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A quadratic parabolic group

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When k is a 2-bridge knot with group πK, there are parabolic representations (p-reps) $\theta\colon \pi K \to \mathrm{PSL}(\mathbb{C}) := \mathrm{PSL}(2,\mathbb{C})$. The most obvious problem that this suggests is the determination of a presentation for an image group $\pi K\theta$. We shall settle the easiest outstanding case in section 2 below, viz. k the figure-eight knot 4_1, which has the 2-bridge normal form $(5,3)$. We shall prove that the (two equivalent) p-reps θ for this knot are isomorphisms of πK on $\pi K\theta$. Furthermore, the universal covering space of $S^3\backslash k$ can be realized as Poincaré's upper half space $\mathscr{U}_3$, and $\pi K\theta$ is a group of hyperbolic isometries of $\mathscr{U}_3$ which is also the deck transformation group of the covering $\mathscr{U}_3 \to S^3\backslash k$. The group $\pi K\theta$ is a subgroup of two closely related groups that we study in section 3. We shall give fundamental domains, presentations, and other information for all these groups.

1. *Preliminaries.* We continue the notation and conventions of (7), (8), (9), e.g. we represent elements of $\mathrm{PSL}(\mathbb{C})$ by matrices in $\mathrm{SL}(\mathbb{C}) := \mathrm{SL}(2,\mathbb{C})$, and we write p-reps θ on the right, thus $x \mapsto x\theta$. However, we make the following change; when $\pi K\theta$ is a matrix group acting on a space X we write transformations on the left, thus $T(x) = Tx$ when $T \in \pi K\theta$, $x \in X$. This is because, in spite of what I wrote in (7), p-reps are anti-homomorphisms.

We begin by recalling Poincaré's upper half space $\mathscr{U}_3$ and the way elements of $\mathrm{PSL}(\mathbb{C})$ act on $\mathscr{U}_3$, cf. (6) or A. Best's unpublished dissertation (2). We define

$$\mathscr{U}_3 := \{(z,t) \in \mathbb{C}\times\mathbb{R} \mid t > 0\}, \quad \mathbb{C}_0 := \mathbb{C}\times\{0\} \subset \mathbb{C}\times\mathbb{R},$$

and identify the projective line $\mathbb{P}^1(\mathbb{C})$ with $\mathbb{C}_0 \cup \{\infty\}$ in the usual way. The space $\mathscr{U}_3$ carries two metrics, Poincaré's hyperbolic metric, and a Euclidean metric induced by the natural map $\mathscr{U}_3 \to \mathbb{R}^3$, $(x+iy,t) \mapsto (x,y,t)$. An object defined with respect to one of these metrics will be prefixed with the appropriate letter E or H, except that the E prefix may be suppressed when not wanted for emphasis. For example, $\mathbb{C}_0$ is the E-boundary of $\mathscr{U}_3$. An *H-plane* $F := \mathscr{U}_3 \cap S$, where S is either an E-sphere with centre on $\mathbb{C}_0$ or S is an E-plane E-perpendicular to $\mathbb{C}_0$. These latter planes are planes in both metrics, so we call them *EH-planes*. The *E-boundary* of the H-plane $F = \mathscr{U}_3 \cap S$ is $\mathbb{C}_0 \cap S$, and is denoted F_0. Two intersecting H-planes meet in an *H-line* (a geodesic in the H-metric), and intersecting EH-planes meet in *EH-lines*.

Let T be a non-trivial element of $\mathrm{PSL}(\mathbb{C})$, say $T = (a\,b|c\,d)$ in flattened notation. The action of T on $\mathbb{P}^1(\mathbb{C})$ is described in (4). Our description of the action of T on $\mathscr{U}_3$ splits into the cases $c \neq 0$, $c = 0$. When $c \neq 0$ T acting on $\mathbb{P}^1(\mathbb{C})$ has an isometric circle $I_0(T) \subset \mathbb{C}_0$ defined as the set of $(z,0) \in \mathbb{C}_0$ such that $|z + c^{-1}d| = |c^{-1}|$. We call the

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H-plane $I(T)$ whose E-boundary is $I_0(T)$ the *isometric sphere of* T. Then T is an E-isometry of $I(T)$ on $I(T^{-1})$, an H-isometry of $\mathscr{U}_3$ on $\mathscr{U}_3$, and the action of T on $\mathscr{U}_3$ is the same as the product of the following three E-operations in the order given.

I. *E-reflection of $\mathscr{U}_3$ in an EH-plane $Ref(T)$ whose E-boundary contains the centre of $I_0(T)$.*

II. *E-inversion of $\mathscr{U}_3$ in $I(T)$.*

III. *E-translation of $\mathscr{U}_3$ carrying $I(T)$ on $I(T^{-1})$.*

Consequently T is determined by $I(T)$, $I(T^{-1})$ and $Ref(T)$, and $Ref(T)$ is determined by $I_0(T^{\pm 1})$ and any single $T(P)$, P a point of $I_0(T)$. Furthermore, the value of the E-Jacobian $Jac(T)$ at $P \in \mathscr{U}_3$ is > 1, $= 1$, or < 1, according as P is E-inside, on, or E-outside $I(T)$. Hence T maps the inside of $I(T)$ on the outside of $I(T^{-1})$.

When $c = 0$, T has no isometric sphere and $Jac(T)$ is constant on $\mathscr{U}_3$. We shall only need the case where $Jac(T) \equiv 1$. Then either T is parabolic of the form $(1\ b|0\ 1)$ and T is the E-translation $(z, t) \mapsto (z + b, t)$, or T is elliptic of the form $(a\, b|0\, a^{-1})$ and T is an E-rotation of $\mathscr{U}_3$ about the EH-axis whose E-end-point is the finite fixed point of T in $\mathbb{C}_0$. The characteristic property of transformations T with $c = 0$ is that $T\infty = \infty$, hence the collection of all such T is a subgroup $\mathrm{PSL}(\mathbb{C})_\infty$. If $\mathfrak{S} \subset \mathrm{PSL}(\mathbb{C})$, we define

$$\mathfrak{S}_\infty := \mathfrak{S} \cap \mathrm{PSL}(\mathbb{C})_\infty.$$

When $\mathfrak{S}$ is a discrete subgroup of $\mathrm{PSL}(\mathbb{C})$, $\mathfrak{S}$ acts discontinuously on $\mathscr{U}_3$ and $\mathfrak{S}_\infty$ acts discontinuously on $\mathbb{C}_0$. Hence $\mathfrak{S}$ has an open fundamental domain $\mathscr{D}(\mathfrak{S}) \subset \mathscr{U}_3$. A *Ford domain* is the portion of $\mathscr{U}_3$ outside all isometric spheres of $\mathfrak{S}$ which E-projects onto a fundamental domain of $\mathfrak{S}_\infty$, cf. (4). All the fundamental domains that we consider will be Ford domains. To verify that a candidate for being a Ford domain actually is a Ford domain, we apply the theorem of Poincaré proved in (5). We assume familiarity with (5), and we use Maskit's notation $\mathscr{D}^*(\mathfrak{S})$ for the identified polyhedron defined in ((5), Section 2).

As further notation, when $\mathfrak{S} = \langle \tau_1, \ldots, \tau_n \rangle \subset \mathrm{PSL}(\mathbb{C})$ and each generator τ_j is defined by a specific matrix $T_j \in \mathrm{SL}(\mathbb{C})$, we write $\mathrm{SL}\mathfrak{S} := \langle T_1, \ldots, T_n \rangle \subset \mathrm{SL}(\mathbb{C})$. When G is a group, $\mathfrak{A}G :=$ the abelianization $G/[G, G]$ of G. Also, $\mathfrak{F}G := G/G^f$, where G^f is the subgroup of G generated by the elements of finite order. There is a theorem of Armstrong which asserts that, in the situation of the preceding paragraph, $\pi_1(\mathscr{D}^*(\mathfrak{S}))$ is isomorphic to $\mathfrak{F}\mathfrak{S}$, cf. (1).

2. *The quadratic parabolic group.* The complement in S^3 of an open tubular neighbourhood of the figure-eight knot k is a compact 3-manifold X_1 whose boundary $\dot{X}_1$ is a torus. The marked knot group $\pi\bar{K}_1$ is $\pi_1(X_1)$ with a particular peripherical subgroup distinguished, and has a presentation

$$\pi\bar{K} = |x_1, x_2 : wx_1w^{-1} = x_2 : \gamma|, \tag{2·1}$$

in which
$$w := x_1^{-1}x_2x_1x_2^{-1}, \quad \tilde{w} := x_1x_2^{-1}x_1^{-1}x_2, \quad \gamma := \tilde{w}^{-1}w, \tag{2·2}$$

cf. Proposition 1 of (7). It follows from (2·1), (2·2) that also

$$\tilde{w}x_1\tilde{w}^{-1} = x_2, \quad \gamma x_1 = x_1\gamma. \tag{2·3}$$

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As in Theorem 2 of (7) we define a p-rep $\theta: \pi K \to \mathrm{PSL}(\mathbb{C})$ by

$$x_1\theta = A := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad x_2\theta = B := \begin{bmatrix} 1 & 0 \\ -\omega & 1 \end{bmatrix}, \tag{2·4}$$

where ω is a root of the rep polynomial $\Lambda(y)$ corresponding to the word w of (2·2). We calculate $\Lambda(y) = 1 + y + y^2$, whence ω is a primitive cube root of unity. Define

$$W := w\theta = \begin{bmatrix} 0 & \omega \\ -\omega^2 & 1-\omega \end{bmatrix}, \quad \tilde{W} := \tilde{w}\theta = \begin{bmatrix} 0 & -\omega \\ \omega^2 & 1-\omega \end{bmatrix}, \tag{2·5}$$

then on writing $A\{t\} := (1\ t|0\ 1)$ we find

$$\gamma^{-1}\theta = W^{-1}\tilde{W} = \begin{bmatrix} -1 & -2-4\omega \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2+4\omega \\ 0 & 1 \end{bmatrix} = A\{2+4\omega\}.$$

Note that $2+4\omega = 2\sqrt{-3}$, so $A\{2+4\omega\}$ is a translation in a direction perpendicular to that of $A = A\{1\}$. We write $\mathfrak{G} := \langle A, B\rangle = \pi K\theta$.

THEOREM 1. *A presentation for* $\mathfrak{G}$ *is*

Generators	$A, B,$
Assistant Generator	$W := A^{-1}BAB^{-1},$
Relation	$WAW^{-1} = B.$

Consequently, the parabolic representation $\theta: \pi K \to \mathfrak{G}$ *of* (2·4) *is an isomorphism.*

Proof. Since $|\omega| = 1$, the isometric spheres of $W^{\pm 1}$, $B^{\pm 1}$ have radius 1. Their centres are $(z, 0)$, where z is

$$I(B) \quad \omega^2 = -1-\omega, \quad I(W) \quad 1+2\omega,$$

$$I(B^{-1}) \quad 1+\omega, \qquad I(W^{-1}) \quad 0.$$

Every $\alpha \in \mathbb{Z}[\omega]$ is congruent to one of these centres mod $(1, 2+4\omega)$, so α is the centre of an isometric sphere of $\mathfrak{G}$ of radius 1. The collection $\mathscr{L}$ of all these spheres is a regular triangular lattice of spheres which is stable under $\langle A, A\{2+4\omega\}\rangle$. Each sphere of $\mathscr{L}$ meets six other spheres of $\mathscr{L}$ along the edges of an H-regular hexagon, and the angles of intersection are all $\frac{1}{3}2\pi$. Furthermore, the interior of the closure of the union of the E-projections on $\mathbb{C}_0$ of the hexagons on $I(B^{\pm 1})$, $I(W^{\pm 1})$ is a fundamental region $\mathscr{D}_\infty$ of $\langle A, A\{2+4\omega\}\rangle$. Let $\mathscr{D}$ be the portion of $\mathscr{U}_3$ lying E-above all spheres in $\mathscr{L}$ which E-projects onto $\mathscr{D}_\infty$. The E-vertical sides of $\mathscr{D}$ meet the spherical sides in the constant angle $\frac{1}{6}2\pi$.

We refer to the Figure, which depicts the projection on $\mathbb{C}_0$ of the objects in $\mathscr{U}_3$ we are discussing. To work out exactly how $\mathfrak{G}$ identifies the sides of $\mathscr{D}$, the only additional information we need is

$$B(0) = 0, \quad W(\omega) = -1.$$

This implies that $Ref(T)$ is the EH-plane bisecting the hexagon on $I(T)$ along the H-line which projects to the dashed line in the Figure, for $T = B^{\pm 1}, W^{\pm 1}$. The elements which identify the vertical sides are A, $A\{2+4\omega\}$ and $AA\{2+4\omega\} = A\{3+4\omega\}$.

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We now check that all the hypotheses $(a)-(i)$ of (5) for Poincaré's theorem are satisfied. The outcome is that $\mathscr{D}$ is a fundamental domain for $\mathfrak{G}$, that

$$\mathfrak{G}_\infty = \langle A, A\{2+4\omega\}\rangle,$$

and that $\mathfrak{G}$ has the following presentation.

Generators $A, B, W, A\{2+4\omega\}, A\{3+4\omega\}$.

Edge cycle relations. First the EH-edge cycles.

(1) *cycles* e_j^1, e_j^2 *for* $1 \leqslant j \leqslant 6$ $\quad A^{-1}A = E,$

(2) *cycle* e_7^1, e_7^2, e_7^3 $\quad A\{3+4\omega\}^{-1}A\{2+4\omega\}A = E,$

$\Rightarrow A\{3+4\omega\} = A\{2+4\omega\}A,$

(3) *cycle* e_8^1, e_8^2, e_8^3 $\quad A\{3+4\omega\}^{-1}AA\{2+4\omega\} = E,$

$\Rightarrow A\{3+4\omega\} = AA\{2+4\omega\}.$

The hexagon edge cycles.

(4) *cycle* e_9^i, $i = 1(1)5$ $\quad B^{-1}W^{-1}A^{-1}BA = E,$

$\Rightarrow W = A^{-1}BAB^{-1},$

(5) *cycle* e_{10}^i, $i = 1(1)6$ $\quad B^{-1}A^{-1}BA\{2+4\omega\}^{-1}W^{-1}A = E,$

$\Rightarrow A\{2+4\omega\} = W^{-1}\tilde{W},$

(6) *cycle* e_{11}^i, $i = 1(1)4$ $\quad WA^{-1}W^{-1}B = E,$

$\Rightarrow WAW^{-1} = B,$

(7) *cycle* e_{12}^i, $i = 1(1)6$ $\quad B^{-1}AW^{-1}A^{-1}WA\{3+4\omega\} = E,$

$\Rightarrow A\{3+4\omega\} = W^{-1}AWA^{-1}B.$

To get the presentation of Theorem 1, we eliminate the redundant relations, beginning with (1). Relation (7) is redundant because

$$W = B^{-1}BW \overset{(6)}{=} B^{-1}WA \overset{(4)}{=} B^{-1}A^{-1}BAB^{-1}A, \quad \Rightarrow WA^{-1}BA^{-1} = B^{-1}A^{-1}B,$$

$$\Rightarrow (W^{-1}A)\,WA^{-1}BA^{-1}(A) = (W^{-1}A)B^{-1}A^{-1}B(A) = W^{-1}\tilde{W}A \overset{(5)}{=}$$

$$= A\{2+4\omega\}A \overset{(2)}{=} A\{3+4\omega\},$$

which is relation (7). Finally, the generators $A\{2+4\omega\}$, $A\{3+4\omega\}$, and the relations involving them are redundant, because (2·2), (2·3) imply that these relations already hold in the knot group πK.

It is obvious from Theorem 1 that SL$\mathfrak{G}$ is isomorphic to $\mathfrak{G}$, i.e. that the non-trivial element $-E$ of the centre of PSL($\mathbb{C}$) is not in SL$\mathfrak{G}$.

COROLLARY. *The identified polyhedron* $\mathscr{D}^*$ *is homeomorphic to the knot complement* $S^3\backslash k$.

Proof. For $t > 0$ define

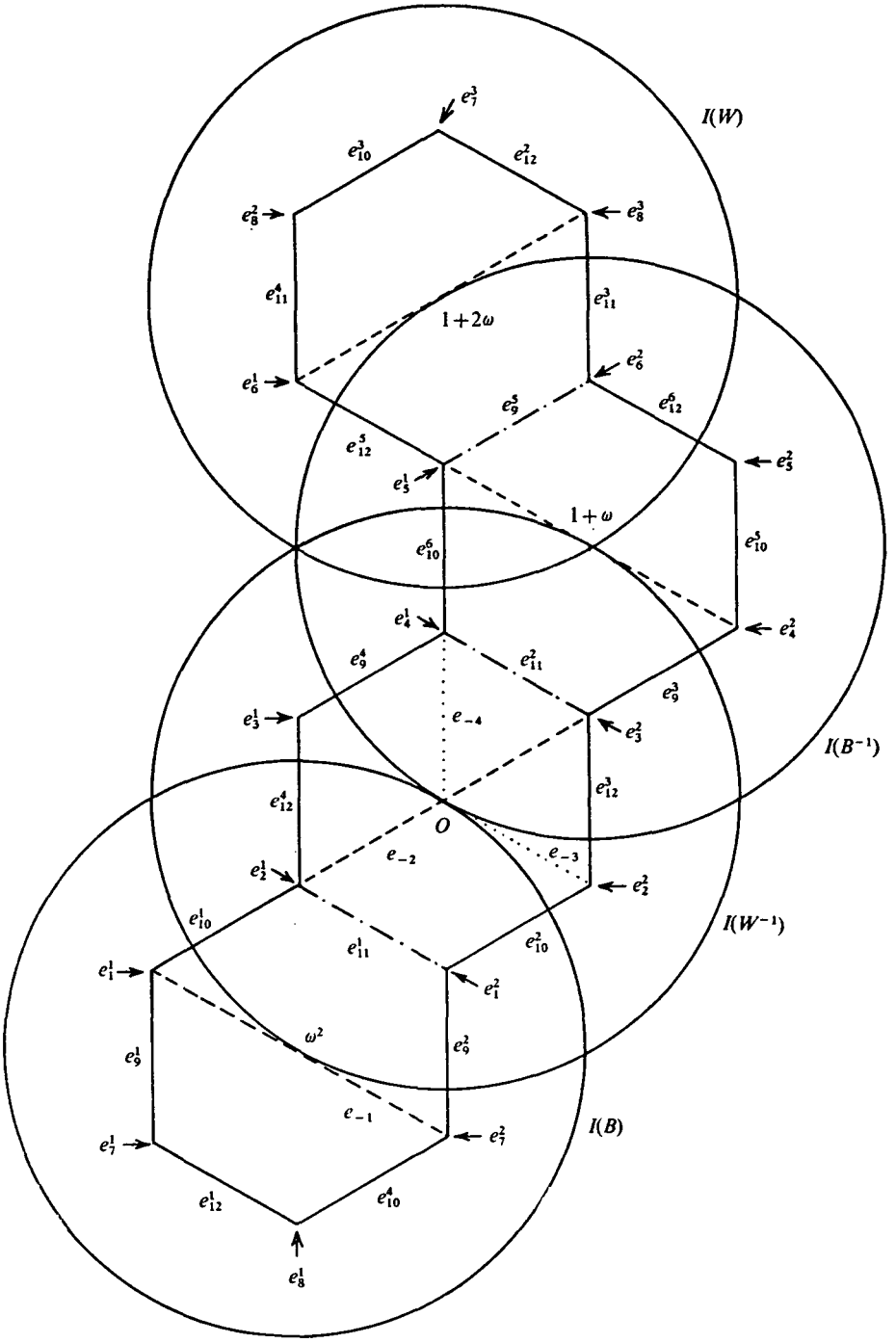
$$\mathscr{U}_3(t) := \{(z,s) \in \mathscr{U}_3 | s \leqslant t\}, \quad \mathscr{D}(t) := \mathscr{D} \cap \mathscr{U}_3(t),$$

and $X_2(t) :=$ the portion of $\mathscr{D}^*$ whose pre-image in $\mathscr{D}$ lies in $\mathscr{D}(t)$. Then for $t > 1$, $X_2(t)$ has a torus $\dot{X}_2(t)$ as boundary. Standard arguments reduce the problem to proving that X_1 is homeomorphic to $X_2 := X_2(2)$. Then, since $\mathfrak{G}^f = \langle E\rangle$,

$$\pi_1(X_2) \approx \pi_1(\mathscr{D}^*) \approx \mathfrak{G} \approx \pi_1(X_1).$$

We select a point $* \in \dot{X}_2$ as basepoint for both $\pi_1(X_2)$ and $\pi_1(\dot{X}_2)$, so that the inclusion $\dot{X}_2 \subset X_2$ induces a homomorphism $\iota: \pi_1(\dot{X}_2) \to \pi_1(X_2)$. The proof of Armstrong's

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Figure

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theorem shows that the elements $\bar{A} = x_1\theta$, $A\{2+4\omega\} = \gamma^{-1}\theta$ of $\mathfrak{G}$ correspond to elements of $\iota(\pi_1(\dot{X}_2))$, whence ι must be an injection. We now appeal to the results of section 6 of (11). In Waldhausen's language, we have shown that there is an isomorphism of $\pi_1(X_1)$ on $\pi_1(X_2)$ which respects the peripherical structure. In addition, X_1 is known to be irreducible, boundary irreducible, and sufficiently large. It is obvious that X_2 is boundary irreducible. Because the universal covering space of $\mathscr{D}^*$ is homeomorphic to $\mathbb{R}^3$, X_2 is irreducible. Hence Waldhausen's Corollary 6·5 shows that X_1 is homeomorphic to X_2.

3. *Two other groups.* Our group $\mathfrak{G}$ is a subgroup of the groups

$$\mathfrak{H} := \langle A, B, A\{1+2\omega\}\rangle, \quad \mathfrak{J} := \langle A, B, A\{\omega\}\rangle \tag{3·1}$$

in PSL($\mathbb{Z}[\omega]$), and because most of the preparatory work has already been done, we can find presentations for these groups very quickly. We begin with $\mathfrak{H}$. Let

$$B_1 := A\{2+2\omega\}^{-1}B = \begin{bmatrix} -1 & 2\omega^2 \\ -\omega & 1 \end{bmatrix}, \tag{3·2}$$

$$V := A\{1+2\omega\}^{-1}W^{-1} = \begin{bmatrix} 0 & -\omega \\ \omega^2 & 0 \end{bmatrix}. \tag{3·3}$$

These elements of $\mathfrak{H}$ are involutions because they have trace zero. It is easy to check that B_1 (respectively V) maps the H-hexagon on $I(B)$ ($I(W^{-1})$) on itself by reflection in the EH-plane bisecting the hexagon along the edge e_{-1} (respectively e_{-2}), cf. the Figure. Let $\mathscr{D}(\mathfrak{H})$ be the portion of $\mathscr{U}_3$ lying above the spheres $I(B)$, $I(W^{-1})$ which projects onto the interior of the closure of the union of the hexagons on $I(B)$ and $I(W^{-1})$.

THEOREM 2. *The domain $\mathscr{D}(\mathfrak{H})$ is a fundamental domain for $\mathfrak{H}$ and*

$$\mathfrak{H}_\infty = \langle A, A\{1+2\omega\}\rangle.$$

A presentation for $\mathfrak{H}$ is

Generators	$A, B_1,$
Assistant Generators	$V := B_1A^{-1}B_1AB_1, \quad A\{1+2\omega\} := VAB_1A^{-1}B_1,$
Relations	$B_1^2 = [A, A\{1+2\omega\}] = E.$

Furthermore, $\mathfrak{J}\mathfrak{H}$ is a free cyclic group generated by the image of A, and $\mathfrak{A}\mathfrak{H}$ has a presentation $|a, b_1: b_1^2 = E, ab_1 = b_1a|$, where a, b_1 are the images of A, B_1 respectively.

Proof. A mechanical check of the conditions of Poincaré's theorem verifies the first two assertions and yields the following presentation for $\mathfrak{H}$.

Generators $A, A\{1+2\omega\}, A\{2+2\omega\}, B_1, V.$

Edge cycle relations. First the vertical edge cycles.

(1h) *cycles* $e_j^1, e_j^2, \quad j = 1, 2, \quad A^{-1}A = E,$

(2h) *cycle* $e_7^1, e_7^2, e_4^1, \quad A\{2+2\omega\}^{-1}A\{1+2\omega\}A = E,$

(3h) *cycle* $e_8^1, e_3^1, e_3^2, \quad A\{2+2\omega\}^{-1}AA\{1+2\omega\} = E.$

The hexagon bisector cycles.

(4h) *cycle* $e_{-1}, \quad B_1^2 = E,$

(5h) *cycle* $e_{-2}, \quad V^2 = E.$

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The hexagon edge cycles.

(6h) *cycle* $e_9^1, e_9^2, e_{10}^4, e_9^4, e_{10}^2, e_{10}^1$, $\quad B_1A^{-1}VA\{1+2\omega\}B_1A = E$,

(7h) *cycle* $e_{12}^1, e_{11}^1, e_{12}^4, e_{12}^3, e_{11}^2$, $\quad A\{2+2\omega\}^{-1}VAVB_1 = E$.

It is easy to boil this down to the advertised presentation. The final two assertions are now immediate.

COROLLARY. *The identified polyhedron* $\mathscr{D}^*(\mathfrak{H})$ *is homeomorphic to the complement of a trivial knot in* S^3.

Proof. Let $X_3 :=$ the image of $\mathscr{U}_3(3) \cap \mathscr{D}(\mathfrak{H})$ under the identifying map

$$\mathscr{D}(\mathfrak{H}) \to \mathscr{D}^*(\mathfrak{H}).$$

It suffices to show that X_3 is a solid torus. Since $\dot{X}_3$ is a torus and $\pi_1(X_3) \approx \mathfrak{H}\mathfrak{H} \approx \mathbb{Z}$, there is a simple closed curve δ on $\dot{X}_3$ representing a generator of $\pi_1(\dot{X}_3)$ which is null-homotopic in X_3. By the loop theorem, δ bounds a non-singular disc $D \subset X_3$ such that $D \cap \dot{X}_3 = \delta$. Let Δ_1 be a closed regular neighbourhood of D in X_3, then Δ_1 is a 3-cell. The closure Δ_2 of $X_3 \backslash \Delta_1$ in X_3 is bounded by a 2-sphere, so, as before, Δ_2 is also a 3-cell. Then $\Delta_1 \cap \Delta_2$ consists of two disjoint closed discs on $\dot{\Delta}_1$, $\dot{\Delta}_2$, and this is a description of $X_3 = \Delta_1 \cup \Delta_2$ as a solid torus.

As generators of the group $\mathfrak{J}$, we shall use

$$L := WA\{\omega\}\,WA\{2+2\omega\}\,WA\{\omega\} = \begin{bmatrix} \omega^2 & 0 \\ 0 & \omega \end{bmatrix}, \quad M := AL. \tag{3·4}$$

Then L is an E-rotation of $\mathscr{U}_3$ of angle $+2\pi/3$ about the EH-axis above $(0,0) \in \mathbb{C}_0$, and M is a similar rotation about the EH-axis containing the edge e_3^2 of $\mathscr{D}$. Let $\mathscr{D}(\mathfrak{J})$ be the portion of $\mathscr{U}_3$ above the interior of the H-quadrilateral on $I(V) = I(W^{-1})$ whose sides are $e_{-3}, e_{12}^3, e_{11}^2, e_{-4}$.

THEOREM 3. *The domain* $\mathscr{D}(\mathfrak{J})$ *is a fundamental domain for* $\mathfrak{J}$, *and* $\mathfrak{J}_\infty = \langle A, L\rangle$. *A presentation for* $\mathfrak{J}$ *is*

Generators $\quad L, M, V,$

Relations $\quad L^3 = M^3 = V^2 = (ML)^3 = (MV)^3 = (VL)^2 = E.$

Also $\mathfrak{H}\mathfrak{J} = \langle E\rangle$, *and* $\mathfrak{A}\mathfrak{J}$ *is cyclic of order* 3 *with the image of* A *as a generator. The identified polyhedron* $\mathscr{D}^*(\mathfrak{J})$ *is homeomorphic to* $\mathbb{R}^3$. *Finally,* $\mathfrak{J} = \mathrm{PSL}(\mathbb{Z}[\omega])$.

Proof. The proofs of all these assertions except the last are similar to arguments used above. To show that $\mathfrak{J} = \mathrm{PSL}(\mathbb{Z}[\omega]) =: \mathfrak{J}'$, we verify without difficulty that (i) all points of $\mathscr{D}(\mathfrak{J})$ are E-outside every isometric sphere of $\mathfrak{J}'$, and (ii) $\mathfrak{J}'_\infty = \mathfrak{J}_\infty$. Then $\mathscr{D}(\mathfrak{J})$ is contained in a Ford domain of $\mathfrak{J}'$, so $\mathfrak{J} \subset \mathfrak{J}'$ implies $\mathfrak{J} = \mathfrak{J}'$.

We remark that $\mathfrak{J}$ has some importance for knot theory because there are infinitely many T-unions k_j of the knots 3_1 and 4_1 such that πK_j has a p-rep on $\mathfrak{J}$, cf. (9). We remark also that the groups $\mathrm{SL}\mathfrak{H}$, $\mathrm{SL}\mathfrak{J}$ both contain $-E$, and that presentations for them can be inferred from Theorems 2, 3 by checking whether a relation $Z = E$ in the projective group corresponds to $Z = +E$ or $Z = -E$ in the special linear group. We find $B_1^2 = V^2 = -E$ in $\mathrm{SL}\mathfrak{H}$ and

$$\mathrm{SL}\mathfrak{J} = |L, M, V: L^3 = M^3 = (ML)^3 = E,\ V^2 = (VL)^2 = (MV)^3 = -E,$$
$$(-E)^2 = E,\ -E \text{ is central}|.$$

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This may be compared with Swan's presentation of $\mathrm{SL}\mathfrak{J} = \mathrm{SL}(\mathbb{Z}[\omega])$ in section 6 of (10). The group $\mathfrak{J}$ was first considered by L. Bianchi, cf. (3).

If the interior of the closure of the union of n fundamental domains of a group $\mathfrak{S} \subset \mathrm{PSL}(\mathbb{C})$ is a fundamental domain for a subgroup $\mathfrak{S}' \subset \mathfrak{S}$, then $|\mathfrak{S}:\mathfrak{S}'| = n$. Therefore

$$|\mathfrak{J}:\mathfrak{G}| = 12, \quad |\mathfrak{J}:\mathfrak{H}| = 6, \quad |\mathfrak{H}:\mathfrak{G}| = 2. \tag{3·5}$$

An alternative method of comparing these three groups is to consider their reductions modulo an ideal of $\mathbb{Z}[\omega] =: R$. However, if $\mathfrak{p}$ is a prime ideal of R, a theorem of L. Dickson (Theorem C of (7)) shows that these three groups have the same reduction, except when $\mathfrak{p} = (2)$. For a subgroup $\mathfrak{S} \subset \mathrm{PSL}(R)$, let $\mathfrak{S}(n)$ denote the image of $\mathfrak{S}$ under the homomorphism induced by $R \to R \pmod{n}$. By elementary calculation we find

$$\mathfrak{G}(2) = \mathfrak{H}(2), \quad |\mathfrak{J}(2):\mathfrak{G}(2)| = 6,$$
$$|\mathfrak{J}(4):\mathfrak{G}(4)| = 12, \quad |\mathfrak{J}(4):\mathfrak{H}(4)| = 6, \quad |\mathfrak{H}(4):\mathfrak{G}(4)| = 2.$$

This gives (3·5), but with $\geqslant$ replacing $=$ throughout. It seems difficult to prove all of (3·5) by continuing this argument.

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