A.A.S. **1**, I, 129-164 (2006)

DEFINED CONTRIBUTION BASED PENSION PLANS

By S. Lindset

ABSTRACT

In this paper we address the problem of valuing (corporate) pension plans, and in particular defined contribution based pension plans. Several pension plans are proposed, both with and without rate of return guarantees. Both maturity and annual guarantees are considered. Emphasis is also given to the risk management of pension plans.

To tie the analysis closer to real-world problems, we allow for both periodic premium and pension payments and mortality risk to be taken into account. Some new results on forward-start guarantees are also derived.

KEYWORDS

Defined Contribution Based Pension Plans; Risk Management; Forward-Start Guarantees

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1. Introduction

The aging of the population in most parts of the western world has led to an increased focus on private pension arrangements, see, e.g., *The Economist*, 11 May 2002, p80. Many employees have pension arrangements through their employers and are members of different corporate pension plans. Traditionally, most of these pension plans have been so-called *defined benefit based pension plans*. In recent years, *defined contribution based pension plans*, where the pensions are a function of (among other things) the return in the financial markets, have increased in popularity. This is likely to be both because of changes in the laws regulating corporate pension plans (for instance, defined contribution based pension plans were not allowed in Norway prior to 2001) and by the fact that the financial markets have historically given a high return on investments.

Pension plans where the employer pays the premiums can be of significant economic value for both the employee and the employer. For example, for a person who considers different potential employers, the values of the pension plans which the employers are offering should be taken into account when evaluating the offers. The goal of this paper is to present a way

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

for employees and employers to value and rank different pension plans. We propose several fairly general pension plans, and the plans are also analysed from a risk management perspective.

Many defined contribution based pension plans are not embedded with any sort of guarantees to reduce the financial risk. This risk can be reduced by embedding the pension plans with *nominal* guarantees; i.e., rate of return guarantees. The theory of arbitrage pricing of contingent claims from financial economics has proved to be a useful tool in the valuation of rate of return guarantees, see e.g., Brennan & Schwartz (1976) and Miltersen & Persson (1999). For an excellent treatment of more complicated guarantees, see Tiong (2000).

A crucial assumption underlying arbitrage pricing is that the payoff of the claim which is to be priced can be replicated with some self-financing trading strategy involving the underlying asset(s) of the claim. The price of the claim can, by arbitrage arguments, be seen to coincide with the initial price of the trading strategy, and is termed *the arbitrage price*. Normally, neither the employer nor the employee has the possibility to replicate the (financial) claim which a pension plan represents. However, it does not seem unreasonable to assume that a pension fund has the possibility, at least to some extent, to replicate the (financial) claims which are present in its customers' pension plans. In a competitive market, this possibility should therefore make the arbitrage price of the pension plan equal to the price which the pension fund can charge for the financial part of a pension plan.

The arbitrage price does not take *mortality risk* into consideration, an important aspect of pension plans. We use the standard assumption about independence between financial and mortality risk. By issuing many similar and statistically independent pension plans, mortality risk can (at least to some extent) be diversified. We therefore also use the assumption about risk neutrality with respect to mortality risk. However, it is not possible for the employees to diversify this risk. Since the employees can neither replicate the arbitrage price of a pension plan nor diversify mortality risk, the *market value*, here defined as the arbitrage price adjusted for mortality risk, will typically not be the only measure which can be used to rank different pension plans, but as a price measure it may still be better than other measures not based on economic arguments.

To justify the use of these two assumptions, assume that the employees can sell (or more realistically, borrow against) the future random cash flow from their pension plans, and use the funds from the sale to construct portfolios which give them the 'optimal' consumption. We can imagine a market for buying these pension plans. If the buyers are sufficiently large in terms of the number of policies bought, they should also be risk neutral with respect to mortality risk. In a competitive market, this and the hedging possibilities for the buyers of the pension plans should make the market value equal to the price for which the employees can sell their pension plans.

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Using these assumptions, the employees can rank different pension plans based on their market values. The same reasoning is also valid for the employers. In a competitive market, they can pay someone the market value of the pension plan, typically a pension fund, to take care of all future premium and pension payments.

This paper differs from most of the existing literature on pension plans and guarantees, not only in that mortality risk is included, but mainly because we consider both periodic premiums and pension payments. Periodic premiums were also considered by Brennan & Schwartz (1976), but they had another approach, based on numerical solution of a partial differential equation. Our approach has the nice feature that it involves closed form solutions based on some extensions of already known pricing results for rate of return guarantees. Using these guarantees and the concepts laid out above, we propose different pension plans, and show how they can be valued.

Our main focus is on defined contribution based pension plans. Both pension plans with and without guarantees embedded are analysed. Two of the main differences between defined contribution and benefit based pension plans are *when* the risk is present and *who* gets exposed to the risk. The defined contribution based pension plan, in its simplest form, invests deterministic amounts (the premiums) in the financial market. The premiums accumulate some uncertain return, leading to uncertain pension payments. Again, in its simplest form, a defined benefit based pension plan has deterministic pension payments. To achieve these pension payments when the premiums are invested in an uncertain financial market, the size of the premiums has to change in accordance with the financial market. Hence, the financial risk is borne by the employees in a defined contribution based pension plan, whereas the employers bear this risk in a defined benefit based pension plan.

A criticism which has been raised against defined contribution based pension plans has been that the employees get exposed to too much risk, risk which they may not be interested in bearing. We show, by numerical examples with realistic parameter values, that these pension plans indeed are risky, even when rate of return guarantees are included. This is supported by findings in Burtless (2000). He showed by an example, using historical data from the United States of America, that the annual pension for a 'typical' U.S. citizen who retired in 1969 would be nearly 100% of his pre-retirement earnings, while a 'typical' U.S. citizen who retired in 1975 would only receive 42%, if they had a defined contribution based pension plan. This illustrates the high risk in these kinds of pension plans and why nominal guarantees are of interest. On the other hand, one of the arguments which has been set in favour for the contribution based pension plans, contra defined benefit based pension plans, is that one should expect a higher rate of return when the premiums are invested in the financial market, leading to a higher expected pension.

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Although going from a defined benefit based to a defined contribution based pension plan shifts much of the risk from the employers to the employees, this does not mean that this relieves the employer from risk management. Including rate of return guarantees imposes a large amount of financial risk on the sponsor, making risk management an important issue. We show how the financial risk can be managed by undertaking trading strategies in the financial market.

The paper is organised as follows. In Section 2 we give a description of our economic model and general set-up. In Section 3 we analyse different defined contribution based pension plans. Section 4 deals with risk management issues. In Section 5 we present numerical examples where the market value of the pension plans and the size of the pension payments are calculated. Some concluding remarks are given in Section 6.

2. The Economic Model and Preliminaries

2.1 *Financial Factors*

We use the model of Black & Scholes (1973), in the following referred to as a *Black and Scholes economy*. This is a rather simplified, though widely used, model of the financial market. We assume that there exists a pension fund which has a given investment policy. This policy generates some random return δ_t^T over the time period from time t to T. We further assume that there exists an equivalent martingale measure Q, under which the return is given by the following equation:[1]

$$\delta_t^T = (r - \tfrac{1}{2}\sigma_S^2)(T-t) + \sigma_S(W_T - W_t)$$

where r is the risk-free interest rate and is assumed constant, σ_S is the instantaneous standard deviation of the return on the pension fund and is also a constant, and W_T is a standard Brownian motion with $W_t = 0$.

A bank account is a risk free asset, where the time T market value is given by:

$$B_T = B_t e^{r(T-t)}$$

where B_t is the initial deposit on the bank account at time t. It will be convenient to set $B_t = 1$.

The customers of the pension fund have accounts which are referred to as *pension accounts*. In the rest of this paper we let the term *level of participation* denote what fraction of the return on the pension *fund* is the underlying

[1] We interchangeably refer to this as the return on the pension fund and the return on the pension fund's investment portfolio.

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

return on the pension *account*. The level of participation is throughout denoted by the parameter γ.

The time t arbitrage price of an investment, maturing at time $T > t$, of one unit of account, which accrues at the same rate of return as the underlying return on the pension account, is given by:[2]

$$\begin{aligned} V_t(e^{\gamma\delta_t^T}) &= E_Q\left[e^{-r(T-t)}e^{\gamma\left((r-\frac{1}{2}\sigma_S^2)(T-t)+\sigma_S(W_T-W_t)\right)}\middle|\mathcal{F}_t\right] \\ &= e^{(\gamma-1)(r+\frac{1}{2}\gamma\sigma_S^2)(T-t)}. \end{aligned} \tag{1}$$

More generally, the time t market value of some cash flow X_T at time T is given by:

$$V_t(X) = E_Q\left[e^{-r(T-t)}X_T\middle|\mathcal{F}_t\right]. \tag{2}$$

2.2 *Mortality Factors*

The remaining lifetime of a person aged x is given by the stochastic variable T_x. The probability that a person aged x survives t more years we denote by $P(T_x > t) = {}_tp_x$. It is assumed that financial and mortality risks are independent.

Given the independence between financial and mortality risks, and an assumption about risk neutrality with respect to mortality risk, it can be shown that the market values (as defined in the fifth paragraph of Section 1) of the claims in equations (1) and (2), if the payment to the investor will be made only if he is alive at time T, are given by:

$$\overline{V}_t(e^{\gamma\delta_t^T}) = V_t(e^{\gamma\delta_t^T})_Tp_x \tag{3}$$

and

$$\overline{V}_t(X) = V_t(X)_Tp_x \tag{4}$$

respectively.

3. Pension Plans

Let p_i, $i \in \{1, 2, \ldots, I\}$, be the size of the premium payment at time t_i if the employee is still alive. Further, let $\hat{p}_i = p_i 1_{\{T_x > t_i\}}$ where $1_{\{T_x > t_i\}}$ be an

[2] We let $E_Q[\cdot]$ denote the expectation under the equivalent martingale measure Q and $E_Q[\cdot \mid \mathcal{F}_t]$ the expectation conditional on $\mathcal{F}_t$, the information available at time t.

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

indicator function, returning the value one if the employee is alive at time t_i and zero otherwise. In the same way, let a_j, $j \in \{1, 2, \ldots, J\}$ be the size of the pension payment at time T_j and $\hat{a}_j = a_j 1_{\{T_x > T_j\}}$. We define a pension plan as a sequence of payments:

$$P = \{-\hat{p}_1, -\hat{p}_2, \ldots, -\hat{p}_I, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_J\}.$$

Thus, the premiums are paid to obtain pension payments at some later points in time. Further, no premium or pension payments are made after the employee's death. For a lifelong pension, we can think of T_J as being infinite.

The terms of the pension plan determine the relationship between the premium payments and the pensions. Because of mortality risk, time-lag between the inception of the pension plan and the premium payments, level of participation $\gamma \neq 1$, and rate of return guarantees which may possibly be embedded, the premiums will typically not coincide with the contributions made to the pension account.

For defined contribution based pension plans, the *contributions* to the employee's pension account are typically a fraction of his or her salary,[3] while, for defined benefit based pension plans, the *pension payments* are typically a fraction of the salary. In real-life situations the salary may fluctuate, but we will implicitly assume that it follows a deterministic function. Since the uncertainty in the financial market is likely to be higher than for the salary, we believe that the error arising from this assumption is relatively small.[4]

3.1 *Defined Contribution Based Pension Plans*

We assume that the employee has a pension account with the pension fund, and at each time of premium payment the account is debited (contributions are being made to the account), and when pension payments are being made the account is credited. The account will also be debited with the rate of return which the terms of the pension plan prescribe. If the account has any positive balance at the time of death, the balance will be distributed to the pension fund (negative balance will not occur in our pension plans). If the employee's death occurs before time T (the time of retirement), no more premium payments will be made.

A crucial assumption for the analysis performed in this paper is that any return on the pension fund which exceeds the return debited on the pension account is not in any way distributed back to either the employee or the

[3] In practice, the contributions may be a more complicated function of the salary.

[4] Pennacchi (1999) assumed that the salary follows a stochastic process with the same source of uncertainty as an (hypothetical) asset, and used this asset in the determination of the market value of the contract.

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

employer. Distribution mechanisms in life insurance contracts are often more involved than for pension contracts, see e.g., Grosen & Jørgensen (2000), Miltersen & Persson (2003), and Hansen & Miltersen (2002).

Maturity guarantee

Since we, in this paper, analyse pension plans where minimum rate of return guarantees can be embedded, in this and the next paragraph we will give a short analysis of two of these guarantees.

Let g be the minimum guaranteed rate of return per unit of time (i.e., per year). If the amount $\bar{\lambda}$ is debited on the pension account at time t_0 and with a maturity guarantee embedded and a level of participation γ, the time T payoff is given by $\bar{\lambda}\max(e^{\gamma\delta_{t_0}^T}, e^{g(T-t_0)})$. If no minimum guaranteed rate of return is included, we can formally set $g = -\infty$, and the payoff then becomes $\bar{\lambda}e^{\gamma\delta_{t_0}^T}$. We denote the time t_0 market value of this contract by $G(\bar{\lambda}, T-t_0, \bar{\lambda}e^{g(T-t_0)}; \gamma)$, and it is given by (see, e.g., Tiong, 2000):

$$G(\bar{\lambda}, T-t_0, \bar{\lambda}e^{g(T-t_0)}; \gamma) = \bar{\lambda}e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma_S^2\right)(T-t_0)}\Phi(d_1) + \bar{\lambda}e^{(g-r)(T-t_0)}\Phi(d_2) \quad (5)$$

where:

$$d_1 = \frac{\left(-g+\gamma\left(r+\left(\gamma-\frac{1}{2}\right)\sigma_S^2\right)\right)(T-t_0)}{\gamma\sigma_S\sqrt{T-t_0}}$$

$$d_2 = \frac{\left(g-\gamma\left(r-\frac{1}{2}\sigma_S^2\right)\right)(T-t_0)}{\gamma\sigma_S\sqrt{T-t_0}}$$

and $\Phi(\cdot)$ is the cumulative normal probability distribution.

Definition 1

For some function $f(x_1, x_2, \ldots, x_n)$, we say that f is homogeneous of degree k if:

$$f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n) \quad t > 0.$$

Lemma 1

The market value of a maturity guarantee in a Black and Scholes economy is homogeneous of degree one in the number of units on which it is written.

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Proof. This follows trivially from equation (5), since both d_1 and d_2 are independent of $\bar{\lambda}$.

Although the above is trivial and can be thought upon as 'price = unit price × quantity', it is an important observation in the case where, for instance, $\bar{\lambda}$ is equal to the time t_0 market value of the underlying asset, something which often is the case for forward-start options and guarantees.

Lemma 2

The market value at time $0 \leq t_0$ of the guarantee with time t_0 market value $G(\bar{\lambda}, T - t_0, \bar{\lambda}e^{g(T-t_0)}; \gamma)$ is given by $G(\lambda, T - t_0, \lambda e^{g(T-t_0)}; \gamma)$, where $\lambda = e^{-rt_0}\bar{\lambda}$.

Proof. From Lemma 1 we have that $G(\bar{\lambda}, T - t_0, \bar{\lambda}e^{g(T-t_0)}; \gamma)$ can be written as $\bar{\lambda}G(1, T - t_0, e^{g(T-t_0)}; \gamma)$. The time zero market value is then given by:[5]

$$\begin{aligned} E_Q\left[e^{-rt_0}\bar{\lambda}G(1, T - t_0, e^{g(T-t_0)}; \gamma)\right] &= \lambda G(1, T - t_0, e^{g(T-t_0)}; \gamma) \\ &= G(\lambda, T - t_0, \lambda e^{g(T-t_0)}; \gamma). \end{aligned}$$

Multi-period guarantee

Let us now consider a multi-period (rate of return) guarantee, or annual guarantee. Again, the amount $\bar{\lambda}$ is debited on the pension account at time t_0, but now with an annual guarantee embedded. The investment lasts for N years (let t_n be the beginning of the $(n+1)$st year and $t_N = T$). The final payoff is given by:

$$\bar{\lambda}\prod_{n=0}^{N-1}\max\left(e^{\gamma\delta_{t_n}^{t_{n+1}}}, e^{g}\right).$$

We denote the time t_0 market value of this contract by $MG(\bar{\lambda}, T - t_0, \bar{\lambda}e^{g}; \gamma)$. It is easily seen that the market value at time t_0 is given by (see, e.g., Tiong, 2000):

$$MG(\bar{\lambda}, T - t_0, \bar{\lambda}e^{g}) = \bar{\lambda}\prod_{n=0}^{N-1}G(1, t_{n+1} - t_n, e^{g}; \gamma). \tag{6}$$

[5] Notice that for $t_0 = 0$ we have a standard maturity guarantee, as analysed by Tiong (2000), and if we also have that $\gamma = 1$, we have the same guarantee as in Brennan & Schwartz (1976). The result also holds in the case where $\bar{\lambda}$ is a linear function of the market value of the pension fund.

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Lemma 3

The time $0 \leq t_0$ market value of the contract with time t_0 market value $MG(\bar{\lambda}, T - t_0, \bar{\lambda}e^g; \gamma)$ is given by $MG(\lambda, T - t_0, e^g\lambda; \gamma)$.

Proof. This follows trivially, since equation (6) is homogenous of degree one in $\bar{\lambda}$.

The contracts in Lemma 2 and Lemma 3 give us the market value of a contract with a rate of return guarantee included, which starts to run at some future point in time. We will call these contracts forward-start guarantee and annual forward-start guarantee, respectively. The result in Lemma 2 with $\gamma = 1$ can be found in, e.g., Pennacchi (1999), while for $\gamma \neq 1$ the annual forward-start guarantee in Lemma 3 seems to be a new result.

Annuity contract with no guarantee

The first pension plan which we analyse is one with no rate of return guarantee included. At each premium payment date t_i, $i \in \{1, 2, \ldots, I\}$, an amount $\bar{\lambda}_i$ is debited on the employee's pension account. The balance of the account earns the fraction γ of the return on the pension fund. At the time when the employee retires (at time $T \geq t_I$), the balance of the account is converted to an annuity. We assume that the annuity only lasts as long as the employee is alive, or possibly only to some final time horizon $T_J \geq T$. For a life pension, we can think of T_J as being infinite. For a finite T_J, the employee can outlive the pension plan and may therefore face financial problems if he or she is still alive after time T_J. Since there is no minimum guaranteed rate of return on the pension account, the employee also faces the risk of a low rate of return on the pension fund. In particular, the pension is very much exposed to the cumulative return on the pension fund at the time of retirement when the balance on the pension account is converted to an annuity. We let $\lambda_i = e^{-rt_i}\bar{\lambda}_i$.

Proposition 1

If at each time of premium payment, an amount $\bar{\lambda}_i$ is debited on the employee's pension account, the time zero market value of the pension payments for a man aged x at time zero is given by:

$$\pi_0 = \sum_{i=1}^{I} \lambda_i e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma_S^2\right)(T-t_i)}{}_T p_x$$

with an annual pension (the first at time T_1):

$$a = \frac{\sum_{i=1}^{I} \bar{\lambda}_i e^{\gamma\delta_{t_i}^T}}{\sum_{j=1}^{J} e^{-r(T_j-T)}{}_{(T_j-T)}p_{(x+T)}}.$$

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Proof. Since the balance on the employee's pension account accrues a fraction γ of the return on the pension fund, the time T market value of the account is given by $\sum_{i=1}^{I} \bar{\lambda}_i e^{\gamma \delta_{t_i}^T}$. The employee will only be able to convert the balance on the pension account to an annuity (which at time T has the same market value as the pension account) if he or she is still alive at time T. Using the results in equations (1) and (3) and the definition of λ_i, it follows that the time zero market value of the pension plan is given by:

$$\begin{aligned}\pi_0 &= E_Q\left[e^{-rT}\sum_{i=1}^{I}\bar{\lambda}_i e^{\gamma\delta_{t_i}^T}\right]{}_Tp_x \\ &= \sum_{i=1}^{I}\lambda_i e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma_S^2\right)(T-t_i)}{}_Tp_x.\end{aligned}$$

In the spirit of the principle of equivalence,[6] we require that the time T market value of the pension payments (i.e., the annuity) must equal the market value of the pension account at time T (i.e. notice that the employee is now aged $x+T$):

$$\sum_{j=1}^{J} a e^{-r(T_j-T)}{}_{(T_j-T)}p_{(x+T)} = \sum_{i=1}^{I}\bar{\lambda}_i e^{\gamma\delta_{t_i}^T}.$$

It then follows that:

$$a = \frac{\sum_{i=1}^{I}\bar{\lambda}_i e^{\gamma\delta_{t_i}^T}}{\sum_{j=1}^{J} e^{-r(T_j-T)}{}_{(T_j-T)}p_{(x+T)}}.$$

Annuity contract with guarantee

Assume that we have the same kind of pension plan as the one above, but now with a minimum rate of return guarantee included. The market value of the pension plan depends on the type of guarantee which is included. A contract with a maturity guarantee is typically less expensive than a contract with an annual guarantee. With a maturity guarantee, the employee's pension is not so sensitive to the accumulated return on the pension fund at the time of retirement as the contract with no guarantee. If the accumulated return on the pension fund is 'low' when the employee retires, the guarantee will become binding and the pension payments will be higher than would have been the case if no guarantee was included. The maturity guarantee

[6] The principle of equivalence roughly states that the insurance premium should be the present value of the expected cash flow from the insurance contract (see, e.g., Persson & Aase, 1994).

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secures that the average rate of return on the pension plan does not become 'low'. However, it is difficult for the employee to make a relatively precise estimate of the size of the pension payments prior to the time of retirement, since the accumulated return on the pension fund can fluctuate much. The annual guarantee, on the other hand, 'locks in' the annual return, securing that the average return within each single year is not too low. This guarantee is sometimes given the descriptive name 'cliquet', which is French for rocket. With this guarantee, the employee can, at the end of each year, monitor the realised return on his pension account, reducing the uncertainty in his estimate of the future pension payments. This is probably the reason why the annual guarantee is used in some countries.

The disadvantage of eliminating the probability of a low rate of return on the pension account is that it comes at a cost. In particular, the annual guarantee can be quite expensive, and if the employer offers a pension plan where each employee is given some amount to spend on premium payments, the idea of having a rate of return guarantee may not be that appealing. Since the guarantee element comes at a cost, a guarantee will necessarily reduce the amount of the premium which can be debited on the pension account. Even in the case of a low rate of return on the pension fund, one may get a higher pension by choosing the contract with no rate of return guarantee, since it allows one to debit a greater amount on the pension account.[7]

For $t_I = T$, let $G(\lambda_i, T - t_I, \lambda_i e^{g(T-t_I)}; \gamma)_T p_x = \lambda_{iT} p_x$.

Proposition 2

If, at each time of premium payment, an amount $\bar{\lambda}_i$ is debited on the pension account and there is a maturity guarantee embedded, then the time zero market value of the pension payments for a man aged x at time zero is given by:

$$\pi_0^1 = \sum_{i=1}^{I} G(\lambda_i, T - t_i, \lambda_i e^{g(T-t_i)}; \gamma)_T p_x$$

[7] Assume that there is only one premium and that it is paid at time zero, and that there is just one pension payment that is made at time T. It is easily seen that the following inequality has to be satisfied for the contract with a maturity guarantee to give a higher pension than the contract with no guarantee:

$$\delta_0^T < \frac{gT - \ln G(1, T, e^{gT}; \gamma)}{\gamma}.$$

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with an annual pension:

$$a^1 = \frac{\sum_{i=1}^{I} \bar{\lambda}_i \max(e^{\gamma \delta_{t_i}^T}, e^{g(T-t_i)})}{\sum_{j=1}^{J} e^{-r(T_j-T)}{}_{(T_j-T)}p_{(x+T)}}.$$

Proof. By using the forward-start guarantee in Lemma 2, the result follows in the exact same manner as in the proof of Proposition 1.

Proposition 3

If, at each time of premium payment, an amount $\bar{\lambda}_i$ is debited on the pension account and there is an annual guarantee embedded, then the time zero market value of the pension payments for a man aged x at time zero is given by:

$$\pi_0^2 = \sum_{i=1}^{I} MG(\lambda_i, T - t_i, \lambda_i e^g; \gamma)_T p_x$$

with an annual pension:

$$a^2 = \frac{\sum_{i=1}^{I} \bar{\lambda}_i \prod_{n=i}^{N-1} \max(e^{\gamma \delta_{t_n}^{t_{n+1}}}, e^g)}{\sum_{j=1}^{J} e^{-r(T_j-T)}{}_{(T_j-T)}p_{(x+T)}}.$$

Proof. By using the forward-start guarantee in Lemma 3, the result follows in the exact same manner as in the proof of Proposition 1.

Alternative contract with guarantee

In the plans analysed above the employee does not enjoy any 'high' return on the pension fund after he has retired. Assume, instead, a pension plan where the balance on the employee's pension account accrues a fraction γ of the return on the pension fund until the time of pension payment. There are many ways in which such a plan can be constructed, but we will here only consider one of the possibilities.

We let each premium be divided into J parts, one for each pension payment.[8] We further equip each part of the premium which is debited on the employee's pension account with a rate of return guarantee maturing at the time of the pension payment, i.e., time T_j, $j \in \{1, 2, \ldots, J\}$.

The advantage of this plan, compared to the one where the balance on the pension account is converted to an annuity at the time of retirement, is that the employee now will benefit from any high returns on the pension fund after retirement. The pension payments are no longer so sensitive to the

[8] To avoid to much uncertainty in the return after the employee has retired, we will only consider plans with rate of return guarantees included.

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accumulated return on the pension fund at time T, the retirement date. This gives a better diversification over time. In addition, there is also a floor for the pension, though the floor may be lower than the pension which the annuity would have given.

We assume that an amount $\bar{\lambda}_i = \sum_{j=1}^{J} \bar{\lambda}_{i,j}$, $i \in \{1, 2, \ldots, I\}$, $j \in \{1, 2, \ldots, J\}$, is debited on the pension account at each time t_i. It is further assumed that each $\bar{\lambda}_{i,j}$ is embedded with a guarantee maturing at time T_j. Let $\lambda_{i,j} = e^{-rt_i}\bar{\lambda}_{i,j}$.

Proposition 4

If, at each time of premium payment, the J amounts $\bar{\lambda}_{i,j}$, $i \in \{1, 2, \ldots, I\}$, $j \in \{1, 2, \ldots, J\}$, are debited on the employee's pension account and there are maturity guarantees embedded, each maturing at each time T_j, then the time zero market value of the pension payments for a man aged x at time zero is given by:

$$\pi_0^3 = \sum_{i=1}^{I}\sum_{j=1}^{J} G(\lambda_{i,j}, T_j - t_i, \lambda_{i,j}e^{g(T_j - t_i)}; \gamma)_{T_j}p_x$$

with a pension at time T_j:

$$a_j^3 = \sum_{i=1}^{I} \bar{\lambda}_{i,j} \max\left(e^{\gamma\delta_{t_i}^{T_j}}, e^{g(T_j - t_i)}\right).$$

Proof. By using the forward-start guarantee in Lemma 2, the result follows in the exact same manner as in the proof of Proposition 1.

Proposition 5

If, at each time of premium payment, the J amounts $\bar{\lambda}_{i,j}$, $i \in \{1, 2, \ldots, I\}$, $j \in \{1, 2, \ldots, J\}$, are debited on the employee's pension account and there are annual guarantees embedded, each maturing at each time T_j, then the time zero market value of the pension payments for a man aged x at time zero is given by:

$$\pi_0^4 = \sum_{i=1}^{I}\sum_{j=1}^{J} MG(\lambda_{i,j}, T_j - t_i, \lambda_{i,j}e^{g}; \gamma)_{T_j}p_x$$

with a pension at time T_j:

$$a_j^4 = \sum_{i=1}^{I} \bar{\lambda}_{i,j} \prod_{n=i}^{N_j - 1} \max\left(e^{\gamma\delta_{t_n}^{t_{n+1}}}, e^{g}\right)$$

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where $t_{N_j} = T_j > t_I$.

Proof. By using the forward-start guarantee in Lemma 3, the result follows in the exact same manner as in the proof of Proposition 1.

The pension contracts proposed here are not attempts to model existing contracts, but instead to propose possible contracts and show how easily contracts with rate of return guarantees embedded can be valued by using forward start guarantees. For instance in the U.S.A., the annuity is calculated in a different way. The annuity is calculated based on the expected remaining lifetime of the retired, leading to different payments in different years. The retired faces the risk of asymptotically running out of funds if he lives to be much older than the average person. The different ways of calculating the pension payments in different countries are often motivated by tax laws and other regulations.

4. Premium Payments and Risk Management

As far as we are able to tell, there is no 'universal' way for how to determine the size and the timing of the premium payments. As argued in the introduction, the market value of the obligation which the employer is taking on by giving an employee a pension contract is equal to the market value for the employee. In practice, the premiums are paid throughout the time when the employee is employed. Paying all the premiums up-front is not advisable, since, in real life, employees may leave to work for other employers prior to the time of retirement, making it hard to assess the premiums, and it would also be a clog on liquidity.

However, there is one requirement which is fundamental. At the time when the pension contract is signed, i.e., at time zero, the market value of the pension payments should equal that of the premium payments, i.e.:

$$V_0\left(\sum_{i=1}^{I}\hat{p}_i\right) = V_0\left(\sum_{j=1}^{J}\hat{a}_j\right).$$

It also seems reasonable, from a liquidity perspective, that the premiums are paid at about the same time as the pension account is debited.

For the framework in this paper, there is no risk with respect to the size of the premiums, except that the employee may die, so that no premium has to be paid. While reserving for the premiums, one does not, therefore, have to be concerned with financial risk. For large employers, mortality risk gets diversified away. However, managing the financial risk of the guarantees *after* they have started to run is more complicated and very important, since

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the risk can be very large. For most employers it is most cost efficient to let the pension fund provide the guarantees, since they can benefit from economies of scale when managing the financial risk inherent in the pension contracts.

In order to manage the risk in the pension contracts, let us abstract from mortality risk. The pension fund can construct dynamic hedging strategies involving two 'assets', the investment portfolio and the bank account, which will offset all financial risk. To this end, we need expressions for the market value of the amount deposited in the pension account at the ith premium payment at any point in time t, $t_i \leq t < T$. If the contract also includes a guarantee, this must be taken into account (the results are the same for the alternative contract, except that T must be replaced by T_j). Let S_t be the time t market value of (one unit of) the pension fund's investment portfolio and $B_{t_i} = 1$. We do not prove these results, but it can be shown that:

$$V_t(e^{\gamma\delta_{t_i}^T}) = \left(\frac{S_t}{S_{t_i}}\right)^{\gamma} e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma^2\right)(T-t)}$$

and

$$V_t\left(\max(e^{\gamma\delta_{t_i}^T}, e^{g(T-t_i)})\right) = \left(\frac{S_t}{S_{t_i}}\right)^{\gamma} e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma^2\right)(T-t)}\Phi(d_{1,t}) + e^{-r(T-t)+g(T-t_i)}\Phi(-d_{2,t})$$

where:

$$d_{1,t} = \frac{-g(T-t_i) + \gamma \ln\frac{S_t}{S_{t_i}} + \gamma\left(r + \left(\gamma - \frac{1}{2}\right)\sigma^2\right)(T-t)}{\gamma\sigma\sqrt{T-t}}$$

and

$$d_{2,t} = d_{1,t} - \gamma\sigma\sqrt{T-t}.$$

For the case with the multi-period guarantee, assume that a given premium is in the pension account for N periods, and that we are now at time t in period P. The realised gross return in the previous $P-1$ periods is then (each period is of length one year):

$$R^{P-1} = \prod_{i=1}^{P-1} \max\left(\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)^{\gamma}, e^{g}\right).$$

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Let further R^{N-P} be the market value at time t_P, i.e., at the beginning of the $(P+1)$th period, of the guaranteed return for the remaining $N-P$ periods. From (6) we have that:

$$R^{N-P} = \prod_{n=P}^{N-1} G(1, t_{n+1} - t_n, e^g; \gamma).$$

Finally, the time t, $t_P \leq t < t_{P+1}$, market value of the multi-period guarantee is given by:

$$V_t\Big(\prod_{k=i}^{N} \max(e^{\gamma\delta_{t_k}^{t_{k+1}}}, e^g)\Big) = R^{P-1}\Bigg(\Big(\frac{S_t}{S_{t_P}}\Big)^{\gamma} e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma^2\right)(t_P-t)}\Phi(d_{3,t}) + e^{-r(t_P-t)+g}\Phi(-d_{4,t})\Bigg)R^{N-P}$$

where:

$$d_{3,t} = \frac{-g + \gamma \ln\frac{S_t}{S_{t_P}} + \gamma\left(r + \left(\gamma - \frac{1}{2}\right)\sigma^2\right)(t_P - t)}{\gamma\sigma\sqrt{t_P - t}}$$

and

$$d_{4,t} = d_{3,t} - \gamma\sigma\sqrt{t_P - t}.$$

For $P = 1$ set $R^{P-1} = 1$ and for $P = N$ set $R^{N-P} = 1$.

Lemma 4

To hedge the claim with no guarantee, we can at each time t, $t_i \leq t < T$, hold:

$$\alpha_t(t_i) = \frac{1}{S_t}\left(\frac{S_t}{S_{t_i}}\right)^{\gamma} e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma^2\right)(T-t)}$$

units of the investment portfolio and

$$\beta_t(t_i) = \frac{1}{B_t}\left(\frac{S_t}{S_{t_i}}\right)^{\gamma} e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma^2\right)(T-t)}(1-\gamma)$$

units of the bank account.

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Proof. This follows by calculating the partial derivatives of the market value of the claim with respect to the value of the investment portfolio S_t and the bank account B_t.

Lemma 5

To hedge the maturity guarantee, we can at each time t, $t_i \leq t < T$, hold:

$$\alpha_t^1(t_i) = \frac{1}{S_t}\left(\frac{S_t}{S_{t_i}}\right)^{\gamma} e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma^2\right)(T-t)}\Phi(d_{1,t})$$

units of the investment portfolio and

$$\beta_t^1(t_i) = e^{(g-r)(T-t_i)}\Phi(-d_{2,t})$$

units of the bank account.

Proof. The proof follows the same lines as the proof of Lemma 4.

Lemma 6

To hedge the multi-period guarantee, we can at each time t, $t_P \leq t < t_{P+1}$, hold:

$$\alpha_t^2(t_i) = R^{P-1}\frac{1}{S_t}\left(\frac{S_t}{S_{t_P}}\right)^{\gamma} e^{(\gamma-1)\left(r+\frac{1}{2}\gamma\sigma^2\right)(t_{P+1}-t)}\Phi(d_{3,t})R^{N-P}$$

units of the investment portfolio and

$$\beta_t^2(t_i) = R^{P-1}e^{-r(t_{P+1}-t_P)+g}\Phi(-d_{4,t})R^{N-P}$$

units of the bank account, where $B_{t_P} = 1$.

Proof. The proof follows the same lines as the proof of Lemma 4.

For a more formal analysis of hedging strategies for rate of return guarantees, see Lindset (2003).

Using the results in Lemmas 4, 5, and 6, the expressions for the hedging strategies for the pension contracts are given in Propositions 6, 7, and 8 (mortality risk is not included). It is assumed that P premiums have been paid.

Proposition 6

To hedge the pension contract with no guarantee, we can hold the following number of units of the investment portfolio:

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$$\alpha_t = \sum_{i=1}^{P} \alpha_t(t_i)$$

and the following number of units of the bank account:

$$\beta_t = \sum_{i=1}^{P} \beta_t(t_i).$$

Proof. The result follows by the construction of the pension contract and Lemma 4.

Proposition 7

To hedge the pension contract with a maturity guarantee embedded, we can hold the following number of units of the investment portfolio:

$$\alpha_t^1 = \sum_{i=1}^{P} \alpha_t^1(t_i)$$

and the following number of units of the bank account:

$$\beta_t^1 = \sum_{i=1}^{P} \beta_t^1(t_i).$$

Proof. The result follows by the construction of the pension contract and Lemma 5.

Proposition 8

To hedge the pension contract with a multi-period guarantee embedded, we can hold the following number of units of the investment portfolio:

$$\alpha_t^2 = \sum_{i=1}^{P} \alpha_t^2(t_i)$$

and the following number of units of the bank account:

$$\beta_t^2 = \sum_{i=1}^{P} \beta_t^2(t_i).$$

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Proof. The result follows by the construction of the pension contract and Lemma 6.

Notice that these strategies require the pension fund to rebalance its hedge portfolio continuously. In practice, weekly rebalancing will remove a considerable amount of the financial risk.

5. Numerical Examples

In this section we present some numerical examples, where we calculate the market value of the pension contracts which we have proposed and the corresponding pensions for different scenarios for the return on the pension fund. For this purpose, we assume that the employees can choose between a high and a low risk portfolio and we look at a high and a low interest rate scenario. For the calculation of the pensions, we have picked three possible scenarios for the return on the investment portfolio: one with a medium performance; one with a very weak performance; and one with a strong performance.

The parameters used in the examples are given in Table 1.

We calculate the market values of pension contracts for employees who are aged 30 to 69 when they enter into the pension plan. It is assumed that they all retire at the age of 70, i.e., $T = 70$. The first premium is paid one year after the employees entered into the pension plan. For the annuity contract, the person aged 30 is exposed to the longest time series of financial returns. The first return is from age 31 to 32 and the last from age 69 to 70. An employee aged 69 on entry into the pension plan will not be exposed to financial risk with the annuity contract. The return in the different years for the three scenarios for the person aged 30 is given in Table 2. If the employee is aged 68 on entry into the pension plan, he or she will be exposed to the same return as the 30 year old does when 31 years old. For example, for the medium return in Table 2 the number -0.1080 means that the return is -10.80% this particular year.

To generate the returns, we assume a CAPM like model (see, e.g., Sharpe (1964) and Mossin (1966)). Let R_m be the expected market return, $\sigma_M = 0.2$ the annual standard deviation of the market return, the risk premium $R_m - r = 0.04$, and $\rho = 0.75$ the correlation coefficient between the return on

Table 1. Parameters used in the numerical examples

r		σ				
High	Low	High	Low	γ	$\bar{\lambda}$	T
8.00%	4.00%	0.20	0.10	0.75	100	70

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Table 2. Yearly return generated by $\delta_{t_i}^{t_{i+1}} = 0.08 + 0.04 \cdot 0.75 \frac{0.2}{0.2} \epsilon_i$, where $\epsilon_i \sim N(0, 1)$

Age	Medium return	Weak return	Strong return	Age	Medium return	Weak return	Strong return
31	−0.1080	−0.0365	0.3691	51	0.2017	−0.1424	0.3027
32	0.2291	−0.1719	0.2792	52	−0.0882	0.0126	0.1607
33	0.2412	−0.3554	−0.1907	53	0.4253	−0.1109	−0.0333
34	−0.2044	0.3037	0.3287	54	0.0591	−0.0514	0.3840
35	−0.0340	−0.2927	0.3789	55	−0.2569	0.1017	0.2863
36	0.1969	−0.1651	0.2629	56	0.1371	−0.1213	0.3888
37	0.0206	0.0055	0.3576	57	−0.0252	0.0362	0.0275
38	0.1267	−0.1291	−0.3112	58	−0.1036	0.1697	0.1162
39	0.0963	−0.2104	0.3220	59	0.0288	0.0479	0.3444
40	0.2310	−0.1329	0.0528	60	0.0177	0.2651	−0.0419
41	0.4041	−0.0647	0.3603	61	0.1817	−0.0698	−0.0645
42	0.2355	0.2381	0.3765	62	0.0393	0.0031	0.2647
43	0.1226	0.1525	0.0009	63	0.2456	0.1122	−0.0458
44	−0.1037	−0.0155	0.6988	64	0.0046	−0.1478	0.2663
45	0.1787	0.0873	0.4169	65	0.3750	−0.0675	0.2939
46	0.1913	−0.1141	0.1712	66	−0.1208	0.0372	−0.1621
47	0.3546	−0.1107	0.1744	67	0.0652	0.0444	0.1822
48	0.1219	−0.0063	−0.1041	68	0.2019	−0.0863	−0.0850
49	0.2958	0.2305	0.1184	69	0.1845	0.0999	−0.0614
50	0.1053	0.1457	0.2789				
				Average return	0.1096	−0.0131	0.1760

the market and the pension fund's investment portfolio. The returns in Table 2 are for a high risk portfolio under a high interest rate scenario. The return in year i is derived as follows (notice that the same ϵ_is are used also for the low risk portfolio and low interest rate scenario):

$$\delta_{t_i}^{t_{i+1}} = r + (R_m - r)\rho \frac{\sigma}{\sigma_m} \epsilon_i \qquad \epsilon_i \sim N(0, 1).$$

Mortality risk is taken into account by using mortality table N1963 (see, e.g., Aase, 1996). This table ends at age 106, thus, for life long contracts, $T_J = 106$ is used as a proxy.

Looking at Table 3 we see that the value of the contracts varies quite a lot, both as a function of when the employee enters into the contract and what kind of contract he has. For the annuity contract with no guarantee, the market value of the contract is highest for the 52 year old, even though 2,200 currency units, say USD, less are debited on his account than for the 30 year old. There are three forces at work here: Firstly, a young employee has more premiums debited on his account, resulting in a higher market value than for an older employee. Secondly, the pension payments for the younger

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Table 3. The time zero market value of the different pension plans, given the age at which the employee enters into the plan; $r = 0.08$, $\sigma = 0.2$, $\gamma = 0.75$, $\bar{\lambda} = 100$, and $\sum_{j=1}^{J} \bar{\lambda}_j = 100$. $T_J = 80$ for π_0, π_0^1, π_0^2, π_0^3, and π_0^4. $\pi_0^3(\infty)$ and $\pi_0^4(\infty)$ are life long pension plans

Age	π_0	π_0^1	π_0^2	π_0^3	π_0^4	$\pi_0^3(\infty)$	$\pi_0^4(\infty)$
30	391.31	431.51	1,659.90	286.94	1,419.60	103.52	590.03
31	398.68	439.83	1,616.62	292.53	1,382.58	105.54	574.65
32	406.03	448.12	1,574.31	298.11	1,346.39	107.56	559.61
33	413.37	456.38	1,532.94	303.67	1,311.01	109.57	544.90
34	420.66	464.58	1,492.46	309.21	1,276.40	111.57	530.51
35	427.90	472.70	1,452.85	314.70	1,242.52	113.56	516.43
36	435.05	480.72	1,414.06	320.13	1,209.35	115.53	502.65
37	442.10	488.60	1,376.06	325.48	1,176.85	117.46	489.14
38	449.02	496.32	1,338.80	330.72	1,144.98	119.36	475.89
39	455.78	503.84	1,302.25	335.85	1,113.72	121.22	462.90
40	462.35	511.12	1,266.36	340.83	1,083.03	123.03	450.14
41	468.70	518.12	1,231.08	345.64	1,052.86	124.77	437.60
42	474.79	524.81	1,196.38	350.25	1,023.18	126.45	425.27
43	480.58	531.12	1,162.20	354.62	993.95	128.04	413.12
44	486.02	537.00	1,128.50	358.73	965.13	129.53	401.14
45	491.05	542.40	1,095.21	362.52	936.66	130.91	389.31
46	495.63	547.24	1,062.29	365.96	908.50	132.16	377.60
47	499.68	551.45	1,029.66	368.99	880.59	133.28	366.00
48	503.13	554.95	997.26	371.57	852.89	134.22	354.49
49	505.92	557.65	965.02	373.63	825.31	134.98	343.03
50	507.93	559.43	932.86	375.11	797.81	135.53	331.60
51	509.08	560.19	900.68	375.92	770.29	135.84	320.16
52	509.24	559.79	868.40	375.98	742.68	135.89	308.68
53	508.29	558.09	835.89	375.19	714.88	135.62	297.13
54	506.08	554.92	803.04	373.44	686.79	135.01	285.45
55	502.43	550.09	769.71	370.61	658.28	134.01	273.60
56	497.15	543.38	735.74	366.54	629.23	132.57	261.53
57	490.01	534.55	700.95	361.07	599.47	130.62	249.16
58	480.74	523.31	665.12	354.01	568.83	128.09	236.42
59	469.04	509.32	628.02	345.14	537.10	124.91	223.24
60	454.54	492.21	589.36	334.17	504.04	120.97	209.49
61	436.81	471.52	548.81	320.82	469.36	116.17	195.08
62	415.35	446.71	505.99	304.70	432.74	110.37	179.86
63	389.53	417.18	460.42	285.39	393.77	103.41	163.66
64	358.64	382.18	411.56	262.36	351.98	95.10	146.30
65	321.77	340.84	358.75	234.98	306.81	85.21	127.52
66	277.85	292.14	301.17	202.51	257.57	73.47	107.05
67	225.56	234.85	237.85	164.02	203.41	59.54	84.55
68	163.26	167.59	167.59	118.40	143.33	43.00	59.57
69	88.92	88.92	88.92	64.28	76.05	23.36	31.61

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employee are farther out in time, and the time value of money effect contributes more negatively on the market value of the younger employee's pension plan than for the elderly. Thirdly, a young employee has a lower probability of surviving until retiral than an older employee, further reducing the market value of the pension contract. Comparing the high interest rate scenario in Table 3 to the low interest rate scenario in Table 5, we see how the reduced effect of the time value of money increases the market value of all the pension plans, and the plans for the younger employees are now relatively much more expensive.

Note that embedding a maturity guarantee in the annuity contract has less impact on the market value in the high than in the low interest rate scenario. This is related to the lower probability for the guarantee becoming binding in the high than in the low interest rate scenario. Protection in form of a rate of return guarantee is more important for a high risk portfolio than for a low risk, and, as we can see from Tables 3 to 6, the market value of the plans with guarantees embedded are highest for the high risk contracts.

Embedding maturity guarantees in the pension plans in order to reduce the risk for the employees comes at a relatively low cost. The multi-period or annual guarantees, on the other hand, are extremely expensive, and should not be used at all in such pension plans as analysed here; they make the pension plans way too expensive.

For the alternative contract, we can see clearly the effect that mortality risk has on the market value of the contracts.[9] For the life long contracts, not only is a substantial amount of the payment to the employee due at later points in time, but perhaps most of the payments are due after the employee is dead, and will therefore never be made. This is best illustrated in Tables 5 and 6, where the interest rate is low. Of course, in practise, one would like to use a more sensible distribution of the premium than the equally weighted distribution which is used here.

As we can see from Tables 7 and 8, embedding a maturity guarantee in the pension plan adds little to the size of the pension payments for the medium scenario (for this scenario, the high risk portfolio under the high interest rate scenario gives an average return very close to the expected yearly return of 11%). This is also somewhat expected from the relatively low increase in price by including the maturity guarantee in the pension plan. As we see, embedding the multi-period guarantee gives significantly higher pensions for the young employees, corresponding nicely with the high cost of providing it. We also notice that taking on the extra risk imposed by the high-risk portfolio has been profitable for most of the employees. Looking at

[9] At each time of premium payment, for the 100 which is deposited on the pension account, each of the two alternative contracts with $T_J = 80$ has ten embedded with the respective guarantees which matures in each of the years from 71 to 80. For the life-long contracts where the approximation $T_J = 106$ is used, each deposit is 'distributed' on $106 - 70 = 36$ years.

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Table 4. The time zero market value of the different pension plans, given the age at which the employee enters into the plan; $r = 0.08$, $\sigma = 0.1$, $\gamma = 0.75$, $\bar{\lambda} = 100$, and $\sum_{j=1}^{J} \bar{\lambda}_j = 100$. $T_J = 80$ for π_0, π_0^1, π_0^2, π_0^3, and π_0^4. $\pi_0^3(\infty)$ and $\pi_0^4(\infty)$ are life long pension plans

Age	π_0	π_0^1	π_0^2	π_0^3	π_0^4	$\pi_0^3(\infty)$	$\pi_0^4(\infty)$
30	421.68	427.46	753.38	287.99	563.21	104.66	216.00
31	428.62	434.68	752.03	292.84	562.20	106.42	215.61
32	435.53	441.87	750.52	297.67	561.07	108.17	215.18
33	442.39	449.01	748.82	302.47	559.81	109.92	214.69
34	449.17	456.10	746.94	307.23	558.40	111.64	214.15
35	455.87	463.10	744.84	311.93	556.83	113.35	213.55
36	462.46	470.00	742.51	316.57	555.09	115.03	212.88
37	468.91	476.77	739.93	321.12	553.16	116.68	212.14
38	475.21	483.39	737.09	325.57	551.03	118.29	211.33
39	481.32	489.83	733.95	329.90	548.69	119.86	210.43
40	487.21	496.05	730.49	334.08	546.10	121.38	209.44
41	492.85	502.02	726.69	338.10	543.26	122.83	208.35
42	498.20	507.70	722.51	341.92	540.14	124.22	207.15
43	503.21	513.05	717.92	345.52	536.70	125.52	205.83
44	507.85	518.02	712.88	348.87	532.93	126.74	204.39
45	512.06	522.55	707.34	351.92	528.80	127.84	202.80
46	515.79	526.59	701.27	354.65	524.26	128.83	201.06
47	518.96	530.06	694.60	357.00	519.27	129.68	199.15
48	521.52	532.90	687.29	358.93	513.81	130.38	197.05
49	523.38	535.03	679.26	360.38	507.80	130.90	194.75
50	524.45	536.34	670.44	361.29	501.21	131.23	192.22
51	524.63	536.73	660.74	361.59	493.96	131.33	189.44
52	523.80	536.09	650.08	361.20	485.99	131.19	186.38
53	521.85	534.27	638.33	360.03	477.21	130.76	183.01
54	518.63	531.13	625.38	357.97	467.53	130.01	179.30
55	513.95	526.49	611.09	354.91	456.84	128.90	175.20
56	507.64	520.14	595.28	350.72	445.02	127.37	170.67
57	499.47	511.85	577.76	345.23	431.92	125.38	165.65
58	489.17	501.34	558.31	338.26	417.38	122.84	160.07
59	476.45	488.30	536.66	329.60	401.20	119.70	153.86
60	460.94	472.36	512.51	319.00	383.14	115.85	146.94
61	442.23	453.08	485.49	306.17	362.94	111.19	139.19
62	419.81	429.94	455.15	290.75	340.26	105.59	130.49
63	393.09	402.32	420.98	272.33	314.72	98.91	120.70
64	361.34	369.48	382.35	250.40	285.84	90.94	109.62
65	323.69	330.53	338.50	224.35	253.06	81.49	97.05
66	279.09	284.42	288.51	193.46	215.68	70.27	82.72
67	226.22	229.85	231.23	156.82	172.86	56.97	66.29
68	163.50	165.28	165.28	113.32	123.56	41.17	47.39
69	88.92	88.92	88.92	61.61	66.48	22.39	25.49

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Table 5. The time zero market value of the different pension plans, given the age at which the employee enters into the plan; $r = 0.04$, $\sigma = 0.2$, $\gamma = 0.75$, $\bar{\lambda} = 100$, and $\sum_{j=1}^{J} \bar{\lambda}_j = 100$. $T_J = 80$ for π_0, π_0^1, π_0^2, π_0^3, and π_0^4. $\pi_0^3(\infty)$ and $\pi_0^4(\infty)$ are life long pension plans

Age	π_0	π_0^1	π_0^2	π_0^3	π_0^4	$\pi_0^3(\infty)$	$\pi_0^4(\infty)$
30	922.82	1,433.83	5,213.41	1,073.29	5,083.66	411.16	2,313.84
31	923.80	1,421.57	4,948.01	1,064.41	4,824.86	407.79	2,196.05
32	924.31	1,408.72	4,695.35	1,055.10	4,578.49	404.25	2,083.91
33	924.32	1,395.26	4,454.77	1,045.34	4,343.91	400.54	1,977.14
34	923.80	1,381.18	4,225.68	1,035.12	4,120.52	396.66	1,875.46
35	922.72	1,366.44	4,007.48	1,024.42	3,907.75	392.59	1,778.62
36	921.06	1,351.02	3,799.62	1,013.23	3,705.06	388.33	1,686.36
37	918.78	1,334.91	3,601.56	1,001.52	3,511.93	383.88	1,598.46
38	915.86	1,318.07	3,412.81	989.28	3,327.87	379.23	1,514.69
39	912.26	1,300.48	3,232.86	976.49	3,152.41	374.36	1,434.83
40	907.94	1,282.11	3,061.28	963.13	2,985.09	369.28	1,358.67
41	902.88	1,262.93	2,897.61	949.17	2,825.49	363.96	1,286.03
42	897.03	1,242.92	2,741.43	934.59	2,673.20	358.41	1,216.71
43	890.35	1,222.03	2,592.34	919.37	2,527.82	352.62	1,150.54
44	882.79	1,200.24	2,449.96	903.47	2,388.98	346.57	1,087.35
45	874.32	1,177.50	2,313.90	886.88	2,256.31	340.25	1,026.97
46	864.88	1,153.78	2,183.82	869.56	2,129.47	333.65	969.23
47	854.41	1,129.03	2,059.37	851.48	2,008.11	326.76	914.00
48	842.86	1,103.20	1,940.21	832.59	1,891.92	319.56	861.11
49	830.17	1,076.25	1,826.03	812.86	1,780.59	312.04	810.44
50	816.25	1,048.09	1,716.51	792.24	1,673.79	304.18	761.83
51	801.03	1,018.68	1,611.35	770.69	1,571.25	295.96	715.16
52	784.43	987.94	1,510.24	748.13	1,472.65	287.35	670.28
53	766.34	955.77	1,412.89	724.51	1,377.72	278.34	627.08
54	746.66	922.09	1,319.00	699.74	1,286.17	268.88	585.40
55	725.25	886.79	1,228.28	673.76	1,197.71	258.96	545.14
56	701.98	849.73	1,140.43	646.44	1,112.05	248.52	506.15
57	676.67	810.77	1,055.15	617.69	1,028.89	237.53	468.30
58	649.14	769.74	972.13	587.36	947.94	225.93	431.46
59	619.16	726.45	891.04	555.31	868.87	213.67	395.47
60	586.46	680.66	811.54	521.34	791.34	200.66	360.18
61	550.75	632.11	733.27	485.24	715.02	186.83	325.44
62	511.65	580.48	655.82	446.75	639.50	172.08	291.07
63	468.72	525.39	578.77	405.57	564.37	156.28	256.87
64	421.45	466.42	501.63	361.33	489.15	139.30	222.64
65	369.21	403.04	423.87	313.60	413.32	120.96	188.12
66	311.24	334.66	344.86	261.86	336.28	101.06	153.06
67	246.60	260.60	263.89	205.47	257.32	79.34	117.12
68	174.17	180.12	180.12	143.67	175.63	55.52	79.94
69	92.55	92.55	92.55	75.54	90.25	29.22	41.08

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 6. The time zero market value of the different pension plans, given the age at which the employee enters into the plan; $r = 0.04$, $\sigma = 0.1$, $\gamma = 0.75$, $\bar{\lambda} = 100$, and $\sum_{j=1}^{J} \bar{\lambda}_j = 100$. $T_J = 80$ for π_0, π_0^1, π_0^2, π_0^3, and π_0^4. $\pi_0^3(\infty)$ and $\pi_0^4(\infty)$ are life long pension plans

Age	π_0	π_0^1	π_0^2	π_0^3	π_0^4	$\pi_0^3(\infty)$	$\pi_0^4(\infty)$
30	984.80	1,333.06	2,407.63	995.72	2,035.56	381.45	839.85
31	983.98	1,321.92	2,340.93	987.48	1,979.18	378.30	816.58
32	982.66	1,310.27	2,275.37	978.86	1,923.75	375.00	793.71
33	980.83	1,298.07	2,210.90	969.85	1,869.24	371.55	771.22
34	978.45	1,285.33	2,147.49	960.42	1,815.63	367.94	749.10
35	975.50	1,272.00	2,085.08	950.56	1,762.86	364.17	727.33
36	971.95	1,258.08	2,023.63	940.27	1,710.91	360.24	705.90
37	967.80	1,243.54	1,963.11	929.52	1,659.74	356.12	684.79
38	962.95	1,228.36	1,903.48	918.29	1,609.32	351.83	663.99
39	957.43	1,212.50	1,844.68	906.56	1,559.61	347.34	643.47
40	951.20	1,195.95	1,786.68	894.32	1,510.58	342.66	623.24
41	944.21	1,178.67	1,729.43	881.55	1,462.18	337.77	603.27
42	936.44	1,160.65	1,672.89	868.22	1,414.37	332.67	583.55
43	927.83	1,141.83	1,617.01	854.30	1,367.13	327.35	564.06
44	918.36	1,122.20	1,561.75	839.78	1,320.41	321.80	544.78
45	907.97	1,101.71	1,507.05	824.63	1,274.16	316.00	525.70
46	896.62	1,080.32	1,452.86	808.81	1,228.35	309.95	506.80
47	884.26	1,058.00	1,399.13	792.29	1,182.92	303.64	488.06
48	870.83	1,034.68	1,345.79	775.04	1,137.82	297.04	469.45
49	856.27	1,010.32	1,292.79	757.02	1,093.01	290.15	450.96
50	840.51	984.87	1,240.06	738.18	1,048.43	282.94	432.57
51	823.48	958.24	1,187.51	718.47	1,004.00	275.40	414.24
52	805.09	930.37	1,135.08	697.84	959.67	267.51	395.95
53	785.24	901.17	1,082.66	676.22	915.36	259.23	377.66
54	763.83	870.54	1,030.17	653.53	870.97	250.55	359.35
55	740.74	838.38	977.48	629.70	826.43	241.44	340.97
56	715.83	804.56	924.47	604.63	781.61	231.84	322.48
57	688.94	768.92	871.00	578.20	736.40	221.73	303.83
58	659.87	731.30	816.90	550.28	690.66	211.04	284.96
59	628.41	691.50	761.98	520.73	644.23	199.73	265.80
60	594.31	649.28	706.02	489.36	596.91	187.72	246.28
61	557.27	604.36	648.75	455.96	548.50	174.93	226.30
62	516.92	556.41	589.89	420.27	498.73	161.26	205.77
63	472.84	505.06	529.08	382.00	447.32	146.60	184.56
64	424.52	449.84	465.88	340.79	393.89	130.81	162.51
65	371.35	390.19	399.80	296.21	338.02	113.72	139.46
66	312.58	325.47	330.24	247.73	279.21	95.13	115.20
67	247.31	254.91	256.47	194.72	216.83	74.79	89.46
68	174.42	177.60	177.60	136.42	150.15	52.42	61.95
69	92.55	92.55	92.55	71.89	78.25	27.63	32.29

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 7. Yearly pension payments for the annuity contract (given the age at which the employee entered into the pension plan) when the investment portfolio of the pension fund has a medium performance; $T_J = 80$, $r = 0.08$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

	$\sigma = 0.2$			$\sigma = 0.1$		
Age	a	a_1	a_2	a	a_1	a_2
30	6,303.90	6,303.90	15,671.9	4,876.64	4,876.64	6,624.52
31	5,472.46	5,472.68	13,630.0	4,382.16	4,382.16	5,959.08
32	4,686.88	4,694.19	11,698.2	3,910.44	3,911.13	5,323.87
33	4,444.70	4,452.40	11,121.4	3,671.03	3,672.04	5,004.54
34	4,844.98	4,844.98	10,666.7	3,694.06	3,694.06	4,789.67
35	3,642.65	3,655.31	8,037.22	3,086.70	3,086.79	4,007.00
36	3,610.86	3,619.22	7,703.44	2,960.43	2,960.43	3,831.25
37	2,987.38	3,013.38	6,391.58	2,593.29	2,595.15	3,361.15
38	2,881.74	2,904.56	6,122.33	2,452.05	2,454.27	3,183.56
39	2,497.69	2,539.49	5,325.45	2,197.01	2,204.38	2,857.66
40	2,445.55	2,479.69	5,097.99	2,091.39	2,098.51	2,725.88
41	2,374.24	2,403.31	4,879.46	1,981.68	1,988.99	2,588.51
42	2,545.11	2,556.55	4,669.49	1,972.31	1,975.43	2,468.38
43	2,573.91	2,580.53	4,467.76	1,906.03	1,908.09	2,352.95
44	2,304.88	2,314.59	4,013.66	1,732.61	1,736.32	2,143.04
45	2,771.15	2,771.15	3,837.65	1,823.87	1,823.87	2,040.37
46	2,632.42	2,632.42	3,652.66	1,706.34	1,706.34	1,911.07
47	1,899.44	1,901.47	2,641.06	1,390.54	1,391.32	1,559.29
48	2,008.57	2,008.57	2,518.87	1,370.20	1,370.20	1,479.51
49	1,709.87	1,709.87	2,148.51	1,210.82	1,210.82	1,308.80
50	1,562.16	1,562.16	1,967.51	1,107.30	1,107.30	1,198.37
51	1,235.83	1,235.83	1,560.54	941.38	941.38	1,020.18
52	1,110.14	1,110.14	1,406.48	851.55	851.55	924.33
53	836.01	836.01	1,063.14	704.35	704.35	765.95
54	707.45	707.94	904.21	616.37	616.37	671.80
55	601.76	605.34	773.85	539.75	540.76	589.88
56	629.47	629.47	724.87	523.04	523.04	548.11
57	556.47	556.47	643.49	465.04	465.04	488.19
58	450.13	450.13	523.06	394.44	394.44	414.93
59	318.11	318.11	371.97	311.61	311.61	328.63
60	251.19	253.28	296.48	259.06	259.06	274.16
61	215.64	218.34	257.78	223.50	223.50	237.57
62	178.46	184.01	216.78	188.17	188.90	201.15
63	156.62	159.91	189.64	161.91	162.65	174.36
64	118.38	129.27	146.87	127.98	130.77	139.16
65	101.54	107.07	122.47	106.32	107.93	115.06
66	95.75	95.75	99.03	90.73	90.73	91.91
67	63.68	63.68	66.41	62.98	62.98	64.02
68	37.29	39.59	39.59	38.67	39.59	39.59
69	19.40	19.40	19.40	19.40	19.40	19.40

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 8. Yearly pension payments for the annuity contract (given the age at which the employee entered into the pension plan) when the investment portfolio of the pension fund has a medium performance; $T_J = 80$, $r = 0.04$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

	$\sigma = 0.2$			$\sigma = 0.1$		
Age	a	a_1	a_2	a	a_1	a_2
30	2,246.06	2,286.96	7,499.05	1,798.26	1,826.35	3,215.66
31	2,000.86	2,059.81	6,714.39	1,656.71	1,694.46	2,974.42
32	1,757.75	1,838.03	5,932.40	1,515.21	1,564.34	2,732.27
33	1,708.92	1,779.84	5,684.21	1,457.31	1,500.47	2,609.56
34	1,909.69	1,938.18	5,445.76	1,502.25	1,520.38	2,491.67
35	1,472.85	1,548.70	4,223.39	1,286.28	1,331.70	2,142.70
36	1,495.87	1,552.78	4,042.21	1,263.16	1,297.20	2,043.11
37	1,268.27	1,348.97	3,450.82	1,133.00	1,182.27	1,841.68
38	1,252.70	1,320.49	3,299.94	1,096.33	1,136.04	1,753.90
39	1,111.83	1,192.95	2,952.67	1,005.20	1,053.76	1,617.16
40	1,114.08	1,178.48	2,821.32	978.76	1,016.00	1,538.18
41	1,107.10	1,159.92	2,695.12	948.64	978.74	1,462.29
42	1,214.94	1,239.86	2,573.87	965.74	979.89	1,389.38
43	1,258.81	1,271.62	2,457.38	954.96	962.30	1,319.33
44	1,155.32	1,170.85	2,269.70	888.37	897.00	1,233.18
45	1,423.24	1,423.67	2,165.13	956.78	957.01	1,169.26
46	1,387.01	1,387.01	2,064.66	916.51	916.51	1,107.84
47	1,026.80	1,030.70	1,534.39	764.71	766.84	927.23
48	1,112.57	1,112.57	1,458.66	770.78	770.78	875.30
49	971.12	971.12	1,277.67	696.97	696.97	793.52
50	909.31	909.31	1,201.22	651.97	651.97	744.44
51	737.21	737.21	978.17	566.89	566.89	649.34
52	678.04	678.04	904.64	524.10	524.10	602.57
53	522.72	522.91	701.69	442.99	443.15	511.43
54	452.16	454.67	611.93	395.74	397.13	459.21
55	392.90	397.64	536.88	353.61	356.19	412.74
56	419.56	419.56	500.26	349.48	349.48	380.99
57	379.12	379.12	454.97	317.10	317.10	347.08
58	313.43	313.43	378.94	274.43	274.43	301.78
59	226.19	232.99	276.04	221.08	225.26	244.50
60	181.96	196.82	225.15	187.17	195.90	208.56
61	158.90	173.42	200.31	164.28	172.65	184.81
62	133.71	150.18	172.52	140.68	149.93	160.18
63	119.23	129.69	150.18	123.05	128.97	138.33
64	91.58	107.72	119.10	98.87	107.72	113.00
65	79.68	87.93	98.86	83.41	87.93	93.00
66	76.24	76.24	79.41	72.29	72.29	73.78
67	51.60	52.44	54.32	51.02	51.49	52.39
68	30.71	33.08	33.08	31.83	33.08	33.08
69	16.21	16.21	16.21	16.21	16.21	16.21

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

the contract with no guarantee in Table 7, we see that the person aged 34 on entry into the pension plan has a higher pension than both the 33 and the 32 year old. This has to do with the high return experienced in the last year before retirement (this is when the 30 year old is aged 65), while the younger colleagues experience a poor return after the older member has retired.

The pensions under the low interest rate scenario in Table 8 are lower than under the high interest rate scenario in Table 7 (this applies for all market conditions). There are two obvious reasons for this. The first effect is that lower interest rates reduce the return on the pension fund, resulting in a lower balance on the pension account at the time of retirement. The second effect comes from the calculation of the annuity.

Moving on to the bad scenario (with high interest rates) in Tables 9 and 10 we see that there is a dramatic reduction in the pension payments. The pensions with a multi-period guarantee embedded are, for some ages, just above 43% of the pensions in the contracts with no guarantee when the market performs more on average. Comparing the pensions with no guarantees included, the pensions under the bad scenario are as low as only 3.5% of the pensions under the good scenario. Although this is based on a portfolio with perhaps unrealistically high risk, it still shows how extremely risky these pension plans can be, and, thus, what a relatively cheap insurance a maturity guarantee can be for avoiding very low pension payments.

Tables 13 and 14 illustrate the effect on the pensions by letting them be life long instead of stopping at the age of 80. The relatively low reduction in the yearly pension payments can be explained by the fact that, when the annuity is calculated at the age of 70, the employee is only expected to receive a few pension payments more than what he gets from the contract in Table 7.

Let us end these examples by showing the pension payments for the alternative contract, assuming the medium performance for the investment portfolio, a high-risk portfolio, and a high interest rate scenario. The employee is assumed to be 30 years old on entry into the pension plan. The returns after retirement are reported in Table 15, while the pension payments are reported in Table 16. As we see, the pension payments are now subject to risk, and can vary quite a lot.

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 9. Yearly pension payments for the annuity contract (given the age at which the employee entered into the pension plan) when the investment portfolio of the pension fund has a weak performance; $T_J = 80$, $r = 0.08$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

	$\sigma = 0.2$			$\sigma = 0.1$		
Age	a	a_1	a_2	a	a_1	a_2
30	913.83	1,879.82	3,599.88	1,734.73	1,909.97	2,976.20
31	829.86	1,786.78	3,322.01	1,597.42	1,800.36	2,753.56
32	864.68	1,698.08	3,173.11	1,575.85	1,730.70	2,626.95
33	817.60	1,612.86	3,030.05	1,479.96	1,637.85	2,479.41
34	776.27	1,530.98	2,892.60	1,392.56	1,551.02	2,345.47
35	796.15	1,452.44	2,760.54	1,361.60	1,488.76	2,234.86
36	867.84	1,391.02	2,633.66	1,371.63	1,459.64	2,128.59
37	779.97	1,310.81	2,403.29	1,253.49	1,354.88	1,955.18
38	758.79	1,245.58	2,290.42	1,191.74	1,286.66	1,859.88
39	779.11	1,193.60	2,181.97	1,163.48	1,240.88	1,768.31
40	622.74	1,106.30	1,772.66	1,001.45	1,119.31	1,530.87
41	582.05	1,044.46	1,684.52	932.56	1,052.83	1,435.30
42	495.43	981.54	1,466.17	828.43	982.60	1,284.53
43	463.29	924.42	1,390.04	771.65	925.03	1,206.67
44	486.16	870.28	1,316.90	761.13	875.27	1,140.72
45	432.47	816.80	1,202.18	690.27	818.25	1,043.53
46	429.30	766.13	1,136.40	661.23	769.95	983.97
47	445.45	718.02	1,073.21	646.87	728.45	926.75
48	422.05	672.17	1,012.49	603.80	681.55	871.77
49	448.04	639.42	954.15	596.00	651.64	818.95
50	384.26	588.09	837.97	527.82	594.70	731.90
51	306.95	541.09	688.63	450.84	542.55	631.82
52	288.91	501.23	642.99	418.11	503.08	588.41
53	292.84	464.43	599.14	401.82	467.86	546.70
54	297.88	434.87	557.01	385.90	436.65	506.62
55	260.83	396.83	503.54	342.93	397.50	455.89
56	244.26	366.29	465.16	314.62	364.40	419.37
57	200.56	328.29	397.57	269.55	327.09	365.20
58	151.53	292.86	316.32	221.19	292.86	305.77
59	138.69	262.73	285.28	199.88	262.73	275.14
60	131.79	233.79	255.45	183.41	233.79	245.71
61	131.61	205.99	226.80	171.58	205.99	217.44
62	123.62	179.27	199.27	154.43	179.27	190.27
63	103.79	153.60	172.82	130.28	153.60	164.17
64	95.51	128.94	147.40	114.05	128.94	139.20
65	94.79	109.42	122.98	102.12	107.54	115.00
66	60.04	82.48	82.48	71.37	82.48	82.48
67	53.05	60.61	60.61	57.41	60.61	60.61
68	38.28	39.59	39.59	39.19	39.59	39.59
69	19.40	19.40	19.40	19.40	19.40	19.40

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 10. Yearly pension payments for the annuity contract (given the age at which the employee entered into the pension plan) when the investment portfolio of the pension fund has a weak performance; $T_J = 80$, $r = 0.04$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

	$\sigma = 0.2$			$\sigma = 0.1$		
Age	a	a_1	a_2	a	a_1	a_2
30	446.90	1,569.94	2,352.47	753.66	1,569.88	1,894.55
31	411.77	1,492.73	2,233.62	707.67	1,492.73	1,802.50
32	434.88	1,418.63	2,130.47	711.54	1,418.63	1,716.25
33	417.30	1,347.43	2,031.36	681.30	1,347.43	1,633.39
34	401.95	1,279.03	1,936.14	653.40	1,279.03	1,553.77
35	418.12	1,213.30	1,844.65	651.11	1,213.30	1,477.27
36	462.72	1,150.40	1,756.75	668.59	1,150.26	1,403.78
37	422.98	1,089.49	1,648.80	623.16	1,089.49	1,325.42
38	418.17	1,031.88	1,568.58	603.97	1,031.40	1,257.88
39	436.47	984.45	1,491.51	601.09	979.43	1,192.99
40	354.98	921.79	1,246.14	527.56	921.54	1,061.44
41	336.77	870.93	1,181.70	500.37	870.23	1,004.25
42	290.86	820.01	1,057.50	452.62	820.01	923.67
43	275.44	772.29	1,000.46	428.92	772.29	871.88
44	292.57	726.54	945.66	430.30	726.47	822.12
45	263.86	682.38	887.40	397.10	682.38	772.84
46	265.22	640.05	837.03	386.85	640.05	726.97
47	278.85	599.38	788.64	384.93	599.38	682.89
48	268.10	560.31	742.15	365.62	560.31	640.54
49	288.83	527.40	697.47	367.20	524.88	599.86
50	251.84	489.03	629.33	331.08	487.77	550.54
51	204.26	452.04	531.50	287.73	452.04	488.26
52	194.70	418.75	495.09	271.14	418.75	453.54
53	199.85	386.75	460.11	264.74	386.75	420.19
54	206.15	357.33	426.50	258.40	356.57	388.14
55	183.32	327.84	394.20	233.55	327.08	357.35
56	174.23	302.42	363.17	217.80	300.07	327.76
57	145.23	272.79	318.88	189.67	271.77	293.14
58	111.21	244.66	260.88	158.05	244.66	252.32
59	102.76	219.50	235.08	144.78	219.50	226.86
60	98.53	195.32	210.29	134.64	195.32	202.39
61	99.33	172.09	186.47	127.67	172.09	178.88
62	94.37	149.77	163.59	116.56	149.77	156.30
63	80.21	128.32	141.60	99.77	128.32	134.60
64	74.64	107.72	120.48	88.56	107.72	113.75
65	75.00	90.81	100.19	80.45	89.29	93.72
66	48.24	68.91	68.91	57.11	68.91	68.91
67	43.09	50.63	50.63	46.56	50.63	50.63
68	31.51	33.08	33.08	32.25	33.08	33.08
69	16.21	16.21	16.21	16.21	16.21	16.21

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 11. Yearly pension payments for the annuity contract (given the age at which the employee entered into the pension plan) when the investment portfolio of the pension fund has a strong performance; $T_J = 80$, $r = 0.08$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

	$\sigma = 0.2$			$\sigma = 0.1$		
Age	a	a_1	a_2	a	a_1	a_2
30	22,062.8	22,076.9	65,748.6	8,646.58	8,650.24	12,511.1
31	23,082.0	23,087.5	63,151.9	8,535.10	8,536.78	12,001.9
32	24,581.3	24,582.7	60,657.0	8,499.86	8,499.96	11,512.7
33	21,425.2	21,428.2	52,893.8	7,657.61	7,658.92	10,378.1
34	24,173.8	24,173.8	50,801.2	7,847.67	7,847.67	9,952.52
35	19,376.7	19,376.7	40,737.3	6,778.77	6,778.77	8,601.42
36	15,852.7	15,855.9	33,345.9	5,913.94	5,914.41	7,508.63
37	16,386.1	16,386.2	32,019.8	5,797.56	5,797.56	7,195.58
38	13,420.1	13,424.9	26,239.1	5,058.59	5,060.27	6,282.50
39	14,064.4	14,065.8	25,191.6	4,991.15	4,991.59	6,017.52
40	14,493.7	14,493.7	24,185.2	4,882.92	4,882.92	5,762.93
41	11,179.1	11,179.1	18,664.2	4,132.35	4,132.35	4,880.07
42	10,228.6	10,229.0	17,089.2	3,806.96	3,806.96	4,499.05
43	10,000.8	10,000.8	16,400.5	3,624.31	3,624.31	4,286.57
44	7,456.61	7,456.61	12,237.4	3,012.39	3,012.39	3,565.79
45	6,000.00	6,000.00	9,856.97	2,599.07	2,599.07	3,079.64
46	4,484.19	4,485.46	7,376.10	2,159.62	2,159.99	2,561.93
47	4,577.79	4,577.79	7,068.24	2,095.21	2,095.21	2,442.84
48	4,040.94	4,040.94	6,248.67	1,889.58	1,889.58	2,206.02
49	3,204.84	3,204.84	4,964.23	1,614.11	1,614.11	1,887.22
50	2,584.17	2,585.57	4,011.46	1,388.67	1,388.82	1,626.50
51	2,346.83	2,349.64	3,652.84	1,266.34	1,267.23	1,486.29
52	2,516.34	2,516.34	3,490.97	1,253.53	1,253.53	1,409.37
53	2,190.81	2,190.81	3,045.95	1,117.64	1,117.64	1,258.77
54	1,909.73	1,909.73	2,661.81	995.76	995.76	1,123.72
55	1,382.78	1,382.78	1,932.93	807.35	807.35	913.16
56	807.24	808.02	1,132.98	586.18	586.30	664.90
57	787.30	787.30	1,069.91	547.79	547.79	620.19
58	578.98	578.98	792.07	443.58	443.58	504.36
59	427.08	428.61	589.71	358.27	358.27	409.61
60	391.84	393.30	547.94	321.20	321.38	369.87
61	292.53	298.30	415.14	258.60	260.93	300.29
62	344.95	344.95	380.23	259.89	259.89	269.88
63	248.97	248.97	275.95	203.33	203.33	211.78
64	188.48	188.48	210.63	161.13	161.13	168.53
65	127.26	127.26	143.93	118.88	118.88	125.08
66	84.29	87.95	97.32	85.022	86.53	90.33
67	74.87	74.87	74.87	68.15	68.15	68.15
68	44.99	44.99	44.99	42.44	42.44	42.44
69	19.40	19.40	19.40	19.40	19.40	19.40

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 12. Yearly pension payments for the annuity contract (given the age at which the employee entered into the pension plan) when the investment portfolio of the pension fund has a strong performance; $T_J = 80$, $r = 0.04$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

	$\sigma = 0.2$			$\sigma = 0.1$		
Age	a	a_1	a_2	a	a_1	a_2
30	6,941.71	6,985.75	28,274.4	2,942.79	2,968.43	5,663.24
31	7,472.65	7,489.41	27,150.2	2,983.53	2,993.56	5,425.61
32	8,189.52	8,192.46	26,070.1	3,052.00	3,053.77	5,197.30
33	7,346.65	7,352.90	23,418.7	2,824.72	2,828.43	4,820.85
34	8,530.46	8,530.46	22,484.9	2,973.40	2,973.40	4,616.25
35	7,038.17	7,039.43	18,573.4	2,638.72	2,639.48	4,104.65
36	5,925.77	5,933.43	15,660.2	2,364.38	2,368.88	3,686.02
37	6,302.16	6,303.75	15,030.6	2,379.88	2,380.76	3,525.92
38	5,311.21	5,318.62	12,686.2	2,132.34	2,136.51	3,166.21
39	5,726.51	5,728.14	12,173.2	2,159.79	2,160.64	3,026.49
40	6,072.17	6,072.17	11,680.3	2,169.40	2,169.40	2,892.25
41	4,819.70	4,820.26	9,282.97	1,885.20	1,885.53	2,518.08
42	4,536.81	4,537.62	8,752.29	1,782.63	1,783.06	2,386.27
43	4,563.22	4,563.22	8,393.54	1,741.76	1,741.76	2,277.13
44	3,500.28	3,500.28	6,448.85	1,485.85	1,485.85	1,946.85
45	2,896.39	2,896.39	5,347.61	1,315.08	1,315.08	1,727.60
46	2,225.30	2,227.59	4,119.17	1,120.49	1,121.73	1,476.35
47	2,333.98	2,333.98	3,942.08	1,113.99	1,113.99	1,402.89
48	2,117.24	2,117.24	3,586.21	1,029.73	1,029.73	1,300.72
49	1,725.35	1,725.35	2,931.65	901.39	901.39	1,142.40
50	1,428.75	1,432.28	2,437.16	794.29	796.30	1,010.55
51	1,331.88	1,336.19	2,282.70	741.50	743.85	947.60
52	1,465.78	1,465.78	2,177.62	751.33	751.33	894.88
53	1,310.59	1,310.59	1,954.18	686.01	686.01	819.96
54	1,173.06	1,173.06	1,756.32	625.80	625.80	750.95
55	872.01	872.01	1,311.66	519.42	519.42	626.06
56	522.14	523.27	790.38	385.76	386.35	467.51
57	520.99	520.99	743.82	368.04	368.04	433.60
58	392.18	392.18	565.31	304.36	304.36	361.08
59	295.69	298.93	431.85	250.80	252.56	300.16
60	276.81	278.98	399.34	229.13	230.29	272.82
61	210.92	217.09	310.10	187.99	191.33	226.57
62	253.39	253.39	282.37	192.33	192.33	202.11
63	186.92	186.92	209.75	153.45	153.45	161.97
64	144.43	144.43	163.74	123.89	123.89	131.58
65	99.44	100.07	114.42	93.068	93.43	99.72
66	67.03	71.51	79.09	67.69	70.09	73.55
67	60.42	60.42	60.42	55.09	55.09	55.09
68	36.95	36.95	36.95	34.89	34.89	34.89
69	16.21	16.21	16.21	16.21	16.21	16.21

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 13. Yearly pension payments for the annuity contract (given the age at which the employee entered into the pension plan) when the investment portfolio of the pension fund has a medium performance; life long pension ($T_J = 106$), $r = 0.08$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

	$\sigma = 0.2$			$\sigma = 0.1$		
Age	a	a_1	a_2	a	a_1	a_2
30	5,369.85	5,369.85	13,349.8	4,154.06	4,154.06	5,642.96
31	4,661.60	4,661.79	11,610.4	3,732.85	3,732.85	5,076.12
32	3,992.43	3,998.65	9,964.90	3,331.03	3,331.61	4,535.03
33	3,786.12	3,792.69	9,473.51	3,127.09	3,127.95	4,263.01
34	4,127.10	4,127.10	9,086.18	3,146.71	3,146.71	4,079.98
35	3,102.92	3,113.70	6,846.34	2,629.34	2,629.42	3,413.28
36	3,075.84	3,082.96	6,562.02	2,521.78	2,521.78	3,263.57
37	2,544.74	2,566.89	5,444.54	2,209.04	2,210.63	2,863.12
38	2,454.75	2,474.19	5,215.18	2,088.73	2,090.62	2,711.85
39	2,127.61	2,163.21	4,536.37	1,871.48	1,877.75	2,434.24
40	2,083.19	2,112.27	4,342.62	1,781.51	1,787.57	2,321.98
41	2,022.45	2,047.21	4,156.47	1,688.05	1,694.28	2,204.97
42	2,168.00	2,177.75	3,977.61	1,680.07	1,682.73	2,102.64
43	2,192.53	2,198.17	3,805.77	1,623.61	1,625.37	2,004.31
44	1,963.36	1,971.63	3,418.96	1,475.88	1,479.05	1,825.50
45	2,360.55	2,360.55	3,269.02	1,553.63	1,553.63	1,738.05
46	2,242.37	2,242.37	3,111.45	1,453.51	1,453.51	1,627.90
47	1,618.00	1,619.73	2,249.74	1,184.50	1,185.16	1,328.25
48	1,710.96	1,710.96	2,145.64	1,167.18	1,167.18	1,260.29
49	1,456.52	1,456.52	1,830.16	1,031.41	1,031.41	1,114.88
50	1,330.70	1,330.70	1,675.98	943.24	943.24	1,020.81
51	1,052.72	1,052.72	1,329.31	801.89	801.89	869.02
52	945.65	945.65	1,198.08	725.37	725.37	787.37
53	712.14	712.14	905.61	599.98	599.98	652.46
54	602.62	603.05	770.23	525.04	525.04	572.26
55	512.60	515.65	659.19	459.78	460.63	502.47
56	536.20	536.20	617.46	445.54	445.54	466.90
57	474.02	474.02	548.14	396.14	396.14	415.86
58	383.44	383.44	445.56	336.00	336.00	353.45
59	270.97	270.97	316.85	265.44	265.44	279.94
60	213.97	215.75	252.55	220.68	220.68	233.54
61	183.69	185.98	219.58	190.38	190.38	202.37
62	152.02	156.74	184.66	160.29	160.92	171.34
63	133.41	136.22	161.54	137.92	138.55	148.53
64	100.84	110.12	125.10	109.01	111.40	118.54
65	86.49	91.20	104.32	90.57	91.94	98.01
66	81.56	81.56	84.35	77.29	77.29	78.29
67	54.24	54.24	56.57	53.65	53.65	54.54
68	31.77	33.73	33.73	32.94	33.73	33.73
69	16.53	16.53	16.53	16.53	16.53	16.53

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 14. Yearly pension payments for the annuity contract (given the age at which the employee entered into the pension plan) when the investment portfolio of the pension fund has a medium performance; life long pension ($T_J = 106$), $r = 0.04$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

	$\sigma = 0.2$			$\sigma = 0.1$		
Age	a	a_1	a_2	a	a_1	a_2
30	1,787.21	1,819.75	5,967.04	1,430.89	1,453.24	2,558.72
31	1,592.10	1,639.00	5,342.69	1,318.26	1,348.29	2,366.77
32	1,398.65	1,462.53	4,720.45	1,205.66	1,244.75	2,174.08
33	1,359.80	1,416.23	4,522.96	1,159.59	1,193.93	2,076.45
34	1,519.55	1,542.22	4,333.23	1,195.35	1,209.78	1,982.64
35	1,171.96	1,232.31	3,360.58	1,023.50	1,059.64	1,704.96
36	1,190.27	1,235.56	3,216.42	1,005.10	1,032.19	1,625.72
37	1,009.17	1,073.38	2,745.84	901.53	940.740	1,465.44
38	996.78	1,050.72	2,625.78	872.36	903.954	1,395.59
39	884.69	949.24	2,349.46	799.84	838.487	1,286.78
40	886.48	937.73	2,244.94	778.81	808.436	1,223.94
41	880.93	922.96	2,144.53	754.84	778.792	1,163.56
42	966.74	986.56	2,048.05	768.45	779.701	1,105.54
43	1,001.64	1,011.83	1,955.35	759.87	765.71	1,049.80
44	919.30	931.65	1,806.01	706.88	713.75	981.25
45	1,132.48	1,132.82	1,722.81	761.31	761.50	930.39
46	1,103.65	1,103.65	1,642.86	729.27	729.27	881.52
47	817.03	820.14	1,220.93	608.49	610.18	737.82
48	885.28	885.28	1,160.66	613.31	613.31	696.49
49	772.73	772.73	1,016.65	554.59	554.59	631.41
50	723.54	723.54	955.82	518.78	518.78	592.35
51	586.60	586.60	778.33	451.08	451.08	516.68
52	539.52	539.52	719.83	417.03	417.03	479.47
53	415.93	416.08	558.34	352.49	352.62	406.95
54	359.79	361.78	486.92	314.90	316.00	365.40
55	312.63	316.40	427.20	281.37	283.42	328.42
56	333.85	333.85	398.06	278.08	278.08	303.15
57	301.67	301.67	362.02	252.32	252.32	276.18
58	249.40	249.40	301.52	218.37	218.37	240.13
59	179.98	185.39	219.65	175.92	179.24	194.55
60	144.78	156.61	179.16	148.93	155.88	165.95
61	126.44	137.99	159.40	130.72	137.38	147.06
62	106.40	119.50	137.28	111.94	119.30	127.46
63	94.87	103.19	119.50	97.91	102.62	110.07
64	72.87	85.71	94.77	78.67	85.71	89.92
65	63.40	69.96	78.67	66.37	69.96	74.00
66	60.67	60.67	63.19	57.52	57.52	58.71
67	41.06	41.73	43.22	40.60	40.97	41.68
68	24.44	26.32	26.32	25.33	26.32	26.32
69	12.90	12.90	12.90	12.90	12.90	12.90

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

Table 15. Return after the employee has retired; $r = 0.08$ and $\sigma = 0.2$

Age	Medium return	Age	Medium return	Age	Medium return
70	0.1585	74	0.2614	78	−0.0062
71	0.0815	75	0.1101	79	−0.0177
72	0.0552	76	−0.0890		
73	0.0774	77	−0.2747		

Table 16. Yearly pension payments for the alternative contract when the investment portfolio of the pension fund has a medium performance; the employee was 30 years old when he entered into the pension plan; $T_J = 80$, $r = 0.08$, $\sigma = 0.2$, $\bar{\lambda} = 100$, and $\gamma = 0.75$

Age	a_3	a_4	Age	a_3	a_4
71	3,659.7	9,098.3	76	5,677.4	14,114
72	3,890.4	9,671.7	77	5,310.9	14,690
73	4,054.7	10,080	78	4,322.4	15,290
74	4,296.9	10,682	79	4,303.0	15,914
75	5,227.4	12,996	80	4,248.3	16,563

6. Conclusions

We have shown how defined contribution based pension plans can be valued. Both contracts with and without guarantees have been constructed and analysed. Forward-start guarantees have been shown to be well-suited for analysing these pension plans when guarantees are embedded. We have also showed that, even when there is a rate of return guarantee included in the pension plan, the contribution based pension plan imposes a lot of risk on the employee upon retirement. We have also shown how the financial risk in pension plans can be managed by undertaking trading strategies in the financial market.

For future research, it could be interesting to investigate how to incorporate allowance for dependence between the time of retirement and the value of the pension plan. If the pension fund experiences a poor return on its investment portfolio just before an employee has planned to retire, it might be that the employee chooses to delay retirement. Also, it would be interesting to let the employee change the risk profile on the pension account, as he or she gets older.

Acknowledgement

The author would like to thank an anonymous referee for useful comments.

https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press

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https://doi.org/10.1017/S1748499500000087 Published online by Cambridge University Press