) \mapsto \mathsf{p}_1^{\Sigma} := [\mathbf{p}_1]_{\approx} : \Sigma(\mathsf{A}, \mathsf{B}(x)) \to \mathsf{A}$$
$$[\mathbf{n}]_{\approx} \mapsto [\Lambda x.\{\mathbf{pair}\}(\{\mathbf{p}_1\}(x), \{\mathbf{n}\}(\{\mathbf{p}_1\}(x), \{\mathbf{p}_2\}(x)))]_{\approx}$$

⊣

COROLLARY 4.13. *$\mathcal{C}_r$ is weakly locally cartesian closed.*

$\mathbf{Col}^r$ is a functorial account of the slice pseudofunctor $\mathcal{C}_r/-$. In fact, it is immediate to verify that for every $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ in $\mathcal{C}^r$ the functor $\mathbf{I}_{\mathsf{A}} \circ \mathbf{Col}^r_{\mathsf{f}} \circ \mathbf{J}_{\mathsf{B}}$ coincides with the functor $\mathcal{C}_r/\mathsf{f}$ defined by representing pullbacks using products and equalizers in the standard way.

4.2. The hyperdoctrine $\widetilde{\mathbf{Prop}^r}$ of realized propositions. Here we define the hyperdoctrine of realized propositions on $\mathcal{C}_r$.

DEFINITION 4.14. The indexed category $\mathbf{Prop}^r : \mathcal{C}_r^{op} \to \mathbf{Pord}$ is defined as the preorder reflection $\mathbf{P}[\mathbf{Col}^r]$ of $\mathbf{Col}^r$ (see Definition 2.2). In this case we write $x' \Vdash \mathsf{P}(x)$ instead of $x' \varepsilon \mathsf{P}(x)$ for an object $\mathsf{P}(x)$ in $\mathbf{Col}^r(\mathsf{A})$ seen as an object of $\mathbf{Prop}^r(\mathsf{A})$.

REMARK 4.15. Notice that if A is an object of $\mathcal{C}_r$ and $\mathsf{P}(x)$ and $\mathsf{Q}(x)$ are objects of $\mathsf{Prop}^r(\mathsf{A})$, then $\mathsf{P}(x) \sqsubseteq_A \mathsf{Q}(x)$ in $\mathsf{Prop}^r(\mathsf{A})$ if and only if there is a numeral $\mathbf{r}$ such that $x' \Vdash \mathsf{P}(x) \vdash \{\mathbf{r}\}(x, x') \Vdash \mathsf{Q}(x)$ in $\widetilde{ID_1}$. So $\mathsf{P}(x)$ entails $\mathsf{Q}(x)$ if there is a recursive way (recursively depending on x in A) to send realizers of $\mathsf{P}(x)$ to realizers of $\mathsf{Q}(x)$. The fact that A is a realized collection of natural numbers allows us to exploit the numerical data from the underlying domain A and include them in the notion of entailment. This does not happen in the effective tripos [7] where the base category is Set and its entailment is uniformly defined with respect to the points in the underlying domain A. Namely, if P and Q are functions from A to $\mathcal{P}(\mathbb{N})$, then in the effective tripos $P \sqsubseteq_A Q$ holds iff $\bigcap_{a \in A} \{n \in \mathbb{N} | \forall x \in P(a)(\{n\}(x) \in Q(a))\} \neq \emptyset$ holds. However (see Remark 4.17) these two notions can be compared.

Now we are ready to prove the following theorem:

THEOREM 4.16. $\mathbf{Prop}^r$ *is a first-order hyperdoctrine with weak comprehensions.*

PROOF. For every object A of $\mathcal{C}_r$, $\mathbf{Prop}^r(\mathsf{A})$ is a Heyting prealgebra. In fact it is sufficient to consider bottom, top, binary infima, binary suprema and Heyting implications given by $\bot_{\mathsf{A}} := 0_{\mathsf{A}}$, $\top_{\mathsf{A}} := 1_{\mathsf{A}}$, $\mathsf{C}(x) \sqcap_{\mathsf{A}} \mathsf{D}(x) := \mathsf{C}(x) \times_{\mathsf{A}} \mathsf{D}(x)$, $\mathsf{C}(x) \sqcup_{\mathsf{A}} \mathsf{D}(x) := \mathsf{C}(x) +_{\mathsf{A}} \mathsf{D}(x)$ and $\mathsf{C}(x) \to_{\mathsf{A}} \mathsf{D}(x) := \mathsf{C}(x) \Rightarrow_{\mathsf{A}} \mathsf{D}(x)$ respectively for all objects $\mathsf{C}(x), \mathsf{D}(x)$ in $\mathbf{Prop}^r(\mathsf{A})$. Moreover from 4.7 it immediately follows that $\mathbf{Prop}^r_{\mathsf{f}}$ is a morphism of Heyting prealgebras from $\mathbf{Prop}^r(\mathsf{B})$ to $\mathbf{Prop}^r(\mathsf{A})$ for every arrow $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ in $\mathcal{C}_r$. From 4.8 and 4.9 one can easily obtain that for every such an arrow, $\exists_{\mathbf{f}}(\mathsf{C}(x)) := \Sigma_{\mathbf{f}}(\mathsf{C}(x))$ and $\forall_{\mathsf{f}}(\mathsf{C}(x)) := \Pi_{\mathbf{f}}(\mathsf{C}(x))$ define left and right adjoints to $\mathbf{Prop}^r_{\mathsf{f}}$ respectively in the category of preorders. One can easily check that these adjoints satisfy Beck–Chevalley condition.

If A is an object of $\mathcal{C}_r$ and $\mathsf{P}(x)$ is in $\mathbf{Prop}^r(\mathsf{A})$, then $\mathsf{p}_1^{\Sigma} : \Sigma(\mathsf{A}, \mathsf{P}(x)) \to \mathsf{A}$ determines a weak comprehension for $\mathsf{P}(x)$. ⊣

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REMARK 4.17. The hyperdoctrine of realized propositions enjoys also another interesting property. To this purpose we give the definition of "separated realized proposition": if A is an object of $\mathcal{C}_r$, a realized proposition $\mathsf{P}(x)$ in $\mathbf{Prop}^r(\mathsf{A})$ is called *separated* if $x' \Vdash \mathsf{P}(x) \wedge x' \Vdash \mathsf{P}(y) \vdash x = y$ in $\widehat{ID_1}$. It is very easy to show that if A is an object of $\mathcal{C}_r$, then every object of $\mathbf{Prop}^r(\mathsf{A})$ is equivalent in $\mathbf{Prop}^r(\mathsf{A})$ to a separated one. In fact if $\mathsf{P}(x)$ is an object of $\mathbf{Prop}^r(\mathsf{A})$, then we can consider the separated object $\mathsf{P}_{sep}(x) := \{x' | \, p_1(x') = x \wedge p_2(x') \Vdash \mathsf{P}(x)\}$ and observe that $\mathsf{P}(x) \sim_{\mathsf{A}} \mathsf{P}_{sep}(x)$ follows in $\widehat{ID_1}$.

DEFINITION 4.18. We denote with $\overline{\mathbf{Prop}^r}$ the posetal reflection (see Definition 2.2) of $\mathbf{Prop}^r$ (which coincides with the posetal reflection of $\mathbf{Col}^r$).

As an immediate consequence of Theorem 4.12 and the fact that $\mathbf{Col}^r$ is a functorial account of the slice pseudofunctor on $\mathcal{C}_r$ we have that:

THEOREM 4.19. *The first-order hyperdoctrine* $\overline{\mathbf{Prop}^r}$ *is naturally isomorphic to the doctrine of weak subobjects* $\mathbf{wSub}_{\mathcal{C}^r}$ *of* $\mathcal{C}^r$.

4.3. Sets and small propositions in $\mathcal{C}_r$. Here, following [6], we define a universe of sets internally in $\mathcal{C}_r$ as a fixpoint of a suitable admissible formula of $\widehat{ID_1}$. This admissible formula describes the elements of such a universe as codes of realized sets (which are defined in turn with their elements) in an *inductive* way. However, since we do not need to use induction on our universe of sets, we do not need to work in the proper theory of inductive definitions with least fixpoints and we can work just in $\widehat{ID_1}$ with fixpoints that are not necessarily the least ones. Following [6], we employ an admissible formula φ to obtain its fixpoint $P_\varphi(x)$ in $\widehat{ID_1}$ by which we define formulas $\mathsf{Set}(x)$, $x \,\overline{\varepsilon}\, y$ and $x \overline{\notin} y$, meaning "x is the code of a set", "x is an element of the set encoded by y" and "x is not an element of the set encoded by y", respectively.

We define the universe of sets in $\mathcal{C}_r$ as $\mathsf{U_S} := \{x | \, \mathsf{Set}(x) \wedge \forall t \, (t \,\overline{\varepsilon}\, x \leftrightarrow \neg \, t \overline{\notin} x)\}$ and we employ it to define a full subcategory of realized sets $\mathcal{S}_r$ of $\mathcal{C}_r$ and a hyperdoctrine $\overline{\mathbf{Prop}^r_s}$ of small proposition over it.

DEFINITION 4.20. The category of *realized sets* is the full subcategory $\mathcal{S}_r$ of $\mathcal{C}_r$ of which the objects are those realized collections of the form $\{x | \, x \,\overline{\varepsilon}\, \mathbf{n}\}$ for some numeral $\mathbf{n}$ for which $\mathbf{n} \varepsilon \mathsf{U_S}$ holds in $\widehat{ID_1}$.

Also here, in order to prove that $\mathcal{S}_r$ is weakly locally cartesian closed in a straightforward way, we introduce an indexed category of realized sets over it.

DEFINITION 4.21. Let A be an object of $\mathcal{S}_r$. The category $\mathbf{Set}^r(\mathsf{A})$ is the full subcategory of $\mathbf{Col}^r(\mathsf{A})$ whose objects are families of realized collections on A of the form $\tau_{\mathsf{A}}(\mathbf{n}) := \{x' | \, x' \,\overline{\varepsilon}\, \{\mathbf{n}\}(x) \wedge x \varepsilon \mathsf{A}\}$ for a numeral $\mathbf{n}$ defining an operation $[\mathbf{n}]_{\approx} : \mathsf{A} \to \mathsf{U_S}$ in $\mathcal{C}_r$. Objects of $\mathbf{Set}^r(\mathsf{A})$ are called *families of realized sets* on A.

The following lemma is an immediate consequence of the previous definition.

LEMMA 4.22. *Let* $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ *and* $\mathsf{g} = [\mathsf{n}]_{\approx} : \mathsf{B} \to \mathsf{U}_S$ *be arrows in* $\mathcal{S}_r$ *and suppose* $\mathbf{m}$ *is a numeral such that* $\mathsf{g} \circ \mathsf{f} = [\mathbf{m}]_{\approx} : \mathsf{A} \to \mathsf{U}_S$. *Then* $\mathbf{Col}^r_{\mathsf{f}}(\tau_{\mathsf{B}}(\mathbf{n})) = \tau_{\mathsf{A}}(\mathbf{m})$.

As a consequence of the previous lemma we can give the following definition.

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DEFINITION 4.23. $\mathbf{Set}^r : \mathcal{S}_r^{op} \to \mathbf{Cat}$ is the indexed category whose fibre over an object A of $\mathcal{S}_r$ is $\mathbf{Set}^r(\mathsf{A})$ and which sends an arrow $f : \mathsf{A} \to \mathsf{B}$ in $\mathcal{S}_r$ to the restriction of $\mathbf{Col}^r_f$ to $\mathbf{Set}^r(\mathsf{B})$, since its image is included in $\mathbf{Set}^r(\mathsf{A})$.

One can prove the following theorem as a consequence of the clauses determining the fixpoint formula giving rise to U_S (see [6]).

THEOREM 4.24. *For every* A *in* $\mathcal{S}_r$, $\mathbf{Set}^r(\mathsf{A})$ *is a finitely complete category with disjoint stable finite coproducts, parameterized list objects and weak exponentials. Moreover, If* $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ *in* $\mathcal{S}_r$, *then* $\mathbf{Set}^r_\mathsf{f}$ *has a left adjoint and for every object* $\mathsf{C}(x)$ *in* $\mathbf{Set}^r(\mathsf{A})$, $\Pi^r_\mathsf{f}(\mathsf{C}(x))$ *is isomorphic in* $\mathbf{Col}^r(\mathsf{B})$ *to an object of* $\mathbf{Set}^r(\mathsf{B})$.

The functor $\mathbf{Set}^r$ *is a functorial account of the slice pseudofunctor* $\mathcal{S}_r/-$ *and the embedding of* $\mathbf{Set}^r$ *in the restriction of* $\mathbf{Col}^r$ *to* $\mathcal{S}_r$ *preserves finite limits, finite coproducts, parameterized list objects and weak exponentials.*

In particular we have the following

COROLLARY 4.25. *$\mathcal{S}_r$ is a finitely complete category with stable disjoint finite coproducts, parameterized list objects and weak exponentials of which the structure is preserved by the embedding in $\mathcal{C}_r$.*

Now we are ready to define the indexed category of small realized propositions.

DEFINITION 4.26. The functor $\mathbf{Prop}^r_s : \mathcal{S}_r^{op} \to \mathbf{Pord}$ is defined as the preorder reflection $\mathbf{P}[\mathbf{Set}^r]$ of the indexed category $\mathbf{Set}^r$. Then, fibre objects of $\mathbf{Prop}^r_s$ are called *small realized propositions*.

Similarly to the case of $\mathbf{Prop}^r$ one can obtain from 4.24 the following theorem.

THEOREM 4.27. *$\mathbf{Prop}^r_s$ is a first-order hyperdoctrine over $\mathcal{S}_r$ with weak comprehensions.*

DEFINITION 4.28. We define $\overline{\mathbf{Prop}^r_s}$ as the posetal reflection (see Definition 2.2) of the doctrine $\mathbf{Prop}^r_s$ (which coincides with the posetal reflection of $\mathbf{Set}^r$) and we use the same notations for order, bottom, top, binary infima and suprema, Heyting implication and left and right adjoints.

Exploiting the definition of the coding and Theorem 4.24 one can easily prove

THEOREM 4.29. *$\mathbf{Prop}^r_s$ is equivalent to the weak subobject doctrine $\mathbf{wSub}_{\mathcal{S}_r}$ over $\mathcal{S}_r$.*

We can hence give the following definition.

DEFINITION 4.30. The pair $(\overline{\mathbf{Prop}^r}, \overline{\mathbf{Prop}^r_s})$ is called the *Kleene effective p-tripos*.

4.4. The internal language of the doctrine $\overline{\mathbf{Prop}^r}$. As for example in [25], we consider the internal language of the doctrine $\overline{\mathbf{Prop}^r}$ in the form of a first-order typed language with equality to show that it extends Kleene realizability interpretation of Heyting arithmetics and hence it validates the Formal Church's thesis.

Following [25], contexts Γ are interpreted as objects $\|\Gamma\|$ of $\mathcal{C}$. Terms in context $t[\Gamma]$ of type A are interpreted as arrows from $\|\Gamma\|$ to A and formulas in context $\phi[\Gamma]$ are interpreted as elements of $\overline{\mathbf{Prop}^r}(\|\Gamma\|)$. In particular, terms $\mathsf{f}(t_1,..,t_n)[\Gamma]$ are interpreted as $\mathsf{f} \circ \langle \|t_1[\Gamma]\|,..., \|t_n[\Gamma]\| \rangle$, formulas $\mathsf{R}(t_1,..,t_n)[\Gamma]$

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are interpreted as $\overline{\mathbf{Prop}^r}_{\langle \|t_1[\Gamma]\|,\ldots,\|t_n[\Gamma]\|\rangle}(\mathsf{R})$ and equality formulas $(t =_{\mathsf{A}} s)[\Gamma]$ as $\overline{\mathbf{Prop}^r}_{\langle \|t[\Gamma]\|,\|s[\Gamma]\|\rangle}(\exists_{\Delta_{\mathsf{A}}}(\top_{\mathsf{A}}))$. Formulas in context obtained with connectives are interpreted using the relative operators in Heyting algebras and formulas in context obtained with quantifiers are interpreted using the relative adjoints in the first-order hyperdoctrine $\overline{\mathbf{Prop}^r}$. If $\phi[\Gamma]$ is a formula in context, we define its validity as follows: $\overline{\mathbf{Prop}^r} \vdash \phi[\Gamma]$ if and only if $\top_{\|\Gamma\|} \sqsubseteq_{\|\Gamma\|} \|\phi[\Gamma]\|$.

The language of HA can be translated into the internal language of $\overline{\mathbf{Prop}^r}$, as every primitive n-ary recursive function f can be represented in $\mathcal{C}_r$ by an arrow $\overline{f} : \mathsf{N}^n \to \mathsf{N}$. The translation which assigns a term $\overline{t}$ of the language of $\overline{\mathbf{Prop}^r}$ to every term t of HA and a formula $\overline{\phi}$ in the internal language of $\overline{\mathbf{Prop}^r}$ to every formula ϕ of HA is defined as follows: $\overline{x_i}$ is x_i^{N} for every $i = 1,\ldots,n,\ldots$ and $\overline{f(t_1,\ldots,t_n)}$ is $\overline{f}(\overline{t_1},\ldots,\overline{t_n})$, $\overline{t = s}$ is $\overline{t} =_{\mathsf{N}} \overline{s}$ and $\overline{\bot}$ is $\bot$, $\overline{\phi \wedge \psi}$ is $\overline{\phi} \wedge \overline{\psi}$, $\overline{\phi \vee \psi}$ is $\overline{\phi} \vee \overline{\psi}$ and $\overline{\phi \to \psi}$ is $\overline{\phi} \to \overline{\psi}$, $\overline{\forall \xi\, \phi}$ is $(\forall \overline{\xi} \in \mathsf{N})\overline{\phi}$ and $\overline{\exists \xi\, \phi}$ is $(\exists \overline{\xi} \in \mathsf{N})\overline{\phi}$.

REMARK 4.31. As a consequence of Theorem 4.19 and results in [14] and [11], the axiom of choice with respect to weak exponentials holds in $\overline{\mathbf{Prop}^r}$, that is for every objects A and B in $\mathcal{C}_r$ and $\mathsf{R} \in \overline{\mathbf{Prop}^r}(A \times B)$

$$\overline{\mathbf{Prop}^r} \vdash (\forall x \in \mathsf{A})(\exists y \in \mathsf{B})\mathsf{R}(x,y) \to (\exists f \in \mathsf{A} \Rightarrow \mathsf{B})(\forall x \in \mathsf{A})\mathsf{R}(x,\mathsf{ev}(f,x)).$$

The next theorem follows easily from the fact that the interpretation of the internal language of $\overline{\mathbf{Prop}^r}$ extends Kleene realizability interpretation $\Vdash$ of HA:

LEMMA 4.32. *Suppose ϕ is a formula of Heyting arithmetic* HA *and* $\mathbf{n}$ *is a numeral. If* $\mathsf{HA} \vdash \mathbf{n} \Vdash \phi$, *then* $\overline{\mathbf{Prop}^r} \Vdash \overline{\phi}$.

THEOREM 4.33. (Formal Church's Thesis)

$$\begin{aligned}\overline{\mathbf{Prop}^r} \vdash\ & (\forall \overline{x} \in \mathsf{N})(\exists \overline{z} \in \mathsf{N})\mathsf{R}(\overline{x},\overline{z}) \\ & \to (\exists \overline{e} \in \mathsf{N})(\forall \overline{x} \in \mathsf{N})(\exists \overline{y} \in \mathsf{N})(\overline{T(e,x,y)} \wedge \mathsf{R}(\overline{x},\overline{U(y)}))\end{aligned}$$

where $T(e,x,y)$ and $U(y)$ are Kleene's predicate and primitive recursive function, respectively.

§5. A predicative version of Hyland's Effective Topos.

DEFINITION 5.1. The *Predicative Effective p-Topos* **pEff** is the category $\mathcal{Q}_{\overline{\mathbf{Prop}^r}}$ obtained by applying the elementary quotient completion in [17] to the doctrine $\overline{\mathbf{Prop}^r}$, that is, it is the category whose objects are the pairs (A,R) where A is an object of the category $\mathcal{C}_r$ and R is an object of $\overline{\mathbf{Prop}^r}(\mathsf{A} \times \mathsf{A})$ for which the following hold:

1. $\overline{\mathbf{Prop}^r} \vdash (\forall x \in \mathsf{A})\mathsf{R}(x,x)$;
2. $\overline{\mathbf{Prop}^r} \vdash (\forall x \in \mathsf{A})(\forall y \in \mathsf{A})(\mathsf{R}(x,y) \to \mathsf{R}(y,x))$;
3. $\overline{\mathbf{Prop}^r} \vdash (\forall x \in \mathsf{A})(\forall y \in \mathsf{A})(\forall z \in \mathsf{A})(\mathsf{R}(x,y) \wedge \mathsf{R}(y,z) \to \mathsf{R}(x,z))$.

An arrow from (A,R) to (B,S) is an equivalence class $[\mathsf{f}]_{\simeq}$ of arrows $\mathsf{f} : \mathsf{A} \to \mathsf{B}$ in $\mathcal{C}_r$ such that $\overline{\mathbf{Prop}^r} \vdash (\forall x \in \mathsf{A})(\forall y \in \mathsf{A})(\mathsf{R}(x,y) \to \mathsf{S}(\mathsf{f}(x),\mathsf{f}(y)))$, where $\mathsf{f} \simeq \mathsf{g}$ if and only if $\overline{\mathbf{Prop}^r} \vdash (\forall x \in \mathsf{A})(\mathsf{R}(x,x) \to \mathsf{S}(\mathsf{f}(x),\mathsf{g}(x)))$.

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As pointed out in [18] the elementary completion coincides with the ex/lex completion by Carboni and Celia Magno [3] when we consider the doctrine of weak subobjects **wSub** of a finitely complete category. As we proved Theorem 4.19, this applies to our case.

THEOREM 5.2. $\mathbf{pEff} \cong (\mathcal{C}_r)_{ex/lex}$. *In particular* **pEff** *is an exact category.*

From Theorems 5.2 and 4.13, using the main result in [4], we get

THEOREM 5.3. **pEff** *is a locally cartesian closed category.*

Even more we can show that **pEff** is actually a list-arithmetic pretopos.

LEMMA 5.4. **pEff** *has disjoint stable finite coproducts and list objects.*

PROOF. The existence of disjoint stable finite coproducts follows essentially from results in [2]. An initial object is $(0, \top_{0\times 0})$. A binary coproduct for (A,R) and (B,S) is determined by the equivalence relation on $\mathsf{A}+\mathsf{B}$ which is the interpretation of the following formula of the internal language of $\overline{\mathbf{Prop}^r}$

$$(\exists t \in \mathsf{A})\,(\exists s \in \mathsf{A})\,(\mathsf{R}(t,s) \wedge \mathsf{p}_1(x) =_{\mathsf{A}+\mathsf{B}} \mathsf{j}_1(t) \wedge \mathsf{p}_2(x) =_{\mathsf{A}+\mathsf{B}} \mathsf{j}_1(s)) \vee$$

$$(\exists t \in \mathsf{B})\,(\exists s \in \mathsf{B})\,(\mathsf{S}(t,s) \wedge \mathsf{p}_1(x) =_{\mathsf{A}+\mathsf{B}} \mathsf{j}_2(t) \wedge \mathsf{p}_2(x) =_{\mathsf{A}+\mathsf{B}} \mathsf{j}_2(s))$$

in the context $[x \in (\mathsf{A}+\mathsf{B}) \times (\mathsf{A}+\mathsf{B})]$, together with the injections $[\mathsf{j}_1]_\simeq$ and $[\mathsf{j}_2]_\simeq$. A list object for (A,R) has as underlying object $\mathsf{List}(\mathsf{A})$ and as equivalence relation that given by the interpretation of the formula

$$\mathsf{lh}(\mathsf{p}_1(x)) = \mathsf{lh}(\mathsf{p}_2(x)) \wedge$$

$$(\forall n \in \mathsf{N})((\exists a \in \mathsf{A})(\exists b \in \mathsf{A})(\mathsf{R}(a,b) \wedge \mathsf{j}_1(a) = \mathsf{cm}(\mathsf{p}_1(x),n) \wedge \mathsf{j}_1(b) = \mathsf{cm}(\mathsf{p}_2(x),n))$$

$$\vee\, (\mathsf{cm}(\mathsf{p}_1(x),n) = \mathsf{j}_2(0) \wedge \mathsf{cm}(\mathsf{p}_2(x),n) = \mathsf{j}_2(0)))$$

in the context $[x \in \mathsf{List}(\mathsf{A}) \times \mathsf{List}(\mathsf{A})]$ where $\mathsf{lh} : \mathsf{List}(\mathsf{A}) \to \mathsf{N}$ is the length arrow and $\mathsf{cm} : \mathsf{List}(\mathsf{A}) \times \mathsf{N} \to \mathsf{A}+1$ is the component arrow[2] in the category $\mathcal{C}_r$. The empty list and the append arrows are given by $[\varepsilon]_\simeq$ and $[\mathsf{cons}]_\simeq$, respectively. ⊣

The previous results about **pEff** can be summarized in the following theorem.

THEOREM 5.5. *The category* **pEff** *is a locally cartesian closed list-arithmetic pretopos.*

Now we give the definition of the hyperdoctrine $\mathbf{pEff}_{prop}$ of propositions associated by construction to **pEff** as an elementary quotient completion in [17] (this doctrine defines the equivalence relations with respect to which the elementary quotient completion is closed under effective quotients):

DEFINITION 5.6. The functor $\mathbf{pEff}_{prop} : \mathbf{pEff}^{op} \to \mathbf{Pos}$ is defined as follows:

1. $\mathsf{P} \in \mathbf{pEff}_{prop}(\mathsf{A},\mathsf{R})$ if $\mathsf{P} \in \overline{\mathbf{Prop}^r}(\mathsf{A})$ and

$$\overline{\mathbf{Prop}^r} \vdash (\forall x \in \mathsf{A})(\forall y \in \mathsf{A})(\mathsf{P}(x) \wedge \mathsf{R}(x,y) \to \mathsf{P}(y));$$

[2]The component arrow sends a pair $((l_0,\ldots,l_n),j)$ to $(0,l_j)$ if $j \le n$ and to $(1,0)$ otherwise.

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2. $\mathsf{P} \sqsubseteq_{(\mathsf{A},\mathsf{R})} \mathsf{Q}$ in $\mathbf{pEff}_{prop}((\mathsf{A},\mathsf{R}))$ iff $\mathsf{P} \sqsubseteq_{\mathsf{A}} \mathsf{Q}$ in $\overline{\mathbf{Prop}^r}(\mathsf{A})$;
3. $\mathbf{pEff}_{prop_{[\mathsf{f}]_\simeq}}(\mathsf{P}) := \overline{\mathbf{Prop}^r}_{\mathsf{f}}(\mathsf{P})$.

From results in [18] and Theorem 4.19 it follows that:

THEOREM 5.7. *The functor* $\mathbf{pEff}_{prop}$ *is a first-order hyperdoctrine over* **pEff** *which is equivalent to its subobject doctrine, that is,* $\mathbf{pEff}_{prop} \cong \mathbf{Sub}_{\mathbf{pEff}}$.

An important property of the doctrine $\mathbf{pEff}_{prop}$ is that it validates the Formal Church's Thesis as a consequence of the fact that the underlying doctrine $\overline{\mathbf{Prop}^r}$ validates it by Lemma 4.33:

THEOREM 5.8. (Formal Church's thesis in $\mathbf{pEff}_{prop}$) $\mathbf{pEff}_{prop}$ *validates*

$$(\forall \overline{x} \in \mathsf{N})(\exists \overline{z} \in \mathsf{N})\mathsf{R}(\overline{x},\overline{z}) \to (\exists \overline{e} \in \mathsf{N}_{\mathbf{pEff}})(\forall \overline{x} \in \mathsf{N})(\exists \overline{y} \in \mathsf{N})(\overline{T(e,x,y)} \wedge \mathsf{R}(\overline{y},\overline{U(y)})).$$

Let us now define a full subcategory of sets of **pEff**.

DEFINITION 5.9. The category $\mathbf{pEff}_{set}$ of sets of **pEff** is the full subcategory of **pEff** of which the objects are those objects (A,R) in **pEff** such that A is an object of $\mathcal{S}_r$ and $\mathsf{R} \in \overline{\mathbf{Prop}^r_s}(\mathsf{A} \times \mathsf{A})$. Equivalently, $\mathbf{pEff}_{set}$ is the base category of the elementary quotient completion in [17] of $\mathbf{pEff}_{prop_s}$.

As we have done above for **pEff**, one can easily adapt the proofs and obtain the following.

THEOREM 5.10. $\mathbf{pEff}_{set}$ *is a locally cartesian closed category with disjoint stable finite coproducts and list objects. Moreover, this structure is preserved by the embedding of* $\mathbf{pEff}_{set}$ *into* **pEff**.

DEFINITION 5.11. The functor $\mathbf{pEff}_{prop_s} : {\mathbf{pEff}_{set}}^{op} \to \mathbf{Pos}$ is the subfunctor of $\mathbf{pEff}_{prop}$ restricted to $\mathbf{pEff}_{set}$ whose value on (A,R) in $\mathbf{pEff}_{set}$ is the full subcategory of $\mathbf{pEff}_{prop}(\mathsf{A},\mathsf{R})$ whose objects are those P which are both in $\overline{\mathbf{Prop}^r_s}(\mathsf{A})$ and in $\mathbf{pEff}_{prop}(\mathsf{A},\mathsf{R})$.

THEOREM 5.12. *There exists an object* Ω *in* **pEff** *which represents* $\mathbf{pEff}_{prop_s}$, *that is, there is a natural isomorphism between* $\mathbf{pEff}_{prop_s}(-)$ *and* $\mathbf{pEff}(-,\Omega)|_{\mathbf{pEff}_{set}}$.

PROOF. The object Ω is $(\mathsf{U_S},[\mathsf{EQ}(x)]_{\sim_{\mathsf{U_S}\times\mathsf{U_S}}})$ where $x' \Vdash \mathsf{EQ}(x)$ is defined as $\forall t\,(t\,\overline{\varepsilon}\,p_1(x) \to \{p_1(x')\}(t)\,\overline{\varepsilon}\,p_2(x)) \wedge \forall s(s\,\overline{\varepsilon}\,p_2(x) \to \{p_2(x')\}(s)\,\overline{\varepsilon}\,p_1(x))$. ⊣

As a consequence of Theorem 4.29, we have the following

THEOREM 5.13. *The functor* $\mathbf{pEff}_{prop_s}$ *is a first-order hyperdoctrine which is equivalent to the subobject doctrine* $\mathbf{Sub}_{\mathbf{pEff}_{set}}$ *over* $\mathbf{pEff}_{set}$.

In particular, from Theorem 5.12 it follows that

THEOREM 5.14. *There exists a natural isomorphism between* $\mathbf{Sub}_{\mathbf{pEff}_{set}}$ *and* $\mathbf{Hom}_{\mathbf{pEff}}(-,\Omega)$ *over* $\mathbf{pEff}_{set}$; *that is, there is a subobject classifier for* $\mathbf{pEff}_{set}$ *in* **pEff**.

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§6. Embedding of pEff in Hyland's Effective Topos. Here we show that the construction of **pEff** performed on the subcategory of recursive functions of **Eff** gives rise to a full subcategory of the Effective Topos **Eff** whose embedding preserves the list-arithmetic locally cartesian closed pretopos structure.

The Effective Topos (see [7]) can be presented as the exact on lex completion $(\mathbf{pAsm})_{ex/lex}$ of the category **pAsm** of partitioned assemblies (see [21]). In particular, **Eff** inherits the validity of Formal Church's thesis from **pAsm** as shown in [15].

The category **pAsm** of partitioned assemblies is the category whose objects are pairs (A,P) where A is a set and $P : A \to \mathbb{N}$ is a function towards the set $\mathbb{N}$ of natural numbers, and whose arrows from such an object (A,P) to another (B,Q) are functions $f : A \to B$ such that there exists a partial recursive function $\varphi : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $Q \circ f = \varphi \circ P$.

The category **Rec** of recursive functions is defined as follows: the objects of **Rec** are subsets $A \subseteq \mathbb{N}$; an arrow from A to B is a function $f : A \to B$ such that there exists a (partial) recursive function $\overline{f} : \mathbb{N} \rightharpoonup \mathbb{N}$ such that $A \subseteq dom(\overline{f})$ and $\overline{f}|_A = f$; composition is given by composition of functions and identities are defined as identity functions.

The category **Rec** is clearly equivalent to the full subcategory of **pAsm** whose objects are of the form $(A, a \mapsto a)$ with $A \subseteq \mathbb{N}$. Moreover, the functor of weak subobjects over **Rec** is equivalent to that of weak subobjects over partitioned assemblies composed with the embedding of **Rec** into it.

If $\mathcal{Q}_{\mathbf{wSub}_{\mathbf{Rec}}}$ is the elementary quotient completion in [17] applied to the doctrine $\mathbf{wSub}_{\mathbf{Rec}}$ of weak subobjects in **Rec**, as a consequence of observations in [17, 18], we have that:

THEOREM 6.1. *The category $\mathcal{Q}_{\mathbf{wSub}_{\mathbf{Rec}}}$ is equivalent to the exact on lex completion of the category of recursive functions* **Rec**. *In particular $\mathcal{Q}_{\mathbf{wSub}_{\mathbf{Rec}}}$ is a full subcategory of the Effective Topos*:

$$\mathcal{Q}_{\mathbf{wSub}_{\mathbf{Rec}}} \cong (\mathbf{Rec})_{ex/lex} \hookrightarrow (\mathbf{pAsm})_{ex/lex} \cong \mathbf{Eff}.$$

The theory $\widehat{ID_1}$ has a *standard* model in set theory: one can in fact consider the set of natural numbers with the interpretation of Peano arithmetic and interpret the fixpoint formulas in $\widehat{ID_1}$ using transfinite induction.

THEOREM 6.2. *The standard interpretation of $\widehat{ID_1}$ in* ZFC *gives rise to a functor* $\mathbf{Int} : \mathcal{C}_r \to \mathbf{Rec}$ *sending each realized collection* A *of $\widehat{ID_1}$ to the subset of* $\mathbb{N}$ *given by the interpretation of the formula* $x \varepsilon \mathsf{A}$ *and sending each arrow* $[\mathbf{n}]_\approx$ *to the recursive function encoded by the corresponding natural number* n. *The interpretation also gives rise to a natural transformation preserving connectives and quantifiers* $\eta : \overline{\mathbf{Prop}^r} \to \mathbf{wSub}_{\mathbf{Rec}} \circ \mathbf{Int}$ *which is defined analogously using Theorem 4.19. In particular this allows one to define a functor* $\mathbf{J} : \mathbf{pEff} \to (\mathbf{Rec})_{ex/lex}$ *and then also the functor* $\mathbf{I} : \mathbf{pEff} \to \mathbf{Eff}$ *obtained by composing* **J** *with the embedding in Theorem 6.1. The functor* **I** *preserves finite limits, exponentials, lists, finite coproducts and quotients.*

PROOF. The first part follows from the definition of the interpretation. The functor **I** preserves finite limits, exponentials, lists, finite coproducts and quotients, because **J** and the embedding from $(\mathbf{Rec})_{ex/lex}$ to **Eff** preserve them. ⊣

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§7. Future work. Our predicative variant **pEff** of the Effective Topos has a richer structure than the one shown here. Indeed, first we can lift $\mathbf{Set}^{r}$ on $\mathcal{C}_r$ in the Kleene effective p-tripos to define dependent sets which can be used to interpret those of the intensional level of the Minimalist Foundation in [9]; second, we can use this lifting to define dependent sets over **pEff** which can be used to interpret those in the extensional level of the Minimalist Foundation in [9].

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