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Vertices of Mather's beta function

OSVALDO OSUNA

IFM, University of Michoacana, Edif. C-3, C.U., C.P. 58040, Morelia, Michoacán, México
(e-mail: osvaldo@ifm.umich.mx)

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Abstract. Given a Lagrangian L, Mather introduced the β-function of L, which is a convex function. Many interesting properties of the Euler–Lagrange flow can be derived from the study of the behaviour of the β-function. In this work we obtain some links between the vertices of the β-function and the Hausdorff dimension of the associated invariant measures, from which we recover a result of differentiability of the β-function.

1. *Introduction*

Let M be a closed connected manifold and TM its tangent bundle. A periodic Lagrangian (of period one) is a smooth function $L : TM \times \mathbb{R} \to \mathbb{R}$ which is convex and superlinear. This means that L restricted to each T_xM has positive definite Hessian and for some Riemannian metric we have

$$\lim_{|v|\to\infty} \frac{L(x, v, t)}{|v|} = \infty,$$

uniformly on x and t. We assume also that the Euler–Lagrange flow φ_t associated to the Lagrangian is complete.

Let us recall the main concepts introduced by Mather in [**7**]. Let $\mathcal{M}(L)$ be the set of probabilities on the Borel σ-algebra on $TM\times\mathbb{S}^1$ that have compact support and are invariant under the Euler–Lagrange flow φ_t. Let $H_1(M, \mathbb{R})$ be the first real homology group of M. Given a closed 1-form ω on M and $\rho \in H_1(M, \mathbb{R})$, let $\langle[\omega], \rho\rangle$ denote the integral of ω on any closed curve in the homology class ρ. If $\mu \in \mathcal{M}(L)$, its homology is defined as the unique $\rho(\mu) \in H_1(M, \mathbb{R})$ such that

$$\langle[\omega], \rho(\mu)\rangle = \int_{TM\times\mathbb{S}^1} \omega \, d\mu,$$

for all closed 1-forms on M. The integral on the right-hand side is with respect to μ with ω considered as the function $\omega : TM \times \mathbb{S}^1 \to \mathbb{R}$. The action of $\mu \in \mathcal{M}(L)$ is defined by

$$A_L(\mu) = \int_{TM\times\mathbb{S}^1} L \, d\mu.$$

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Given $h \in H_1(M, \mathbb{R})$ we say that $\mu \in \mathcal{M}(L)$ is a *minimizing measure* of L with homology h if $\rho(\mu) = h$ and

$$A_L(\mu) = \min\{A_L(v) \mid v \in \mathcal{M}(L), \rho(v) = h\}.$$

Recall that the L-action of an absolutely continuous curve $\gamma : [a, b] \to M$ is defined by

$$A_L(\gamma) := \int_a^b L(\gamma(t), \dot{\gamma}(t), t)\, dt.$$

Following Mather [**7**] we define the β-function of L as the map $\beta : H_1(M, \mathbb{R}) \to \mathbb{R}$ given by

$$\beta(h) := \min\{A_L(\mu) \mid \mu \in \mathcal{M}(L),\ \rho(\mu) = h\}$$

this function is convex, superlinear and clearly $\mu \in \mathcal{M}(L)$ is minimizing if and only if $\beta(\rho(\mu)) = A_L(\mu)$.

Many interesting properties of the Euler–Lagrange flow can be derived from the study of the behaviour of the β-function, for instance if h is an *extremal point* of β, i.e. if $h_1, h_2 \in H_1(M, \mathbb{R})$, $t \in (0, 1)$ are such that $h = th_1 + (1 - t)h_2$ implies

$$\beta(th_1 + (1 - t)h_2) < t\beta(h_1) + (1 - t)\beta(h_2),$$

then there exist ergodic measures with homology h see [**6**].

Recall that a supporting affine map of β at the point $(\rho, \beta(\rho))$ with slope $[\lambda]$, is an affine map $\psi : H_1(M, \mathbb{R}) \to \mathbb{R}$ with $\psi(h) := \langle [\lambda], h \rangle + k$, where λ is a closed 1-form and k a constant, such that

$$\beta(\rho) = \psi(\rho) \quad \text{and} \quad \beta(h) \geq \psi(h) \quad \text{for all } h \in H_1(M, \mathbb{R}).$$

Given $\rho \in H_1(M, \mathbb{R})$ we call ρ a *vertex* of β if there exists an open set $U \subset H^1(M, \mathbb{R})$ of slopes representing supporting affine maps of β at the point $(\rho, \beta(\rho))$.

It is clear that if h is a vertex of β then it is an extremal point of β.

We shall fix a basis $h_1, h_2, \ldots, h_m$ of the free factor of $H_1(M, \mathbb{Z})$, clearly this is also a basis for $H_1(M, \mathbb{R})$ (the image in $H_1(M, \mathbb{R})$ of an integer homology class).

Definition. We will say that a homology class $\rho \in H_1(M, \mathbb{R})$ is *rational* and write $\rho \in H_1(M, \mathbb{Q})$, if $\rho = \sum_{i=1}^m \rho_i h_i$ with $\rho_i \in \mathbb{Q}$ for every $i = 1, \ldots, m$.

Observe that when $\dim H_1(M, \mathbb{R}) = 1$ then β is differentiable at the homology ρ if and only if ρ is not a vertex of β. In fact, when $M = \mathbb{S}^1$ Mather [**8**] and Bangert [**1**] showed that:

(a) the β-function is differentiable at every irrational $\rho \in H_1(\mathbb{S}^1, \mathbb{R}) \simeq \mathbb{R}$;

(b) if $\rho \in H_1(\mathbb{S}^1, \mathbb{R})$ is a vertex of β, then ρ is rational and the ergodic measures with homology ρ are supported on periodic orbits.

Motivated by these results we obtain the following theorem.

THEOREM 1. *Let L be a periodic Lagrangian on M and let $\rho \in H_1(M, \mathbb{R})$ be a vertex of β; if ρ is not rational then* $\dim_H(\mu) \geq 3$ *for all minimizing measures μ with homology ρ.*

Here $\dim_H(\mu)$ denotes the *Hausdorff dimension of the measure* μ (see the definition in [**9**]). An analogous result was announced in [**3**] and never published.

From Theorem 1, we can recover the differentiability result due to Mather and Bangert.

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COROLLARY 1. *Let L be a periodic Lagrangian on $\mathbb{S}^1$, then the β-function is differentiable at every irrational homology ρ.*

Proof. Observe that, when $M = \mathbb{S}^1$, all the minimizing measures of L have support with dimension less than or equal to the dimension of $\mathbb{S}^1 \times \mathbb{S}^1$ (see the graph theorem in [**7**]); thus an irrational ρ cannot be a vertex (by Theorem 1) and then it is a point of differentiability because $\dim H_1(\mathbb{S}^1, \mathbb{R}) = 1$. □

2. *Proof of Theorem 1*

Let us consider a Lagrangian L, and ρ and μ as in the statement of Theorem 1. Let $\psi : H_1(M, \mathbb{R}) \to \mathbb{R}$ be a supporting affine map of β at the point $(\rho, \beta(\rho))$ with slope $[\lambda]$, and write $\psi(h) := \langle [\lambda], h\rangle + k$. By taking the Lagrangian $L_0(x, v, t) := L(x, v, t) - \langle [\lambda], v\rangle - k$ instead of L we can assume that $\beta(\rho) = 0$ and $\beta \geq 0$, it is clear that the new Lagrangian has the same solutions and minimizers as L. Since ρ is a vertex of β then it is an extremal point and using the ergodic decomposition theorem, we can suppose that μ is ergodic.

To prove that $\dim_H(\mu) \geq 3$ we will show that there exists a constant $k > 0$ depending on μ and L alone, such that, if $\epsilon > 0$ is small enough, then

$$\mu(B_\epsilon(z)) \leq k\epsilon^3, \tag{1}$$

for every ϵ-ball around $z \in \operatorname{supp}(\mu)$. Now the theorem follows by applying the next lemma.

LEMMA 1. [**9**, Theorem 7.1] *Let μ be a finite Borel measure on $\mathbb{R}^n$. If for μ-almost every point z we have that*

$$\liminf_{r \to 0} \frac{\log \mu(B_r(z))}{\log r} \geq \alpha,$$

then $\dim_H(\mu) \geq \alpha$.

To prove (1) we begin by considering the measure $\widehat{\mu}$ induced by μ on the Borel σ-algebra of $TM \times \{0\} \simeq TM$, defined as

$$\widehat{\mu}(A) := \mu(\varphi_{[0,1]}(A \times \{0\})),$$

for all measurable sets $A \subset TM$. It is clear that $\widehat{\mu}$ is invariant under $\psi := \varphi_1$, the time-one map of the flow of L. We have the following claim.

CLAIM. *For every $\theta \in \operatorname{supp}(\mu) \cap TM \times \{0\} =: \operatorname{supp}(\widehat{\mu})$, we have*

$$\widehat{\mu}(B_\epsilon(\theta)) \leq C\epsilon^2, \tag{2}$$

for a constant $C > 0$ (where $B_\epsilon(\theta)$ denotes the ϵ-ball around θ in TM).

It is not hard to see that this claim implies equation (1).

Proof of the claim. To prove the claim we begin by taking a μ-generic point $\theta = (x, v, 0) \in \operatorname{supp}(\mu) \cap TM \times 0$ with dense positive orbit $O^+(\theta)$ in $\operatorname{supp}(\mu)$. It is sufficient to prove (2) for the points $\theta_1 = (x_1, v_1, 0) \in O^+(\theta) \cap TM \times 0$.

Let $T > 0$ be such that $\theta_2 := \psi^T(\theta_1) \in B_\epsilon(\theta_1)$. Denote by $x : \mathbb{R} \to M$ the solution of L with initial conditions $x(0) = x_1$ and $\dot{x}(0) = v_1$. Take $\epsilon > 0$ such that

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if $d(z_1, z_2) < \epsilon$ then z_1, z_2 belong to the same coordinate chart of M. Now we choose a constant $0 < \tau < T$ such that $d(x(\tau), x(T)) < \epsilon/2$. By working in local coordinates we consider the curve $y(t) := x(t) + [(t-\tau)/(T-\tau)](x(0) - x(T))$ if $t \in [\tau, T]$. Now, let us modify the curve $x(t)$, defining a closed curve $y_T : [0, T] \to M$ by

$$y_T(t) := \begin{cases} x(t) & \text{if } t \in [0, \tau], \\ y(t) & \text{if } t \in [\tau, T]. \end{cases}$$

We have the following lemma.

LEMMA 2. *Let y_T be the above closed curve, then there exists a constant $C := C(L, \mu) > 0$ which does not depend on θ_1 and which satisfies*

$$A_L(y_T) \leq C\epsilon^2.$$

Before proving Lemma 2, let us show how it implies the claim. From Lemma 2 we have

$$\beta_L\left(\frac{1}{T}[y_T]\right) \leq \frac{C\epsilon^2}{T}, \tag{3}$$

where $[y_T] \in H_1(M, \mathbb{Z})$ is the class of the closed curve $y_T([0, T]) \subset M$.

Since ρ is a vertex of β we can find an open set $U \in H^1(M, \mathbb{R})$ of slopes such that

$$\beta(\rho) = \langle [\lambda], \rho \rangle + c(\lambda) \text{ and } \beta(h) \geq \langle [\lambda], h \rangle + c(\lambda),$$

for all $h \in H_1(M, \mathbb{R})$ and every $[\lambda] \in U$. By taking the Lagrangian $L_0(x, v, t) := L(x, v, t) - \langle [\omega], v \rangle - k$ instead of L with $[\omega] \in U$, we can assume (by shrinking U if it is necessary) that U is a symmetric set; this means that $[\lambda] \in U$ if and only if $-[\lambda] \in U$. Thus we have

$$|\langle [\lambda], h - \rho \rangle| \leq \beta(h) \quad \text{for all } h \in H_1(M, \mathbb{R}). \tag{4}$$

It follows from (3) and (4) that

$$\left| \left\langle [\lambda], \frac{1}{T}[y_T] - \rho \right\rangle \right| \leq \frac{C\epsilon^2}{T}. \tag{5}$$

Now, since ρ is not rational, we can find an integer N_0 and $[\lambda_0] \in H^1(M, \mathbb{Z})$ such that $[\lambda_0]/N_0 \in U$ and $\langle [\lambda_0], \rho \rangle \in \mathbb{R} \backslash \mathbb{Q}$.

Taking $m_T := \langle [\lambda_0], [y_T] \rangle \in \mathbb{Z}$ and $\xi := \langle [\lambda_0], \rho \rangle$, thus equation (5) yields

$$|m_T - T \cdot \xi| \leq N_0 C\epsilon^2. \tag{6}$$

We consider the irrational rotation $R_\xi : \mathbb{S}^1 \to \mathbb{S}^1$,

$$R_\xi(z) := z \cdot \exp(2\pi i \xi),$$

and let $\tilde{R}_\xi : \mathbb{R} \to \mathbb{R}$, $\tilde{R}_\xi(t) := t + \xi$ be its lifting. As R_ξ is uniquely ergodic with respect to the Lebesgue measure m on $\mathbb{S}^1$, by taking $B_{N_0 C\epsilon^2}(1) \subset \mathbb{S}^1$ and using Birkhoff's ergodic

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theorem, we obtain

$$\begin{aligned}
2C\epsilon^2 = m(B_{N_0C\epsilon^2}(1)) &= \lim_{N\to\infty}\frac{1}{N}\#\{T\in[0,N]\cap\mathbb{Z}\mid d(R_\xi^T(1),1)\le N_0C\epsilon^2\}\\
&= \lim_{N\to\infty}\frac{1}{N}\#\{T\in[0,N]\cap\mathbb{Z}||k-\tilde{R}_\xi^T(0)|\le N_0C\epsilon^2 \text{ for some } k\in\mathbb{Z}\}\\
&\ge \lim_{N\to\infty}\frac{1}{N}\#\{T\in[0,N]\cap\mathbb{Z}||m_T-T\xi|\le N_0C\epsilon^2\}\quad \text{by (6)}\\
&\ge \lim_{N\to\infty}\frac{1}{N}\#\{T\in[0,N]\cap\mathbb{Z}|\psi^T(\theta_1)\in B_\epsilon(\theta_1)\}\\
&= \hat{\mu}(B_\epsilon(\theta_1)).
\end{aligned}$$

This proves (2) for the point θ_1 and therefore the claim. □

3. *Proof of Lemma 2*

In order to prove Lemma 2, we need some preliminary results. For any k in $\mathbb{R}$ define the $(L+k)$-action of an absolutely continuous curve $\gamma:[a,b]\to M$ as

$$A_{L+k}(\gamma)=\int_a^b (L+k)(\gamma(\tau),\dot{\gamma}(\tau),\tau)\,d\tau.$$

For t in $\mathbb{R}$ we denote by $[t]$ the corresponding point in $\mathbb{S}^1$. For any pair of points $(x,[s]),(y,[t])$ on $M\times\mathbb{S}^1$ and n a non-negative integer, define $C((x,[s]),(y,[t]);n)$ as the set of absolutely continuous curves $\gamma:[a,b]\to M$ with $\gamma(a)=x$, $\gamma(b)=y$ such that $[a]=[s]$, $[b]=[t]$, and the integer part of $b-a$ is n.

Let Φ_k^n be the real function defined on $M\times\mathbb{S}^1\times M\times\mathbb{S}^1$ as

$$\Phi_k^n((x,[s]),(y,[t]))=\min_{\gamma\in C((x,[s]),(y,[t];n))}\{A_{L+k}(\gamma)\}.$$

Then the *action functional* is defined by

$$\Phi_k=\inf_{n\in\mathbb{N}}\Phi_k^n.$$

We have the following proposition.

PROPOSITION 1. [**2**] *There is a critical value c(L) given by*

$$c:=c(L)=-\min\left\{\int L\,d\mu : \mu \textit{ is an invariant probability}\right\}.$$

Following Fathi (see [**2, 4**]) we say that $u:M\times\mathbb{S}^1\to\mathbb{R}$ is a *Forward weak KAM solution* if:

- u is $L+c$ dominated, i.e.

$$u(y,[t])-u(x,[s])\le\Phi_c((x,[s]),(y,[t]));$$

- for every $(x,[s])\in M\times\mathbb{S}^1$ there exists a curve $\gamma:(s,\infty)\to M$ such that

$$u(\gamma(t),[t])-u(x,[s])=A_{L+c}(\gamma|_{[s,t]}),$$

and in this case we say that γ is a (u,c)-calibrated curve.

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A point $(x, v, [s]) \in TM \times \mathbb{S}^1$ is a *positive semistatic point* if the solution $\gamma = \gamma_{(x,v,s)}$ of the Euler–Lagrange equations with initial conditions $(x, v, [s])$ satisfies, for all t,

$$A_{L+c}(\gamma|_{[s,t]}) = \Phi_c((x, [s]), (\gamma(t), [t])).$$

A point $(x, v, [s]) \in TM \times \mathbb{S}^1$ is a static point if it is semistatic and satisfies

$$A_{L+c}(\gamma|_{[s,t]}) = -\Phi_c((\gamma(t), [t]), (\gamma(s), [s])).$$

We denote by $\widehat{\Sigma}$ the set of static points and by $\mathcal{A}_0 := \pi(\widehat{\sum})$ where π is the canonical projection.

The relevance of weak KAM solutions is that they have several interesting properties, such as the following.

PROPOSITION 2. [**2**] *If $u : M \times \mathbb{S}^1 \to \mathbb{R}$ is a weak KAM solution, then:*

(i) *u is Lipschitz;*

(ii) *u is differentiable on $\mathcal{A}_0$;*

(iii) *let $(x, t) \in \mathcal{A}_0$, γ a (u, c)-calibrated curve such that $\gamma(t) = x$, then $\partial_{(x,t)}u = L_v(x, \dot{\gamma}(t), t)$.*

Proof of Lemma 2. Recall that we defined the closed curve $y_T : [0, T] \to M$ by

$$y_T(t) := \begin{cases} x(t) & \text{if } t \in [0, \tau], \\ y(t) & \text{if } t \in [\tau, T], \end{cases}$$

where (working in a local chart) the curve $y(t)$ is given as

$$y(t) := x(t) + \frac{(t - \tau)}{(T - \tau)}(x(0) - x(T)) \quad \text{if } t \in [\tau, T].$$

Consider a Forward weak KAM solution $u : M \times \mathbb{S}^1 \to \mathbb{R}$. We have that $\beta \geq 0$ and $\beta(\rho) = 0$; therefore, without loss of generality we may suppose that $c = 0$. Thus by calibration we have

$$\int_0^T L(x(t), \dot{x}(t), t)\, dt = u(x(T), 0) - u(x(0), 0). \tag{7}$$

From Fathi's regularity result [**5**, Theorem 4.5.1] we have

$$u(x(T), 0) - u(x(0), 0) \leq \partial_x u_{(x(T),T)}(x(T) - x(0)) + K_1 \|x(T) - x(0)\|^2, \tag{8}$$

where K_1 only depends on the chart and L.

Now the L-action of y_T is given by

$$\begin{aligned} A_L(y_T) &= \int_0^\tau L(x(t), \dot{x}(t), t)\, dt + \int_\tau^T L(y(t), \dot{y}(t), t)\, dt \\ &= A_L(x|_{[0,T]}) + \left\{ \int_\tau^T L(y(t), \dot{y}(t), t)\, dt - \int_\tau^T L(x(t), \dot{x}(t), t)\, dt \right\}. \end{aligned}$$

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To estimate the difference between keys, we have

$$L(y(t), \dot{y}(t), t) \leq L(x(t), \dot{x}(t), t) + \partial_x L_{(x(t),\dot{x}(t),t)}(y(t) - x(t)) + \partial_v L_{(x(t),\dot{x}(t),t)}(\dot{y}(t) - \dot{x}(t)) + K_2\|x(T) - x(0)\|^2.$$

where K_2 only depends on the chart and L.

By integration, taking into account the fact that the Euler–Lagrange equations are satisfied, we get that the difference between keys is less than or equal to

$$\partial_v L_{(x(t),\dot{x}(t),t)}(x(0) - x(T)) + K_2(T - \tau)\|x(0) - x(T)\|^2. \tag{9}$$

But we have $\partial_v L_{(x(t),\dot{x}(t),t)} = \partial_{(x(t),t)}u$. Therefore summing, we obtain

$$A_L(y_T) \leq K_1\|x(T) - x(0)\|^2 + d_{(x(T),0)}u(x(T) - x(0)) + d_{(x(T),0)}u(x(0) - x(T)) + K_2(T - \tau)\|x(0) - x(T)\|^2.$$

This implies the lemma as we wished to show. □

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