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Book Ramsey Numbers and Quasi-Randomness

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For Béla Bollobás on his 60th birthday

A set of n triangles sharing a common edge is called a book with n pages and is denoted by B_n. It is known that the Ramsey number $r(B_n)$ satisfies $r(B_n) = (4 + o(1))n$. We show that every red–blue edge colouring of $K_{\lfloor(4-\varepsilon)n\rfloor}$ with no monochromatic B_n exhibits quasi-random properties when ε tends to 0. This implies that there is a constant $c > 0$ such that for every red–blue edge colouring of $K_{r(B_n)}$ there is a monochromatic B_n whose vertices span at least $\lfloor cn^2 \rfloor$ edges of the same colour as the book.

As an application we find the Ramsey number for a class of graphs.

1. Introduction

Our notation and terminology are standard (see, *e.g.*, [1]). Thus, $G(n)$ is a graph of order n, and $G(n, m)$ is a graph of order n and size m. Given a graph G and a vertex $u \in V(G)$, we write $\Gamma(u)$ for the set of vertices adjacent to u; $d_G(u) = |\Gamma_G(u)|$ is the degree of u. Departing from common usage, we set $d_G(U) = |\bigcap_{x\in U}\Gamma_G(x)|$; in particular, $d_G(uv) = |\Gamma_G(u) \cap \Gamma_G(v)|$. We shall write $d(u)$ and $\Gamma(u)$ instead of $d_G(u)$ and $\Gamma_G(u)$ when the graph G is understood.

If the graphs G_1 and G_2 are vertex-disjoint then $G_1 + G_2$ denotes the union of G_1 and G_2 together with the edges joining vertices of G_1 to vertices of G_2. We write $k_3(G)$ for the number of triangles of G. Given a graph G, the Ramsey number $r(G)$ is the minimal n such that every red–blue colouring of K_n contains a monochromatic copy of G.

For $q \geqslant 1$, $k \geqslant 1$, a *k-book* $B_q^{(k)}$ of size q is a graph consisting of q distinct $(k+1)$-cliques, sharing a common k-clique. We denote by $bk^{(k)}(G)$ the size of the largest k-book in a graph G. We call 2-books simply books, and write $bk(G)$ for $bk^{(2)}(G)$.

Books have attracted considerable attention in Ramsey graph theory (see, *e.g.*, [8], [4] and [6]). In particular, Erdős, Faudree, Rousseau, and Schelp [3] proved that, for every fixed k,

$$r\left(B_n^{(k)}\right) > \left(2^k + o(1)\right)n.$$

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On the other hand, Rousseau and Sheehan in [8] proved that

$$r(B_n) \leqslant 4n + 2,$$

and observed that the Paley graphs imply $r(B_n) = 4n + 2$ whenever $4n + 1$ is a prime power. Hence,

$$r(B_n) = (4 + o(1))n.$$

The main results of this note, stated and proved in Section 3, continue this line of investigation. In particular, we show that red–blue edge colourings of $K_{\lfloor(4-\varepsilon)n\rfloor}$ with no monochromatic B_n exhibit quasi-random properties when ε tends to 0.

We deduce the following Ramsey result: there is a constant $c > 0$ such that every red–blue edge colouring of $K_{r(B_n)}$ contains a monochromatic book B_n whose vertices span at least $\lfloor cn^2 \rfloor$ edges of the same colour as the book.

This, in turn, implies that, for n sufficiently large and every $0 < \beta < 1$, there exist

$$\alpha = \alpha(\beta) > 0, \qquad a \geqslant \lfloor \alpha \log n \rfloor, \qquad b \geqslant \lfloor n^{1-\beta} \rfloor$$

such that

$$r(H) = r(B_n),$$

where the graph H is obtained by adding $K_{a,b}$ to B_n, *i.e.*,

$$H = K_2 + \left(K_{a,b} \cup \overline{K}_{n-a-b}\right).$$

We show that this result is tight in the following sense: $r(H) > r(B_n)$ for any graph H defined as

$$H = K_2 + \left(F \cup \overline{K}_{n-k}\right),$$

where F is a nonbipartite graph of order $k \leqslant n$.

2. Some preliminary results

In this section we provide some necessary background for proofs of the aforementioned results in Ramsey theory. Much of this can be readily understood in the context of quasi-randomness and experts in this area will find no surprises. This material is included in order to make the paper more self-contained and accessible to readers with limited background in quasi-randomness.

Among the many results that Chung, Graham and Wilson state in [2] is the following one, attributed to Thomason [10, 11].

Theorem 2.1. *For every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that, if G is a graph of sufficiently large order n and*

$$\sum_{u,v \in V(G)} \left| d_G(uv) - \frac{1}{4}n \right| < \varepsilon n^3, \tag{2.1}$$

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then

$$\left|e(G[X]) - \frac{1}{4}|X|^2\right| < \delta n^2$$

for every $X \subset V(G)$.

It is not an easy task to extract a short proof of Theorem 2.1 from [2]; for the reader's benefit we present one at the end of this section.

We shall start with a simple condition for 'almost-regularity' of a graph.

Lemma 2.2. *For every* ε *and sufficiently large* n, *if* $G = G(n)$ *satisfies*

$$\sum_{uv \in E(G)} \left|d_G(uv) - \frac{n}{4}\right| + \sum_{uv \in E(\overline{G})} \left|d_{\overline{G}}(uv) - \frac{n}{4}\right| < \varepsilon n^3,$$

then

$$\sum_{u \in V(G)} \left|d_G(u) - \frac{n}{2}\right| < \sqrt{\varepsilon} n^2. \tag{2.2}$$

Proof. We calculate $k_3(G) + k_3(\overline{G})$ in two ways. First, by a well-known relation of Lorden [5],

$$\begin{aligned} k_3(G) + k_3(\overline{G}) &= \binom{n}{3} - \frac{1}{2} \sum_{u \in V(G)} d_G(u)(n - 1 - d_G(u)) \\ &= \frac{n(n-1)(n-5)}{24} + \frac{1}{2} \sum_{u \in V(G)} \left(d_G(u) - \frac{n-1}{2}\right)^2 \end{aligned} \tag{2.3}$$

and secondly,

$$k_3(G) + k_3(\overline{G}) = \frac{1}{3} \sum_{uv \in E(G)} d_G(uv) + \frac{1}{3} \sum_{uv \in E(\overline{G})} d_{\overline{G}}(uv).$$

Equating the two expressions for $k_3(G) + k_3(\overline{G})$, we obtain

$$\sum_{u \in V(G)} \left(d_G(u) - \frac{n}{2}\right)^2 = \frac{2}{3} \left(\sum_{uv \in E(G)} \left(d_G(uv) - \frac{n}{4}\right) + \sum_{uv \in E(\overline{G})} \left(d_{\overline{G}}(uv) - \frac{n}{4}\right) \right) + O(n^2).$$

Hence the hypothesis implies

$$\sum_{u \in V(G)} \left(d_G(u) - \frac{n}{2}\right)^2 < \varepsilon n^3,$$

and, using Cauchy's inequality, (2.2) follows for sufficiently large n. □

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Lemma 2.3. *For every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that if G is a graph of sufficiently large order n, and*

$$\sum_{uv \in E(G)} \left| d_G(uv) - \frac{n}{4} \right| + \sum_{uv \in E(\overline{G})} \left| d_{\overline{G}}(uv) - \frac{n}{4} \right| < \varepsilon n^3, \tag{2.4}$$

then

$$\sum_{u,v \in V(G)} \left| d_G(uv) - \frac{n}{4} \right| < \delta n^3.$$

Proof. Set $\varepsilon = \min\{(\delta/3)^2, 1\}$. Observe that for $uv \in E(\overline{G})$,

$$d_G(uv) = d_{\overline{G}}(uv) + d_G(u) + d_G(v) - (n-2).$$

Consequently,

$$\begin{aligned}
\sum_{u,v \in V(G)} \left| d_G(uv) - \frac{n}{4} \right| &= \sum_{uv \in E(G)} \left| d_G(uv) - \frac{n}{4} \right| \\
&\quad + \sum_{uv \in E(\overline{G})} \left| d_{\overline{G}}(uv) - \frac{n}{4} + d_G(u) + d_G(v) - (n-2) \right| \\
&\leqslant \sum_{uv \in E(G)} \left| d_G(uv) - \frac{n}{4} \right| + \sum_{uv \in E(\overline{G})} \left| d_{\overline{G}}(uv) - \frac{n}{4} \right| \\
&\quad + \sum_{uv \in E(\overline{G})} \left| d_G(u) - \frac{n}{2} \right| + \sum_{uv \in E(\overline{G})} \left| d_G(v) - \frac{n}{2} \right| + 2\binom{n}{2}.
\end{aligned}$$

Applying Lemma 2.2 we find that

$$\begin{aligned}
\sum_{uv \in E(\overline{G})} \left| d_G(u) - \frac{n}{2} \right| + \sum_{uv \in E(\overline{G})} \left| d_G(v) - \frac{n}{2} \right| &\leqslant n \sum_{u \in V(G)} \left| d_G(u) - \frac{n}{2} \right| \\
&< \sqrt{\varepsilon} n^3.
\end{aligned}$$

Hence,

$$\sum_{u,v \in V(G)} \left| d_G(uv) - \frac{n}{4} \right| < \varepsilon n^3 + \sqrt{\varepsilon} n^3 + 2\binom{n}{2} < 3\sqrt{\varepsilon} n^3 = \delta n^3. \qquad \square$$

Lemma 2.4. *For every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that if G is a graph of of sufficiently large order n, and*

$$bk(G) < \left(\frac{1}{4} + \varepsilon \right) n \qquad \text{and} \qquad bk(\overline{G}) < \left(\frac{1}{4} + \varepsilon \right) n,$$

then

$$\sum_{u,v \in V(G)} \left| d_G(uv) - \frac{n}{4} \right| < \delta n^3.$$

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Proof. From (2.3) we have

$$3\big(k_3(G)+k_3(\overline{G})\big) \geqslant \frac{n(n-1)(n-5)}{8}.$$

Hence

$$\sum_{uv\in E(G)}\left(d_G(uv)-\frac{n}{4}\right)+\sum_{uv\in E(\overline{G})}\left(d_{\overline{G}}(uv)-\frac{n}{4}\right)\geqslant -\frac{5n(n-1)}{8},$$

and, by hypothesis, the terms in the above sums are each less than εn. It follows that

$$\sum_{uv\in E(G)}\left|d_G(uv)-\frac{n}{4}\right|+\sum_{uv\in E(\overline{G})}\left|d_{\overline{G}}(uv)-\frac{n}{4}\right|<\varepsilon n^3,$$

and the desired result follows from Lemma 2.3. □

Observe that it just as easy to show that (2.4) also implies a necessary condition for quasi-randomness as follows from the next lemma whose proof is omitted.

Lemma 2.5. *For every $\delta>0$ there exists $\varepsilon=\varepsilon(\delta)>0$ such that if G is a graph of sufficiently large order n, and*

$$\sum_{u,v\in V(G)}\left|d_G(uv)-\frac{1}{4}n\right|<\varepsilon n^3,$$

then

$$\sum_{uv\in E(G)}\left|d_G(uv)-\frac{1}{4}n\right|+\sum_{uv\in E(\overline{G})}\left|d_{\overline{G}}(uv)-\frac{1}{4}n\right|<\delta n^3.$$ □

From Lemma 2.4 and Theorem 2.1 we obtain the following.

Corollary 2.6. *For every $\delta>0$ there exists $\varepsilon=\varepsilon(\delta)>0$ such that if G is a graph of sufficiently large order n, and*

$$bk(G)<\left(\frac{1}{4}+\varepsilon\right)n,\qquad bk(\overline{G})<\left(\frac{1}{4}+\varepsilon\right)n,$$

then

$$\left|e(G[X])-\frac{1}{4}|X|^2\right|<\delta n^2.$$

for every $X\subset V(G)$. □

Applying a well-known inequality of Nordhaus and Stewart [7] we obtain the following simple lower bound on the booksize of a graph.

Lemma 2.7. *For every graph $G=G(n,m)$,*

$$bk(G)\geqslant\left(\frac{4m}{n^2}-1\right)n.$$

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Proof. For the sake of completeness we start with a brief proof of Nordhaus–Stewart inequality.

For every edge $uv \in E(G)$ we have

$$|\Gamma(u) \cap \Gamma(v)| = |\Gamma(u)| + |\Gamma(v)| - |\Gamma(u) \cup \Gamma(v)| \geqslant d(u) + d(v) - n.$$

Summing this inequality over all $uv \in E(G)$ we obtain

$$3k_3(G) \geqslant \sum_{uv \in E(G)} (d(u) + d(v) - n) = \sum_{u \in V(G)} d^2(u) - nm \geqslant \frac{4m^2}{n} - nm.$$

Hence, from

$$bk(G) \geqslant \frac{3k_3(G)}{m},$$

the desired result follows. □

2.1. Proof of Theorem 2.1

In our proof we borrow some ideas of [11], Theorem 1.

Proof. Set $\varepsilon = \min\{\delta^4, 1\}$ and choose $X \subset V(G)$, $X \neq \varnothing$, $X \neq V$.

For every $u \in V(G)$, let $x(u) = |\Gamma_G(u) \cap X|$, and set

$$|X| = k, \qquad e_1 = e(G[X]), \qquad e_2 = e(X, V(G)\backslash X).$$

We immediately see that

$$2e_1 = \sum_{u \in X} x(u), \qquad e_2 = \sum_{u \in V(G)\backslash X} x(u).$$

Also, from (2.1), we have

$$\sum_{u \in X} \binom{x(u)}{2} + \sum_{u \in V(G)\backslash X} \binom{x(u)}{2} = \sum_{u,v \in X} d_G(uv) < \frac{n}{4}\binom{k}{2} + \varepsilon n^3.$$

The convexity of $\binom{x}{2}$ implies

$$k\binom{2e_1/k}{2} + (n-k)\binom{e_2/(n-k)}{2} < \frac{n}{4}\binom{k}{2} + \varepsilon n^3,$$

and hence

$$\frac{(2e_1)^2}{k} + \frac{(e_2)^2}{n-k} < \frac{nk^2}{4} + 2\varepsilon n^3. \tag{2.5}$$

From (2.1) and Lemma 2.2 we obtain

$$\sum_{u \in X}\left(d_G(u) - \frac{n}{2}\right) \geqslant -\sum_{u \in V(G)}\left|d_G(u) - \frac{n}{2}\right| \geqslant -\sqrt{\varepsilon} n^2.$$

Hence

$$e_2 - \left(\frac{kn}{2} - 2e_1\right) = 2e_1 + e_2 - \frac{kn}{2} = \sum_{u \in X}\left(d_G(u) - \frac{n}{2}\right) \geqslant -\sqrt{\varepsilon} n^2,$$

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and therefore

$$(e_2)^2 > \left(\frac{kn}{2} - 2e_1\right)^2 - \sqrt{\varepsilon}n^2\left(e_2 + \frac{kn}{2} - 2e_1\right) > \left(\frac{kn}{2} - 2e_1\right)^2 - \sqrt{\varepsilon}n^4.$$

This, in view of (2.5), implies

$$\begin{aligned}(n-k)(2e_1)^2 + k\left(\frac{kn}{2} - 2e_1\right)^2 &< \frac{nk^3(n-k)}{4} + 2\varepsilon n^3 k(n-k) + \sqrt{\varepsilon}n^4 k \\ &< \frac{nk^3(n-k)}{4} + 3\sqrt{\varepsilon}n^5.\end{aligned}$$

Rearranging terms and dividing by $4/n$, we obtain

$$e_1^2 - \frac{k^2 e_1}{2} + \frac{k^4}{16} < \frac{3}{4}\sqrt{\varepsilon}n^4 < \sqrt{\varepsilon}n^4,$$

and hence

$$\left|e(G[X]) - \frac{1}{4}|X|^2\right| = \left|e_1 - \frac{1}{4}k^2\right| < \sqrt[4]{\varepsilon}n^2 = \delta n^2,$$

completing the proof. □

3. Some Ramsey-type results

In this section we shall prove some consequences for Ramsey theory.

Theorem 3.1. *For every $\delta > 0$ there exist $\varepsilon = \varepsilon(\delta) > 0$ and $n_0 = n_0(\delta)$ such that if $n > n_0$, $l \geqslant (4-\varepsilon)n$, and the graph $G = G(l)$ satisfies*

$$bk(G) < n, \qquad bk(\overline{G}) < n \tag{3.1}$$

then

$$\left|e(G[X]) - \frac{1}{4}|X|^2\right| < \delta l^2.$$

for every $X \subset V(G)$.

Proof. Observe that

$$n \leqslant \frac{l}{(4-\varepsilon)} < \left(\frac{1}{4} + \varepsilon\right)l$$

for every sufficiently small $\varepsilon > 0$. Hence, by hypothesis, we obtain

$$bk(G) < \left(\frac{1}{4} + \varepsilon\right)l, \qquad bk(\overline{G}) < \left(\frac{1}{4} + \varepsilon\right)l,$$

and by Corollary 2.6 the proof is completed. □

Theorem 3.2. *There exists a constant $c > 0$ such that every red–blue edge colouring of $K_{r(B_n)}$ contains a monochromatic book B_n whose vertices span at least $\lfloor cn^2 \rfloor$ edges of the same colour as the book.*

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Proof. Clearly we need to prove the theorem only for n sufficiently large, so assume n is as large as needed. Let $\varepsilon(\delta)$ be the function from Corollary 2.6 and set

$$c = \min\left\{\frac{1}{10}, \frac{1}{16}\varepsilon\left(\frac{1}{128}\right)\right\}. \tag{3.2}$$

Given a red–blue edge colouring of $K_{r(B_n)}$, let the graph G be on the same vertex set as $K_{r(B_n)}$, and let $E(G)$ be the set of red edges. The order of G is $r(B_n)$, so either $B_n \subset G$ or $B_n \subset \overline{G}$. Assume $B_n \subset G$ – say $uv \in E(G)$ – and $uv + X \subset G$ is a book such that $|X| = n$. Then, if

$$\left|e(G[X]) - \frac{1}{4}|X|^2\right| < \frac{1}{128}(r(B_n))^2 \tag{3.3}$$

holds, we have

$$e(G[X]) > \frac{1}{4}|X|^2 - \frac{1}{128}(r(B_n))^2 \geqslant \frac{n^2}{4} - \frac{(4n+2)^2}{128} > \frac{n^2}{10},$$

and in this case the desired result follows in view of (3.2).

Suppose now that (3.3) does not hold. Applying Corollary 2.6 with $\delta = 1/128$, we see that for $\varepsilon = \varepsilon(1/128)$ we have either

$$bk(G) \geqslant (1/4 + \varepsilon)r(B_n)$$

or

$$bk(\overline{G}) \geqslant (1/4 + \varepsilon)r(B_n).$$

Assume $bk(G) \geqslant (1/4 + \varepsilon)r(B_n)$ – say $uv \in E(G)$ – and $uv + X \subset G$ is a book such that $|X| \geqslant (1/4 + \varepsilon)r(B_n)$.

For n sufficiently large we may assume $r(B_n) > (4 - \varepsilon)n$, and so

$$|X| > \left(\frac{1}{4} + \varepsilon\right)(4 - \varepsilon)n > (1 + \varepsilon)n.$$

If $e(G[X]) \geqslant \varepsilon|X|^2/16$ then the proof is completed by (3.2), so assume $e(G[X]) < \varepsilon|X|^2/16$. Hence, setting $H = \overline{G}[X]$, we find that

$$e(H) > \binom{|X|}{2} - \frac{\varepsilon}{16}|X|^2.$$

From Lemma 2.7 applied to H, it follows that

$$\begin{aligned} bk(H) &\geqslant \left(1 - \frac{2}{|X|} - \frac{\varepsilon}{4}\right)|X| > \left(1 - \frac{\varepsilon}{2}\right)|X| \\ &> \left(1 - \frac{\varepsilon}{2}\right)(1 + \varepsilon)n > n. \end{aligned}$$

Therefore, $B_{n+1} \subset H$, *i.e.*, X induces a blue book of size $(n+1)$ – say $zw \in E(H)$ – and $zw + Y \subset H$ is a book such that $|Y| = n + 1$. Clearly Y induces at least $\binom{n+1}{2}/2 > n^2/4$ edges of the same colour, so taking either the red book $uv + Y$ or the blue book $zw + Y$, the proof is completed in view of $c < 1/4$. □

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Recalling the well-known result (see, *e.g.*, [1, p. 113]) that for every $c > 0$ and $0 < \beta < 1$, there exist

$$\alpha = \alpha(c, \beta) > 0, \qquad a \geqslant \lfloor \alpha \log n \rfloor, \qquad b \geqslant \lfloor n^{1-\beta} \rfloor,$$

such that every $G(n, \lfloor cn^2 \rfloor)$ contains $K_{a,b}$, we obtain the result announced in the Introduction.

Theorem 3.3. *For n sufficiently large and every $0 < \beta < 1$ there exist*

$$\alpha = \alpha(\beta) > 0, \qquad a \geqslant \lfloor \alpha \log n \rfloor, \qquad b \geqslant \lfloor n^{1-\beta} \rfloor$$

such that

$$r(H) = r(B_n),$$

where the graph H is defined by

$$H = K_2 + \left(K_{a,b} \cup \overline{K}_{n-a-b}\right).$$

In particular, if $n = 4k + 1$ is a sufficiently large prime power then $r(H) = 4n + 2$. □

Observe that Theorem 3.3 is, in a sense, best possible, since for every nonbipartite graph F of order $k \leqslant n$, the graph H defined by

$$H = K_2 + \left(F \cup \overline{K}_{n-k}\right)$$

satisfies $r(H) > 4n + 4 \geqslant r(B_n)$. Indeed, consider the complete 4-partite graph $K_4(n + 1)$. Its complement consists of 4 disjoint cliques of order $n + 1$ and consequently contains no B_n; for every book $uv + X \subset K_4(n + 1)$, the set X induces a bipartite graph.

4. Concluding remarks

Despite its simplicity, Theorem 3.1 corroborates the expectation that for some sequences of graphs $\{H_n\}$, every sequence of graphs $\{G_n\}$ such that

$$|G_n| = (1 + o(1))r(H_n), \qquad H_n \not\subseteq G_n, \qquad H_n \not\subseteq \overline{G}_n,$$

is quasi-random.

It is also curious that techniques borrowed from the theory of quasi-random graphs could be useful in finding exact Ramsey numbers.

We shall conclude with three conjectures, the first one being due to Thomason [9].

Conjecture 4.1. *For every fixed integer $k > 2$,*

$$r\left(B_n^{(k)}\right) = (2^k + o(1))n.$$

Conjecture 4.2. *Let $k > 2$ be a fixed integer. For every $\delta > 0$, there exist $\varepsilon = \varepsilon(\delta) > 0$ and $n_0 = n_0(\delta)$ such that if $n > n_0$, $l \geqslant r\left(B_n^{(k)}\right) - \varepsilon n$, and $G = G(l)$ satisfies*

$$B_n^{(k)} \not\subseteq G, \qquad B_n^{(k)} \not\subseteq \overline{G},$$

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then

$$\left| e(G[X]) - \frac{1}{4}|X|^2 \right| < \delta l^2,$$

for every $X \subset V(G)$.

Conjecture 4.3. *For every fixed integer* $k > 2$, *there exists a constant* $c = c(k) > 0$ *such that every red–blue edge colouring of* $K_{r(B_n^{(k)})}$ *contains a monochromatic* $B_n^{(k)}$ *whose vertices span at least* $\lfloor cn^2 \rfloor$ *edges of the same colour as the book.*

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