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Deterministic epidemic waves

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1. *Introduction.* In the well-known deterministic model for the spread of an epidemic, one considers a population of uniform density along a line and divides the population into three classes: susceptible but uninfected, infected and infectious, infected but removed. If we denote space and time variables by s, t and let $x(s,t)$, $y(s,t)$, $z(s,t)$ be the proportions of the population at (s,t) in these three classes, then $x+y+z=1$ and we suppose that

$$\left.\begin{aligned} \frac{\partial}{\partial t}x(s,t) &= -x(s,t)\,\bar{y}(s,t), \\ \frac{\partial}{\partial t}y(s,t) &= x(s,t)\,\bar{y}(s,t) - by(s,t), \\ \frac{\partial}{\partial t}z(s,t) &= by(s,t). \end{aligned}\right\} \tag{1}$$

Here $\bar{y}(s,t)$ denotes a space average $\int y(s+\sigma)\,p(\sigma)\,d\sigma$, where p is a probability density function; b is the removal rate; the scale of t has been adjusted to remove a constant that would otherwise occur in (1).

We shall be interested in wave solutions of (1), so that $x(s,t)=x(s-ct)$ (and similarly for y,z) for a wave travelling towards $+\infty$ with velocity c. Then (1) becomes, writing $u=s-ct$, and x' for dx/du,

$$cx' = x\bar{y}, \quad cy' = -x\bar{y}+by, \quad cz' = -by. \tag{2}$$

The system (2) has been considered by Kendall (2) and Mollison (3). Kendall made (2) more tractable by replacing the space average

$$\bar{y}(u) = \int y(u+\sigma)\,p(\sigma)\,d\sigma \tag{3}$$

by a 'diffusion approximation' $\bar{y}(u) = y(u)+ky''(u)$. Mollison considered the special case when the weighting factor $p(\sigma)$ in (3) is the double exponential $\frac{1}{2}e^{-|\sigma|}$ and when $b=0$ (there are no removals). Each of them was then able to convert the problem into one involving a system of ordinary differential equations with two or three unknowns, and each of them found that wave solutions exist if and only if the velocity c is not less than some minimum value c_0.

We shall make very general assumptions about $p(\sigma)$ and prove an existence theorem, treating the simpler case of no removals ($b=0$) first in §2 and the more complicated case $b>0$ later in §4. Our results are more general than those of Kendall and Mollison,

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but less precise because we have no uniqueness theorem and, when there is a critical velocity c_0, we cannot prove existence for $c = c_0$. We shall also, in §3, look at the special cases of negative exponential weighting:

$$p(\sigma) = e^{-\sigma} \quad \text{for} \quad \sigma \geqslant 0, \quad = 0 \quad \text{for} \quad \sigma < 0,$$

$$p(\sigma) = e^{\sigma} \quad \text{for} \quad \sigma \leqslant 0, \quad = 0 \quad \text{for} \quad \sigma > 0;$$

this can be treated by looking at a plane system of first order differential equations. Some numerical results, and open questions connected with the results in §2 and §4 will be discussed in §5.

References to earlier work can be found in Kendall (2).

2. *The simple epidemic* ($b = 0$). When $b = 0$, we have $x + y = 1$ and equations (2) reduce to the single equation

$$cx'(u) = x(u)\,(1 - \bar{x}(u)), \tag{4}$$

where

$$\bar{x}(u) = \int x(u+\sigma)\,p(\sigma)\,d\sigma.$$

We look for a solution of (4) such that $0 \leqslant x \leqslant 1$. Since $x(u)$ is increasing, it must tend to limits $x(\infty)$ and $x(-\infty)$ at $\pm\infty$, and necessarily

$$x(\infty) = 1, \quad x(-\infty) = 0. \tag{5}$$

[If $x(\infty) = L < 1$, then $\bar{x}(\infty) = L$ and $x'(u)$ would tend to a positive limit as $u \to +\infty$, impossible because $0 \leqslant x(u) \leqslant 1$ for all u; similarly $x(-\infty) = L > 0$ is impossible. Of course we have excluded the trivial solutions $x(u) \equiv 0$, $x(u) \equiv 1$.] We now write $x = e^X$, so that $X \leqslant 0$, and for convenience write $a = c^{-1}$. Then (4) and (5) become

$$X' = a(\overline{1 - e^X}), \quad X(-\infty) = -\infty, \quad X(+\infty) = 0. \tag{6}$$

The differential equation in (6) is

$$\left.\begin{aligned} X'(u) &= a\int (1 - e^{X(u+v)})\,p(v)\,dv, \\ &= a\int (1 - e^{X(w)})\,p(w-u)\,dw. \end{aligned}\right\} \tag{7}$$

If we integrate this from u to ∞, using the condition $X(+\infty) = 0$, and invert the order of integration, we get

$$\left.\begin{aligned} X(u) &= -a\int (1 - e^{X(u+v)})\,Q(v)\,dv, \\ &= -a\int (1 - e^{X(w)})\,Q(w-u)\,dw, \end{aligned}\right\} \tag{8}$$

where $Q(v) = \int_{-\infty}^{v} p(\sigma)\,d\sigma$ is the left-hand tail of the distribution with density $p(\sigma)$. Conversely if X satisfies (8) it will satisfy the differential equation (7) and the boundary conditions $X(-\infty) = -\infty$, $X(+\infty) = 0$: we omit the straightforward proof.

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Next, we observe that if (6) has a solution X then $X(u)$ must be exponentially small as $u \to +\infty$. To see this, we use the first equation in (8). Since $Q(v) \to 1$ as $v \to +\infty$ we can find k so that $ka > 4$ and $Q(v) \geqslant \frac{1}{2}$ for $v \geqslant k$. So

$$\begin{aligned} -X(u) &= a\int (1-e^{X(u+v)})\,Q(v)\,dv \\ &\geqslant \tfrac{1}{2}a\int_k^{2k} (1-e^{X(u+v)})\,dv \\ &\geqslant \tfrac{1}{2}ka(1-e^{X(u+2k)}) \quad \text{(because } X \text{ increases)} \\ &\geqslant \tfrac{1}{4}ka(-X(u+2k)) \quad \text{for large } u \end{aligned}$$

because $$(1-e^{X(u+2k)})/(-X(u+2k)) \to 1 \quad \text{as} \quad u \to +\infty.$$

So $$-X(u+2k) \leqslant K(-X(u)) \quad \text{for large } u,$$

where $K < 1$ because $ka > 4$, which implies easily that $X(u) = O(e^{-Cu})$ as $u \to \infty$ for some positive C. Also (7) shows that $X'(u) \leqslant a$, so that $X(u) = O(|u|)$ as $u \to -\infty$. Thus the integral

$$\int_{-\infty}^{\infty} -X(u)\,e^{\lambda u}\,du$$

will be finite for sufficiently small $\lambda > 0$. It follows that the transform

$$F(\lambda) \equiv \int_{-\infty}^{\infty} Q(w)\,e^{-\lambda w}\,dw \tag{9}$$

will also be finite for sufficiently small $\lambda > 0$. To see this, multiply the equation

$$-X(u) = a\int (1-e^{X(u+v)})\,Q(v)\,dv$$

by $e^{\lambda u}$ and integrate to obtain

$$\begin{aligned} \int -X(u)\,e^{\lambda u}\,du &= a\int du\,e^{\lambda u}\left(\int (1-e^{X(u+v)})\,Q(v)\,dv\right) \\ &= a\int dv\,Q(v)\left(\int (1-e^{X(u+v)})\,e^{\lambda u}\,du\right) \\ &= a\left(\int Q(v)\,e^{-\lambda v}\,dv\right)\left(\int (1-e^{X(w)})\,e^{\lambda w}\,dw\right). \end{aligned}$$

The second integral on the right is finite because $1-e^X \leqslant -X$ for $X < 0$, hence the first integral $\int Q(v)\,e^{-\lambda v}\,dv$ must also be finite. Of course the right-hand half $\int_0^{\infty}$ of this integral will converge for all $\lambda > 0$ because $Q \leqslant 1$; convergence of the left-hand half $\int_{-\infty}^{0}$ implies, using monotonicity of Q, that $Q(u) = O(e^{\lambda u})$ as $u \to -\infty$. In other words, if problem (6) has a solution then the tail Q *must* be exponentially small at $-\infty$, the transform F at (9) must exist for small $\lambda > 0$, and the solution itself must be exponentially small at $+\infty$. (Similar conclusions appear to have been reached by Mollison (3).)

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In beginning to look for a solution it is therefore reasonable to hope that it will be asymptotically a negative exponential at $+\infty$, so that we may as well look for $X(u)$ such that $X(u) \sim -e^{-\alpha u}$ as $u \to +\infty$ with $\alpha > 0$; a multiplicative constant has been removed by shifting the origin of u. From (8) we have

$$-X(u)\,e^{\alpha u} = a\int [(1-e^{X(u+v)})\,e^{\alpha(u+v)}]\,e^{-\alpha v}\,Q(v)\,dv.$$

The factor in square brackets tends to 1 as $u \to \infty$, and by monotone convergence the right-hand side tends to $a\int e^{-\alpha v}\,Q(v)\,dv = aF(\alpha)$ as $u \to +\infty$, with $F(\alpha)$ necessarily finite: so α is a positive root of the equation $aF(\alpha) = 1$. We note in passing that $F(\alpha)$ is convex for the range of α where it is finite, so that $aF(\alpha) = 1$ can have at most two positive roots. If there are two, say $\alpha_1 < \alpha_2$, then $aF(\alpha)$ will be less than unity for $\alpha_1 < \alpha < \alpha_2$. Taking any such α, we can rule out the possibility of a solution $X(u) \sim -e^{-\alpha_2 u}$ because

$$\begin{aligned} -X(u)\,e^{\alpha u} &= a\int [1-e^{X(u+v)}\,e^{\alpha(u+v)}]\,e^{-\alpha v}\,Q(v)\,dv \\ &< a\int [-X(u+v)\,e^{\alpha(u+v)}]\,e^{-\alpha v}\,Q(v)\,dv \end{aligned}$$

and integration over u gives

$$\begin{aligned} \int -X(u)\,e^{\alpha u}\,du &< aF(\alpha)\int -X(w)\,e^{\alpha w}\,dw \\ &< \int -X(w)\,e^{\alpha w}\,dw \end{aligned}$$

because $aF(\alpha) < 1$, with the integral finite, which is impossible.

We have now provided enough motivation for stating the theorem which we shall prove below:

THEOREM 1. *Suppose that the equation $aF(\lambda) = 1$, i.e. $F(\lambda) = c$, has a positive root $\lambda = \alpha$, and that $aF(\lambda)$ exists and is < 1 in some interval $\alpha < \lambda \leqslant \alpha + \delta$. Then the equation*

$$X(u) = -a\int [1-e^{X(u+v)}]\,Q(v)\,dv$$

has a solution such that $X(u) < 0$ for all u, $X(u) \sim -e^{-\alpha u}$ as $u \to +\infty$, $X(u) \sim au$ as $u \to -\infty$.

Proof. Put $aF(\alpha+\delta) = m < 1$ and consider the iterative scheme

$$X_0(u) = -e^{-\alpha u}, \quad X_{n+1}(u) = -a\int [1-e^{X_n(u+v)}]\,Q(v)\,dv. \tag{10}$$

Then

$$\begin{aligned} X_1(u) &= -a\int [1-e^{X_0(u+v)}]\,Q(v)\,dv \\ &\geqslant -a\int [-X_0(u+v)]\,Q(v)\,dv \\ &= -a\int e^{-\alpha(u+v)}\,Q(v)\,dv \\ &= -a\,e^{-\alpha u}\,F(\alpha) = -e^{-\alpha u} = X_0(u). \end{aligned}$$

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It is then easy, because $1-\exp X$ decreases as $X<0$ increases, to show inductively that

$$X_0(u) \leqslant X_1(u) \leqslant \ldots \leqslant X_n(u) \leqslant X_{n+1}(u) \leqslant \ldots < 0, \tag{11}$$

so that $X_n(u)$ increases to a limit $X(u)$ as $n \to \infty$.

We now want to show that $X(u) \not\equiv 0$, in fact that $X(u) < 0$ for all u. It is easy to see from (10) that each $X_n(u)$, and therefore $X(u)$, is an increasing function of u, so it will be enough to show $X(u) < 0$ in some range $u \geqslant u_0$. We shall do this by showing that we can find constants $M > 0$ and $\delta > 0$ such that

$$X_n(u) \leqslant -e^{-\alpha u} + M e^{-(\alpha+\delta)u} \quad \text{for all } u \quad (n = 0, 1, 2, \ldots). \tag{12}$$

The proof is by induction. (12) holds, whatever M we choose, when $n = 0$. Assume that it holds for n. We may assume (decreasing δ if necessary) that $0 < \delta \leqslant \alpha$, and we will use the fact that

$$X < e^X - 1 < X + \tfrac{1}{2}X^2 \quad \text{for} \quad X < 0.$$

Now

$$\begin{aligned} X_{n+1}(u) &= a\int_{-\infty}^{\infty} [e^{X_n(w)} - 1]\, Q(w-u)\, dw \\ &\leqslant a\int_{0}^{\infty} [e^{X_n(w)} - 1]\, Q(w-u)\, dw \\ &\leqslant a\int_{0}^{\infty} [X_n(w) + \tfrac{1}{2}X_n^2(w)]\, Q(w-u)\, dw. \end{aligned}$$

Using (12) for X_n, and $-e^{-\alpha w} \leqslant X_n(w) < 0$ from (11) so that $X_n^2(w) \leqslant e^{-2\alpha w} \leqslant e^{-(\alpha+\delta)w}$

$$\begin{aligned} X_{n+1}(u) &\leqslant a\int_{0}^{\infty} (-e^{-\alpha w} + (M+\tfrac{1}{2})\, e^{-(\alpha+\delta)w})\, Q(w-u)\, dw \\ &= a\int_{-\infty}^{\infty} (-e^{-\alpha w} + (M+\tfrac{1}{2})\, e^{-(\alpha+\delta)w})\, Q(w-u)\, dw \\ &\qquad + a\int_{-\infty}^{0} e^{-\alpha w}[1 - (M+\tfrac{1}{2})\, e^{-\delta w}]\, Q(w-u)\, dw \\ &= -aF(\alpha)\, e^{-\alpha u} + a(M+\tfrac{1}{2})\, F(\alpha+\delta)\, e^{-(\alpha+\delta)u} \\ &\qquad + a\int_{-\infty}^{0} e^{-\alpha w}[1 - (M+\tfrac{1}{2})\, e^{-\delta w}]\, Q(w-u)\, dw. \end{aligned}$$

Since $aF(\alpha) = 1$ and $aF(\alpha+\delta) = m < 1$, the first two terms are

$$= -e^{-\alpha u} + m(M+\tfrac{1}{2})\, e^{-(\alpha+\delta)u},$$

which is $\leqslant -e^{-\alpha u} + M e^{-(\alpha+\delta)u}$ if we can pick M so that $m(M+\frac{1}{2}) \leqslant M$. In the third term, the integrand is $\leqslant 0$ provided that $1-(M+\frac{1}{2})\, e^{-\delta w} \leqslant 0$ for $w \leqslant 0$, which will hold if $M+\frac{1}{2} \geqslant 1$. We can satisfy both requirements for M by making

$$M = \max\left(\tfrac{1}{2}m/(1-m),\ \tfrac{1}{2}\right),$$

and if we do this we can justify the inductive step from n to $n+1$ in (12). Then (12) is proved, and $n \to \infty$ gives

$$X(u) \leqslant -e^{-\alpha u} + M e^{-(\alpha+\delta)u} \quad \text{for all } u.$$

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This proves that $X(u) < 0$ for u large enough to make $M e^{-\delta u} < 1$ and therefore (as noted earlier) that $X(u) < 0$ for all u. Since $X_n(u) \geqslant X_0(u) = -e^{-\alpha u}$ we also have

$$-e^{-\alpha u} \leqslant X(u) \leqslant -e^{-\alpha u} + M e^{-(\alpha+\delta)u}$$

for all u, which shows that $X(u) \sim -e^{-\alpha u}$ as $u \to +\infty$. It is now easy to show, by letting $n \to \infty$ in the recurrence relation (10), that the equation (8) holds. Also from the second form of (8), i.e.

$$X(u) = -a \int [1 - e^{X(w)}] Q(w-u)\, dw,$$

one can easily deduce that $X'(u)$ exists and equals

$$a \int [1 - e^{X(w)}] p(w-u)\, dw = a \int [1 - e^{X(u+v)}] p(v)\, dv.$$

Since $X(u)$ must tend to a limit $l < 0$ as $u \to -\infty$, we now see that $X'(u) \to a(1-e^l) > 0$. This shows that $X(u) \to -\infty$ as $u \to -\infty$ (in fact we now see that $X'(u) \to a$, so that $X(u) \sim au$, as $u \to -\infty$), which completes the proof of Theorem 1.

In most simple examples the 'characteristic equation' $F(\alpha) = c$ in Theorem 1 has two roots (perhaps coincident) or none, because $F(\alpha)$ is convex and has a minimum value c_0 at $\alpha = \alpha_0$ and then either becomes infinite at some larger values of α or exists for all $\alpha > \alpha_0$ and $\to \infty$ as $\alpha \to \infty$. For example, the double exponential $p(s) = \frac{1}{2}e^{-|s|}$ has $F(\alpha) = 1/(\alpha - \alpha^3)$, $\alpha_0 = 1/\sqrt{3}$, $\min F(\alpha) = c_0 = \frac{1}{2}(3\sqrt{3})$, as one would expect from Mollison's result in (3). The uniform density $p(s) = \frac{1}{2}$, $-1 \leqslant s \leqslant 1$, has $F(\alpha) = \alpha^{-2} \sinh \alpha$ with minimum at the root $\alpha_0 > 0$ of $\frac{1}{2}\alpha \cosh \alpha = \sinh \alpha$; $\alpha_0 \cong 1{\cdot}91$ and $F(\alpha_0) = c_0 \cong 0{\cdot}91$. In such cases Theorem 1 proves the existence of waves for any velocity $c > c_0$ but cannot deal with the critical velocity $c = c_0$. There is one exception to this general description. If the average $\bar{X}$ is entirely lop-sided, i.e. if $p(s) = 0$ for $s < 0$, then $F(\lambda)$ exists for all $\lambda > 0$ and decreases from ∞ to 0 as λ increases from 0 to ∞. So there is a solution α of $F(\alpha) = c$ for every $c > 0$, and Theorem 1 proves existence of waves with any positive velocity c. In this case the proof of Theorem 1, in which we can now take $\delta = \alpha$, provides an inequality $-e^{-\alpha t} \leqslant X(t) \leqslant -e^{-\alpha t} + M e^{-2\alpha t}$ for the solution. Some further argument, which we omit, even shows that we may take $M = \frac{1}{2}$.

This raises the question whether $c_0 = \min F(\alpha)$ is a genuine lower bound for wave velocities, i.e. whether waves are impossible for $c < c_0$. This question is answered to some extent by:

THEOREM 2. *Suppose that* $F(\lambda) \equiv \int_{-\infty}^{\infty} Q(u) e^{-\lambda u}\, du$ *is finite for* $0 < \lambda \leqslant A$, *and that* $F(\lambda)$ *has a minimum* c_0 *at* $\lambda = \alpha_0 < A$. *Then for* $0 < c < c_0$ *there is no solution to* $cX(u) = -\int [1 - e^{X(u+v)}] Q(v)\, dv$ *with* $X(-\infty) = -\infty$, $X(+\infty) = 0$.

Proof. We shall write $\phi(\lambda) = \int_{-\infty}^{\infty} X(u) e^{\lambda u}\, du$ when this integral is finite and first show that it is finite for all $0 < \lambda < A$. Since the part $\int_{-\infty}^{0}$ is finite for all $\lambda > 0$ (remember $X(u) = O(|u|)$ as $u \to -\infty$), it is $\int_{0}^{\infty}$ that may become infinite. Since we

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know that $X(u)$ is exponentially small at $+\infty$, $\phi(\lambda)$ is finite for sufficiently small $\lambda > 0$, so that if $\phi(\lambda)$ is not finite for all $\lambda < A$, there will exist $B < A$ such that $\phi(\lambda)$ is infinite for $\lambda > B$, finite for $0 < \lambda < B$. For complex λ, $\phi(\lambda)$ is regular for $0 < re\lambda < B$ but (see Widder (6), Theorem 5*b*, p. 58) has a singularity at $\lambda = B$. Now the equation $cX(u) = -\int [1-e^{X(w)}]\, Q(w-u)\, dw$, multiplied by $e^{\lambda u}$ and integrated over u, leads to

$$c\phi(\lambda) = -\left(\int [1-e^{X(w)}]\, e^{\lambda w}\, dw\right) F(\lambda)$$

$$= F(\lambda)\left\{\phi(\lambda) + \int (e^{X(w)} - 1 - X(w))\, e^{\lambda w}\, dw\right\} \quad \text{if } \phi(\lambda) \text{ is finite.}$$

But for $X < 0$ we have $0 < e^X - 1 - X < \frac{1}{2}X^2$, which for $X = X(w)$ is $O(w^2)$ as $w \to -\infty$ and $O(|X(w)|\, e^{-kw})$ for some $k > 0$ as $w \to +\infty$. So if $\phi(\lambda)$ is finite for $\lambda < B$, the integral $\int (\exp X(w) - 1 - X(w))\, e^{\lambda w}\, dw$ will be finite for $\lambda < B + k$ and will be a regular function $\psi(\lambda)$ of complex λ with $re\lambda < B + k$. Now look at the equation

$$(c - F(\lambda))\, \phi(\lambda) = F(\lambda)\, \psi(\lambda), \tag{13}$$

with $0 < c < c_0$. The right-hand side is regular in a sufficiently small disc $|\lambda - B| < r$, and $c - F(\lambda)$, non-zero for real λ, will also be regular and non-zero in this disc. Hence $\phi(\lambda)$ cannot, after all, have a singularity at $\lambda = B$. We have now proved our claim that $\phi(\lambda)$ is finite for all $\lambda < A$.

Now take the equation

$$c\phi(\lambda) = -F(\lambda) \int [1-e^{X(w)}]\, e^{\lambda w}\, dw, \tag{14}$$

and multiply it by $e^{-\lambda w_0}$, with w_0 to be chosen later, to give

$$\int [cX(w) + F(\lambda)\,(1-e^{X(w)})]\, e^{\lambda(w-w_0)}\, dw = 0.$$

Differentiating with respect to λ at $\lambda = \alpha_0$ and recalling that $F(\alpha_0) = c_0 > c$ and $F'(\alpha_0) = 0$, we get

$$\int (w - w_0)\, \{cX(w) + c_0(1-e^{X(w)})\}\, e^{\alpha_0(w-w_0)}\, dw = 0. \tag{15}$$

But $(1-e^X)/(-X)$ increases from 0 to 1 as X increases from $-\infty$ to 0, so that we can find w_0 such that

$$(1-e^{X(w_0)})/(-X(w_0)) = \frac{c}{c_0} < 1.$$

Then $cX(w) + c_0(1-e^{X(w)})$ will be positive for $w > w_0$, negative for $w < w_0$, so that the integrand in (15) is positive for $w \neq w_0$, and the integral will be positive, contradicting (15).

We have reached a contradiction by assuming that a solution exists, and this proves Theorem 2.

3. *Two examples* (*Negative exponential weighting*). Our positive results (the existence Theorem 1 and non-existence Theorem 2) were suggested not only by Mollison's

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complete analysis (3) of the double exponential weighting case but also by an analysis, easier than Mollison's, of negative exponential weighting concentrated on a half-line. It is easier because we have to deal with a 2-dimensional system of ordinary differential equations instead of Mollison's 3-dimensional system. We assume the reader to be familiar with the classical theory of such systems, see, for example, (5), chapter II.

Example 1. $p(s) = e^{-s}$, $s \geqslant 0$, 0 otherwise.

To find x such that $x'(u) = ax(u)(1-\bar{x}(u))$, $x(-\infty) = 0$, $x(\infty) = 1$. Here

$$\bar{x}(u) = \int x(u+s)\,p(s)\,ds = \int_0^\infty x(u+s)\,e^{-s}\,ds = e^u \int_u^\infty x(v)\,e^{-v}\,dv,$$

so that

$$\bar{x}'(u) = e^u \int_u^\infty \ldots - x(u) = \bar{x}(u) - x(u).$$

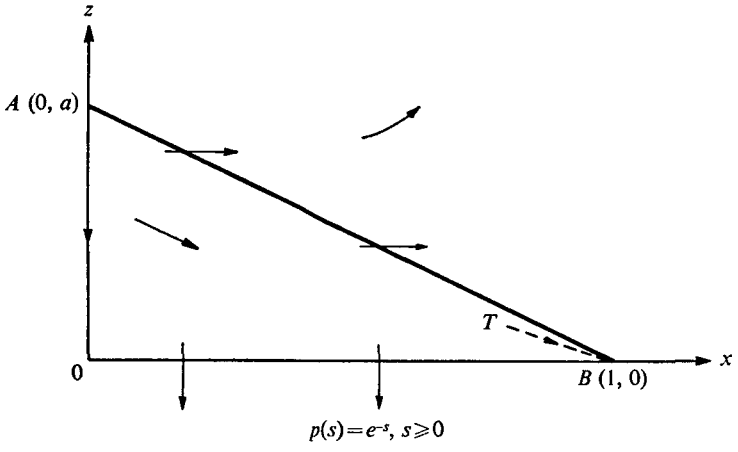


Fig. 1

Writing $x'/x = z$, we have $z = a(1-\bar{x})$, $z' = -a\bar{x}' = -a(\bar{x}-x) = z + a(x-1)$, so that we are looking for a solution of the system

$$x' = xz, \quad z' = z + a(x-1) \tag{16}$$

such that $0 < x < 1$, $z > 0$. Also $x(-\infty) = 0$ implies $\bar{x}(-\infty) = 0$, so that $x'/x = a(1-\bar{x}) \to a$ at $-\infty$ whereas $x(+\infty) = 1$ similarly implies $x'/x \to 0$ at $+\infty$. That is, the solution $(x(u), z(u))$ of system (16) should lead from $(0, a)$ to $(1, 0)$. We leave it to the reader to verify that, conversely, such a solution of system (16) leads to a solution $x(u)$ of our original equation $x' = ax(1-\bar{x})$.

Now (16) is a standard 2-dimensional system, with critical points at $A(0, a)$ and $B(1,0)$. Here $(1, 0)$ is a saddle point, and routine calculations show that there is a trajectory T approaching $(1, 0)$ as $u \to +\infty$, lying in $0 < x < 1$, $z > 0$ with slope $-a/[\frac{1}{2} + \sqrt{(\frac{1}{4} + a)}]$, so that $0 >$ slope $> -a$. Trajectories, sketched on Fig. 1 with arrows indicating directions as $u \to +\infty$, cross the sides of triangle OAB as shown (AB crossed horizontally, OB vertically). This makes it clear that the trajectory T, followed backwards (with decreasing u) must stay inside the triangle OAB, and it will have to approach the critical point $A(0, a)$ as $u \to -\infty$. (In fact A is a node, but we shall

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not pursue behaviour near A.) This shows that, for any $a > 0$, there is a unique trajectory leading from A to B as u goes from $-\infty$ to $+\infty$. Hence there is a unique solution $x(u)$, up to change of u-origin, of our original problem. We can also find out how it behaves near B, as $u \to +\infty$, by linearizing (16) near B. With $x = 1+\xi$, ξ small, the linearized equations are

$$\xi' = z, \quad z' = a\xi + z$$

and solutions having $\xi \to 0$, $z \to 0$, as $u \to \infty$ behave like

$$\xi \sim C \exp\{\tfrac{1}{2} - \sqrt{(\tfrac{1}{4}+a)}\}\, u = C e^{-\alpha u}, \quad \text{say},$$

and we will want $C < 0$ to keep $x < 1$.

Although we did this work before we proved Theorems 1 and 2, we should check that Theorem 1 ensures existence of a solution for any $a > 0$. Now $F(\alpha) = 1/[\alpha(1+\alpha)]$, so that the characteristic equation $aF(\alpha) = 1$ is $\alpha(1+\alpha) = a$ with the positive solution $\alpha = -\frac{1}{2} + \sqrt{(\frac{1}{4}+a)}$. Thus Theorem 1 tells us, as it should, that for any $a > 0$ there is a solution with $X(u) = \log x(u) \sim -e^{-\alpha u}$, hence with $x(u) \cong 1 + X(u) \cong 1 - e^{-\alpha u}$ as we found above.

Example 2. $p(s) = e^s$, $s \leqslant 0$, 0 otherwise.

Here $\bar{x}(u) = e^{-u} \int_{-\infty}^{u} x(v)\, e^v\, dv$ and similar calculations to those for Example 1 lead to the system

$$x' = xz, \quad z' = -z + a(1-x), \tag{17}$$

and again we look for a trajectory leading from $A(0, a)$ to $B(1, 0)$. This time the eigenvalues $-\frac{1}{2} \pm \sqrt{(\frac{1}{4}-a)}$ for the linearized system near B are complex if $a > \frac{1}{4}$, solutions near B are spirals and cannot stay inside the triangle OAB. Hence we must assume $a \leqslant \frac{1}{4}$ (i.e. $c \geqslant 4$). Then B is a stable node and we may hope to find a trajectory reaching B. Turning our attention to A, this is a saddle point, and there is a trajectory T leaving it from $u = -\infty$ with slope $-a/(a+1)$. As shown in Fig. 2, T initially lies above AB but below the curve Γ: $z = a - ax^2$. The arrows in Fig. 2 (AB is crossed horizontally) show that T must stay above AB; if T were to cross Γ, its slope

$$\frac{dz}{dx} = \frac{a(1-x) - z}{xz}$$

would, since $z = a(1-x^2)$ on Γ, be

$$\frac{a(1-x) - a(1-x^2)}{xz} = \frac{-ax(1-x)}{ax(1-x^2)} = \frac{-1}{1+x} < -\frac{1}{2},$$

while Γ has slope $-2ax > -2 \cdot \frac{1}{4} \cdot 1 = -\frac{1}{2}$ (because $a \leqslant \frac{1}{4}$ and $x < 1$). That is, slope $T <$ slope Γ at any crossing of Γ by T. But T starts below Γ so that at the first crossing, if any, we should have slope $T \geqslant$ slope Γ. Hence T does not cross Γ, but stays below Γ and above AB. It must therefore approach the critical point B as $u \to +\infty$. Hence:

there is no solution when $a > \frac{1}{4}$,

there is exactly one solution when $a \leqslant \frac{1}{4}$,

i.e. there is a critical velocity $c = 4$.

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This is confirmed by Theorems 1 and 2 except that Theorem 1 does not cover the critical case $a = \frac{1}{4}$, $c = 4$. Here $F(\alpha) = \dfrac{1}{\alpha(1-\alpha)}$ for $0 < \alpha < 1$; $\min F(\alpha) = 4$, attained at $\alpha = \frac{1}{2}$; for $a < \frac{1}{4}$, the characteristic equation $\alpha(1-\alpha) = a$ has two solutions $\alpha = \frac{1}{2} \pm \sqrt{(\frac{1}{4}-a)}$ and we should get a solution with $X(u) \sim \exp\{-(\frac{1}{2}-\sqrt{(\frac{1}{4}-a)})u\}$. We could confirm this by looking at the linearized form of system (17) near $B(1, 0)$.

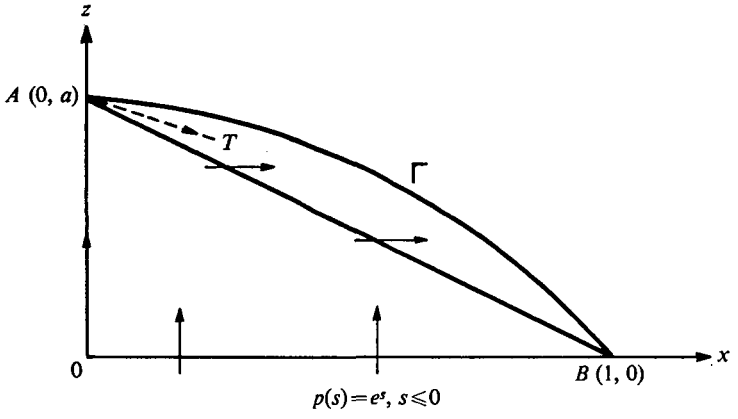


Fig. 2

4. *The general epidemic* ($b > 0$). We now want to consider the system (2) described in § 1 when $b > 0$, so that there are removals. The parameter b is given (it is the removal rate, usually denoted by ρ), and we again try to find out what values of a are possible. The system reads

$$x' = ax\bar{y}, \quad y' = -ax\bar{y} + aby \tag{18}$$

(we do not need $z' = -aby$ because $x+y+z = 1$). Here $a = c^{-1}$, c being the wave velocity, and $b > 0$. As regards boundary conditions, we shall want $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$; $x(+\infty) = 1$ because the wave travels towards $+\infty$ into an uninfected population; $x(u)$ is increasing so that $x(-\infty) = L$ must exist; it seems appropriate to suppose $y(-\infty) = y(+\infty) = 0$ because there should be no infection ahead of the wave and none in the wake because it should ultimately be removed.

Now $L = x(-\infty)$ can be determined in terms of b as follows. From $(x+y)' = aby$, we can deduce $(\bar{x}+\bar{y})' = ab\bar{y} = b(x'/x)$, so that

$$\bar{x} + \bar{y} = b \log x + C.$$

Since $x(+\infty) = 1$ and $y(+\infty) = 0$, we have $\bar{x}(+\infty) = 1$, $\bar{y}(+\infty) = 0$, so that $C = 1$. But $\bar{x}(-\infty) = L$, $\bar{y}(-\infty) = 0$, $\log x(-\infty) = \log L$. Hence $L = b \log L + 1$. It is easy to see that $L = 1$ is the only solution when $b \geqslant 1$, i.e. when $b \geqslant 1$ the problem has only the trivial solution $x \equiv 1, y \equiv 0$. When $0 < b < 1$, the equation $L = b \log L + 1$ still has the solution $L = 1$ but also has a solution L_0 with $0 < L_0 < 1$, and we might hope to prove existence of a solution with $x(-\infty) = L_0$. So let us assume that $0 < b < 1$ from now on.

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Assume now that (18) has a solution. Then

$$y' - aby = -x',$$

$$(y\,e^{-abu})' = -x'\,e^{-abu} = -(x\,e^{-abu})' - abx\,e^{-abu}$$

so that (integrating this equation)

$$y(u)\,e^{-abu} = -x(u)\,e^{-abu} + ab\int_u^\infty x(v)\,e^{-abv}\,dv,$$

$$y(u) = -x(u) + ab\int_u^\infty x(v)\,e^{-ab(v-u)}\,dv$$

$$= -x(u) + ab\int_0^\infty x(u+w)\,e^{-abw}\,dw$$

$$= ab\int_0^\infty (x(u+w) - x(u))\,e^{-abw}\,dw.$$

Writing $X(u) = \log x(u)$ as in §2, we have

$$X'(u) = -a\bar{y}(u)$$

and by integration, using $X(+\infty) = 0$, we get

$$X(u) = -a\int y(v)\,Q(v-u)\,dv.$$

Now insert $y(v) = ab\int_0^\infty (x(v+s) - x(v))\,e^{-abs}\,ds$ to get

$$X(u) = -a\int dv\,Q(v-u)\left(\int_0^\infty (x(v+s) - x(v))\,ab\,e^{-abs}\,ds\right)$$

$$= -a\int_0^\infty ds\,ab\,e^{-abs}\left(\int Q(v-u)\,[(1-x(v)) - (1-x(v+s))]\,dv\right)$$

$$= -a\int_0^\infty ds\,ab\,e^{-abs}\left(\int (1-x(v))\,Q(v-u)\,dv - \int (1-x(v'))\,Q(v'-u-s)\,dv'\right),$$

i.e.
$$X(u) = -a\int_0^\infty ds\,ab\,e^{-abs}\left(\int (1-e^{X(v)})\,(Q(v-u) - Q(v-u-s))\,dv\right). \tag{19}$$

(We have written the expressions in this particular form so as to improve the prospects of getting convergent integrals.)

We now try, following our method in §2, to find a solution of (19) by iteration, starting with $X_0(u) = -e^{-\alpha u}$ for suitable α and continuing with

$$X_{n+1}(u) = -a\int_0^\infty ds\,ab\,e^{-abs}\left(\int (1-e^{X_n(v)})\,(Q(v-u) - Q(v-u-s))\,dv\right). \tag{20}$$

Now
$$X_1(u) = -a\int_0^\infty ds\,ab\,e^{-abs}\left(\int (1-e^{X_0(v)})\,(Q(v-u) - Q(v-u-s))\,dv\right),$$

and
$$Q(v-u) - Q(v-u-s) \geqslant 0 \quad \text{because } Q \text{ is increasing,}$$

and
$$1 - e^{X_0(v)} \leqslant -X_0(v) = e^{-\alpha v} \quad \text{because } X_0(v) < 0.$$

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So
$$X_1(u) \geqslant -a\int_0^\infty ds\, ab\, e^{-abs}\left(\int (e^{-\alpha v})\,(Q(v-u)-Q(v-u-s))\,dv\right)$$
$$= -a\int_0^\infty ds\, ab\, e^{-abs}\,(e^{-\alpha u}-e^{-\alpha(u+s)})\,F(\alpha)$$
$$= -aF(\alpha)\,e^{-\alpha u}\int_0^\infty (ab\,e^{-abs}-ab\,e^{-(ab+\alpha)s})\,ds$$
$$= -aF(\alpha)\frac{\alpha}{ab+\alpha}\,e^{-\alpha u},$$

where, as in §2, $F(\alpha)=\int e^{-\alpha w}Q(w)\,dw$. If α satisfies $aF(\alpha)\,\alpha/(ab+\alpha)\leqslant 1$, we shall get $X_1(u)\geqslant -e^{-\alpha u}=X_0(u)$ and as before this leads (by induction) to

$$X_0(u)\leqslant X_1(u)\leqslant \ldots \leqslant X_n(u)\leqslant \ldots < 0. \tag{21}$$

From (21) we see that $X_n(u)$ tends to a limit $X(u)\leqslant 0$. To show that $X(u)<0$, it is enough (because $X(u)$ is increasing) to prove this for $u\geqslant u_0$, and to achieve this we shall assume that

$$\left.\begin{aligned} aF(\alpha)\frac{\alpha}{ab+\alpha} &= 1 \\ \text{and, for some } \delta>0,\qquad aF(\alpha+\delta)\frac{\alpha+\delta}{ab+\alpha+\delta} &= m<1. \end{aligned}\right\} \tag{22}$$

Assuming without loss of generality that $\delta\leqslant\alpha$, we try to choose A so that

$$X_n(u)\leqslant -e^{-\alpha u}+A\,e^{-(\alpha+\delta)u}\quad (n=0,1,2,\ldots) \tag{23}$$

can be proved by induction. (23) certainly holds when $n=0$. If it holds for n, then

$$X_{n+1}(u) = -a\int_0^\infty ds\, ab\, e^{-abs}\left(\int (1-e^{X_n(v)})\,(Q(v-u)-Q(v-u-s))\,dv\right)$$
$$\leqslant -a\int_0^\infty ds\, ab\, e^{-abs}\left(\int_0^\infty (1-e^{X_n(v)})\,(Q(v-u)-Q(v-u-s))\,dv\right).$$

Since $Q(v-u)-Q(v-u-s)\geqslant 0$ and

$$1-e^{X_n(v)} > -X_n(v)-\tfrac{1}{2}X_n^2(v),$$

we deduce

$$X_{n+1}(u)\leqslant a\int_0^\infty ds\, ab\, e^{-abs}\left(\int_0^\infty (X_n(v)+\tfrac{1}{2}X_n^2(v))\,(Q(v-u)-Q(v-u-s))\,dv\right).$$

By the inductive assumption,

$$X_n(v)\leqslant -e^{-\alpha v}+A\,e^{-(\alpha+\delta)v};$$

also, since $0>X_n(v)>-e^{-\alpha v}$,

$$\tfrac{1}{2}X_n^2(v)\leqslant \tfrac{1}{2}e^{-2\alpha v}\leqslant \tfrac{1}{2}e^{-(\alpha+\delta)v}\quad\text{for}\quad v\geqslant 0,$$

we find that

$$X_{n+1}(u)\leqslant a\int_0^\infty ds\, ab\, e^{-abs}\left(\int_0^\infty [-e^{-\alpha v}+(A+\tfrac{1}{2})\,e^{-(\alpha+\delta)v}]\,[Q(v-u)-Q(v-u-s)]\,dv\right)$$
$$= a\int_0^\infty ds\, ab\, e^{-abs}\left(\int_{-\infty}^\infty [-e^{-\alpha v}+(A+\tfrac{1}{2})\,e^{-(\alpha+\delta)v}]\,[Q(v-u)-Q(v-u-s)]\,dv\right)$$
$$-a\int_0^\infty ds\, ab\, e^{-abs}\left(\int_{-\infty}^0 [-e^{-\alpha v}+(A+\tfrac{1}{2})\,e^{-(\alpha+\delta)v}]\,[Q(v-u)-Q(v-u-s)]\,dv\right).$$

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The first term here is

$$a\int_0^\infty ds\,ab\,e^{-abs}\{-F(\alpha)\,e^{-\alpha u}+(A+\tfrac{1}{2})\,F(\alpha+\delta)\,e^{-(\alpha+\delta)u}+F(\alpha)\,e^{-\alpha(u+s)} - (A+\tfrac{1}{2})\,F(\alpha+\delta)\,e^{-(\alpha+\delta)(u+s)}\}$$

$$= -aF(\alpha)\,e^{-\alpha u}+a(A+\tfrac{1}{2})\,F(\alpha+\delta)\,e^{-(\alpha+\delta)u} + a\frac{ab}{ab+\alpha}F(\alpha)\,e^{-\alpha u}-a(A+\tfrac{1}{2})\frac{ab}{ab+\alpha+\delta}F(\alpha+\delta)\,e^{-(\alpha+\delta)u}$$

$$= -a\frac{\alpha}{ab+\alpha}F(\alpha)\,e^{-\alpha u}+a(A+\tfrac{1}{2})\frac{\alpha+\delta}{ab+\alpha+\delta}e^{-(\alpha+\delta)u}$$

$$\leqslant -e^{-\alpha u}+m(A+\tfrac{1}{2})\,e^{-(\alpha+\delta)u}.$$

The second term is

$$a\int_0^\infty ds\,ab\,e^{-abs}\left(\int_{-\infty}^0 e^{-\alpha v}(1-(A+\tfrac{1}{2})\,e^{-\delta v})\,(Q(v-u)-Q(v-u-s))\,dv\right)$$

which will be <0 if we have $A+\frac{1}{2}\geqslant 1$ so that $1-(A+\frac{1}{2})\,e^{-\delta v}<0$ for $v<0$. So the inductive step from n to $n+1$ will work provided that $A\geqslant\frac{1}{2}$ and $m(A+\frac{1}{2})\leqslant A$, and $A=\max(\frac{1}{2},\frac{1}{2}m/(1-m))$ will do this.

The rest of the argument is straightforward: $X_n(u)$ increases to a limit $X(u)<0$, satisfying the equation (19), and $X(u)\sim -e^{-\alpha u}$. Since $x(u)=\exp X(u)$ is <1, it decreases to a limit L_0 as $u\to-\infty$, where L_0 must be the unique positive root <1 of $L+b\log L=1$. To sum up, we have proved

THEOREM 3. *Suppose that the equation* $aF(\lambda)\,\lambda/(ab+\lambda)=1$ *has a positive root* $\lambda=\alpha$, *and that* $F(\lambda)$ *exists in some interval* $\alpha<\lambda\leqslant\alpha+\delta$ *with* $aF(\lambda)\,\lambda/(ab+\lambda)<1$ *in this interval. Then there exists a solution* $X(u)$ *to the equation*

$$X(u)=-a\int_0^\infty ds\,ab\,e^{-abs}\left(\int(1-e^{X(v)})\,(Q(v-u)-Q(v-u-s))\,dv\right)$$

such that $X(u)<0$ *for all* u *and* $X(u)\sim -e^{-\alpha u}$ *as* $u\to+\infty$.

To discuss the significance of the 'characteristic equation' $aF(\lambda)\,\lambda/(ab+\lambda)=1$ in Theorem 3, it is convenient to rewrite it as

$$F(\lambda)=c\left(\frac{\lambda+ab}{\lambda}\right)=c+\frac{b}{\lambda},$$

or

$$F(\lambda)-\frac{b}{\lambda}=c;$$

we shall then want

$$F(\lambda)-\frac{b}{\lambda}<c \quad \text{in the range} \quad \alpha<\lambda\leqslant\alpha+\delta$$

in order to apply Theorem 3. Writing $G(\lambda)=F(\lambda)-(b/\lambda)$, normally either $G(\lambda)$ has a positive minimum c_0 or takes all positive values (it can then even become negative). Then Theorem 3 can be used to prove existence of wave solutions, either for all $c>c_0$ or for all $c>0$. The second case cannot occur when $p(s)$ is symmetrical, because

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$F(\lambda) = f(\lambda)/\lambda$, where $f(\lambda) = \int p(s)\, e^{-\lambda s}\, ds$ and by symmetry of p, $f(\lambda) = \int p(s) \cosh \lambda s\, ds$. But then

$$f(\lambda) \geqslant \int p(s)\,(1 + \tfrac{1}{2}\lambda^2 s^2)\, ds = 1 + \tfrac{1}{2}\sigma^2\lambda^2,$$

where σ^2 is the variance of p, so that

$$G(\lambda) = \frac{f(\lambda) - b}{\lambda} \geqslant \frac{1-b}{\lambda} + \tfrac{1}{2}\sigma^2 \geqslant \sigma\sqrt{[2(1-b)]}$$

for all λ for which $F(\lambda)$ is finite. Incidentally this argument shows that $c_0 \geqslant \sigma\sqrt{[2(1-b)]}$, but a better lower bound, $c_0 \geqslant 1{\cdot}51\sigma$ when $b = 0$, was obtained by Mollison ((4), Lemma 5, (iii)).

It remains to show, under suitable conditions, that waves with velocity

$$c < c_0 = \min G(\lambda)$$

are impossible. This can be done by modifying the arguments in the proof of Theorem 2. Suppose that $G(\lambda) = F(\lambda) - (b/\lambda)$ is finite for $0 < \lambda \leqslant A$, and has a minimum $c_0 > 0$ at $\lambda = \alpha_0 < A$, and that $c < c_0$. Multiply equation (19) by $e^{\lambda u}$ and integrate over u to get, after a little manipulation,

$$c\phi(\lambda) = \frac{\lambda}{ab + \lambda} F(\lambda)\,(\phi(\lambda) + \psi(\lambda)),$$

where $\phi(\lambda) = \int X(u) \exp[\lambda u]\, du$, $\psi(\lambda) = \int (\exp X(u) - 1 - X(u)) \exp[\lambda u]\, du$,

as in the proof of Theorem 2. This equation can be rewritten as

$$\left(1 + \frac{ab}{\lambda}\right) c\phi(\lambda) = F(\lambda)\,(\phi(\lambda) + \psi(\lambda)),$$

$$(c - G(\lambda))\,\phi(\lambda) = F(\lambda)\,\psi(\lambda). \tag{24}$$

One can then use this in place of (13) to deduce that $\phi(\lambda)$ is finite for all $\lambda < A$. The second part of the proof of Theorem 2 can now be modified: write out the transforms occurring in (24) in full, and multiply by $e^{-\lambda w_0}$ to get

$$\int [cX(w) + G(\lambda)\,(1 - e^{X(w)}) + (b/\lambda)\,(X(w) + 1 - e^{X(w)})]\, e^{\lambda(w - w_0)}\, dw = 0.$$

Now differentiate w.r.t. λ at $\lambda = \alpha_0$, and use the facts that $G(\alpha_0) = c_0$, $G'(\alpha_0) = 0$, to give

$$\int \left\{(w - w_0)\left[cX(w) + c_0(1 - e^{X(w)}) + \frac{b}{\alpha_0}\,(X(w) + 1 - e^{X(w)})\right] - \frac{b}{\alpha_0^2}\,(X(w) + 1 - e^{X(w)})\right\} e^{\alpha_0(w - w_0)}\, dw = 0.$$

Rewrite this as

$$\int (w - w_0)\left[\left(c + \frac{b}{\alpha_0}\right) X(w) + \left(c_0 + \frac{b}{\alpha_0}\right)(1 - e^{X(w)})\right] e^{\alpha_0(w - w_0)}\, dw = \frac{b}{\alpha_0^2} \int (X(w) + 1 - e^{X(w)})\, e^{\alpha_0(w - w_0)}\, dw, \tag{25}$$

and now choose w_0 so that

$$-\left(c + \frac{b}{\alpha_0}\right) X(w_0) = \left(c_0 + \frac{b}{\alpha_0}\right)(1 - X(w_0)).$$

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Then the integral on the left-hand side of (25) is positive; but the integral on the right is negative, because $X+1-\exp X < 0$ for $X < 0$. This is a contradiction, so that no solution can exist for $c < c_0$, under the conditions on $G(\lambda)$ stated above.

We leave it to the reader to check the details of the proof, to state a formal theorem analogous to Theorem 2, and to examine some examples for himself.

5. *Concluding remarks.* Apart from the simple examples of section 2 which can be considered as supplementary to Mollison's results:

(*A*) We do not have uniqueness theorems, nor can we treat $c = c_0$. The best we can do in the general case is to show, in the case described by Theorem 1, that any solution X must 'behave like $-e^{-\alpha t}$ as $t \to \infty$' in the weak sense that the transform

$$\phi(\lambda) = \int X(u)\, e^{\lambda u}\, du$$

must be finite for all $\lambda < \alpha$. It cannot be finite for $\lambda = \alpha$ because if it were we should have

$$0 = (c - F(\alpha))\, \phi(\alpha) = F(\alpha) \int (\exp X(w) - 1 - X(w))\, e^{\alpha w}\, dw,$$

which is impossible since the integral on the right is positive.

(*B*) We have not considered the behaviour of the general system (1), subject to given initial conditions. But clearly one should try to find out how the development of a deterministic epidemic is related to the wave solutions that we have studied.

(*C*) In our early attempts at the wave problem, written as an integral equation (8) we tried unsuccessfully to write this as an operator equation $X = T(X)$, with the non-linear operator T acting on a suitable space of functions. We could not find a normed space for which T was a contraction mapping. i.e.

$$\|T(X_2) - T(X_1)\| \leqslant k \|X_2 - X_1\| \quad \text{with} \quad k < 1.$$

However, if α is a solution of $aF(\alpha) = 1$ and we define

$$\|X\| = \sup_u \|X(u)\, e^{\alpha u}\|,$$

it is easy to show that T is weakly contracting in the sense that

$$\|T(X_2) - T(X_1)\| < \|X_2 - X_1\|.$$

While this did not lead to an existence proof, it suggested that numerical solution by iteration based on (8), i.e.

$$X_{n+1}(u) = -a \int (1 - e^{X_n(w)})\, Q(w-u)\, dw,$$

might be possible. Tentative attempts to do this were quite promising, but we did not push them far enough to make the results worth reporting. One point that is perhaps worth mentioning is that instead of the lower bound $-e^{-\alpha u}$ used in the proof of Theorem 1, we could use

$$\begin{aligned} X_0 &= -e^{-\alpha u} \quad (u \geqslant u_0) \\ &= a(u - u_0) - e^{-\alpha u_0} \quad (u \leqslant u_0), \end{aligned}$$

where $a = \alpha e^{-\alpha u_0}$ determines u_0.

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It is straightforward to show that X_0 has the property that $X_1 = T(X_0)$ satisfies $X_1 \geqslant X_0$ for all u and this provides a closer lower bound than that used in the proof of Theorem 1.

(*D*) Since writing our paper, we have learned that Professor H. E. Daniels has studied similar problems. He suggests a technique for finding approximate solutions and reaches conclusions similar to ours. His results are to appear in (1).

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