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A PARAMETRIC CHARACTERIZATION OF INTEGRATED VECTOR AUTOREGRESSIVE (VAR) PROCESSES

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This paper provides a polynomial factorization theorem that is then used to extend the characterization parts of the parametric representation theorems of Johansen (1992, *Econometric Theory* 8, 188–202) for vector autoregressive processes integrated of up to order 2. A characterization theorem is provided in the general case of an $I(d)$ process. For the discussion of the complicated polynomial cointegration properties of such processes, the case of an $I(3)$ process is considered as an example.

1. INTRODUCTION

Research in the topic of cointegration is past its first infancy. Whereas the interest at the beginning primarily was concentrated on processes integrated of order 1 (see, e.g., Granger, 1981; Granger and Weiss, 1983; Engle and Granger, 1987), the effort has more recently been concentrated on processes integrated of orders higher than 1 and specifically on $I(2)$ processes (see, e.g., Johansen, 1992; Granger and Lee, 1991; Engle and Yoo, 1991; Gregoir and Laroque, 1993). Also, Park and Phillips (1989), Kitamura (1995), Chang and Phillips (1995), and Chang (1996) are important papers in this area. The present study is concerned only with a minor—but not the least important—part of the analysis of higher order integrated processes. Using a polynomial factorization theorem provided in the paper, the approach in Johansen (1992) is extended to set up a specific theorem for the parametric characterization of a vector autoregressive process integrated of order d. The special case of an $I(3)$ process is considered in the last part of the paper.

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2. THE BASIC MODEL

The basic model studied in this paper is the p-dimensional vector autoregression (VAR) of order k with normal errors assumed to be independently and identically distributed with mean zero and covariance matrix Ω_p:

$$A_0 Z_t + A_1 Z_{t-1} + \cdots + A_k Z_{t-k} = \epsilon_t, \quad t = 1,2,\ldots,T.$$

$$\Leftrightarrow A(L)Z_t = \epsilon_t \tag{1}$$

Here Z_t is the p-dimensional vector process and the A_i's are the matrices $(p \times p)$ of coefficients. The initial values $Z_{-k+1},\ldots,Z_0$ are treated as fixed values. It is possible to extend the model by allowing for deterministic terms on the right-hand side of (1), but such extensions are not critical for the analysis in the following material and are therefore left out for expositional ease.

Because economic time series almost always show some kind of trending behavior, some definitions concerning the time series properties of the series are needed. Following Johansen (1992) a nonstationary process will be called *integrated of order j*, $I(j)$, if the process transformed by the operator Δ^j becomes stationary ($\Delta^j = (1-L)^j Z_t$, where $L^j Z_t = Z_{t-j}$) while the process transformed by Δ^{j-1} still is nonstationary. A stationary process is often referred to as an $I(0)$ process. As indicated by the definition of integration, stationarity of this kind of processes can always be obtained by appropriate differencing. Furthermore, a vector process is said to be *cointegrated* if a linear combination of the series is integrated of an order lower than the process itself. The concept of cointegration is very interesting from an econometric point of view because of the interpretation of cointegration relations as economic long-run or equilibrium relationships. In the case of cointegration an $I(1)$ process can be transformed to stationarity in two different ways: (1) ΔZ_t and (2) $\omega^{\mathrm{T}} Z_t$ where ω is a $(p \times r)$ matrix, $r < p$, such that r stationary linear combinations (cointegration relations) of the process exist. (A superscript T means that the vector or matrix preceding it is transposed.)

If the process is integrated of an order higher than 1, the possibility of cointegration between differences and linear combinations of the process arises. This phenomenon is called *multicointegration* (see Granger and Lee, 1991) or *polynomial cointegration* (see Engle and Yoo, 1991). Hence, $I(2)$ processes can be transformed to stationarity as follows: (1) $\Delta^2 Z_t$, (2) $\omega^{\mathrm{T}} \Delta Z_t$, or (3) $\omega_1^{\mathrm{T}} \Delta Z_t + \omega_2^{\mathrm{T}} Z_t$.

The parametric characterization of an $I(d)$ process is important because this makes it possible to develop statistical tests to determine the order of integration, and knowledge of the integration and cointegration properties of a process is extremely important for model formulation and inference.

The basic assumption concerning the stochastic process, an assumption that works throughout the paper, is as follows.

Basic Assumption. The roots of $|A(z)| = 0$ are either outside the unit disc or equal to one.

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This means for one thing that explosive processes and seasonally integrated processes are not considered.

3. A POLYNOMIAL FACTORIZATION THEOREM[1]

Before we turn to the discussion of the parametric characterization of integrated processes it is convenient to set up a theorem for a certain factorization of a polynomial matrix.

THEOREM 1. *Polynomial factorization theorem. Let* $A(z)$ *be a* $(p \times p)$ *polynomial matrix of finite degree such that* $|A(z)| = (1 - z)^s d(z)$ *where* $d(z)$ *has all its roots outside the unit circle, then four* $(p \times p)$ *matrices* $G(z)$, $T(z)$, B, *and* C *exist such that* (i) $|G(z)|$ *has the same roots as that of* $d(z)$; (ii) $T(z)$ *is a block upper triangular matrix whose block diagonal matrices are diagonal and equal to a monomial in* $(1 - z)$ *such that the determinant of* $T(z)$ *is equal to* $(1 - z)^s$; (iii) B *and* C *are full-rank constant matrices*; (iv) $CA(z) = G(z)T(z)B$.

Proof of Theorem 1. The proof is made by induction, but before the proof is given some necessary notation[2] is introduced.

First notice that a polynomial matrix, $A(z)$, always can be written in the following form:

$$A(z) = A(1) + A_1(z)(1 - z).$$

Now, consider a sequence (indexed by i) of matrices α_i, i.e., $(\alpha_i)_i$, with α_i of dimension $(p \times r_i)$, $0 \leq r_i \leq p$ and of full-column rank. Now define other sequences of matrices $(C_i)_i$ of dimensions $(p \times R_i)$ by $C_i = (C_{i-1}, \alpha_i^-)$ where $R_i = \sum_0^i r_j$, $\alpha_i^- = \alpha_i(\alpha_i^{\mathrm{T}}\alpha_i)^{-1}$, and $(\tilde{C}_i)_i$ by $\tilde{C}_i = (C_i, \tilde{\alpha}_{i+1}^-)$ where $\tilde{\alpha}_{i+1}^-$ is defined such that $\tilde{C}_i$ is a $(p \times p)$ full-rank matrix. There exist many possible choices for $(\tilde{\alpha}_i)_i$, but for our purpose we define them by imposing some orthogonality conditions. Actually, it is not the sequence $(\alpha_i)_i$ that is given; rather it is another sequence (called $(\alpha(i))_i$ with $\alpha(i)$ of dimension $((p - R_{i-1}) \times r_i)$ and of full rank), but from which the matrices $(\alpha_i)_i$ are easily found: $\alpha_i = C_{i-1,\perp}\alpha(i)$, where $C_{i-1,\perp}$ is the orthogonal complement to C_{i-1}. Furthermore, we associate to a sequence of matrices β_j the same kind of matrices denoted B_j and $\tilde{B}_j$. A sequence, $\tilde{B}_j^*$, is also defined that is similar in structure to the $\tilde{B}_j$ sequence but with β_j^- exchanged with β_j. Notice that $\tilde{B}_j\tilde{B}_j^{*\mathrm{T}} = I_p$. Finally, we define a $((p - R_i) \times (p - R_i))$ matrix $K(C_i, \alpha(i + 1))$ such that $(\alpha_{i+1}^-, \tilde{\alpha}_{i+2}^-) = \tilde{\alpha}_{i+1}^- K(C_i, \alpha(i + 1))$ and a $(p \times p)$ block diagonal matrix $\tilde{K}_{i+1}(\alpha)$ whose upper block is an identity matrix and whose lower block is equal to $K(C_i, \alpha(i + 1))$. We also need to introduce sequences of polynomial matrices. We need some recursive definitions and start by the initializing equations:

$$F^0(A)(z) = A(z)$$

$$H_0(A)(z) = F^0(A)_1(z)\beta_0^-$$

$$U_0(A) = I_{r_0}.$$

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For step j in the recursive relations we have (matrices with negative indices are equal to zero):

$$F^j(A)(z) = F^{j-1}(A)_1(z) - H_{j-2}(A)(z)U_{j-2}(A)^{-1}C_{j-2}^{\mathrm{T}}F^{j-1}(A)(1)$$

$$U_j(A) = \begin{pmatrix} U_{j-1}(a) & C_{j-1}^{\mathrm{T}}F^j(A)(1)\beta_j^- \\ 0 & I_{r_j} \end{pmatrix}$$

$$H_j(A)(z) = (H_{j-1}(A)(z), \quad F^j(A)_1(z)\beta_j^-).$$

Back to the proof. As mentioned, the proof is made by induction, but the induction works a bit atypically in the sense that what we prove is that for a given factorization structure developed for $i = m$ (here i is the largest index of the α's [or β's] appearing in a given factorization) for which the theorem is not fulfilled, one can always (by extracting unit roots from the first factor) find a factorization with the same structure for $i = m + j, j > 0$ (but less than some upper limit determined by the maximum possible number of unit roots in the process) for which the theorem is fulfilled. Whether the factorization theorem is fulfilled is easily decided from the suggested factorization structure by checking the rank of a certain matrix. Actually, what the induction offers is a procedure that describes the move from the factorization structure suggested for $i = m$ (assumed to violate the theorem) to the same kind of factorization but for $i = m + 1$. If the theorem is valid for $i = m + 1$ we are done. Otherwise, a violation implies that there are still unit roots in the first factor. We know that there is an upper limit to the number of unit roots of the kind of finite order VAR processes considered, and this means that by further iterations of the factorization procedure all unit roots can be removed from the first factor and the theorem will eventually be fulfilled. In summary: reduced rank of a certain matrix for a given iteration in the factorization procedure means that we need to continue with the procedure. Eventually, the matrix of interest will have full rank such that the wanted factorization has been derived and the procedure stops.

Let us first assume that there are indeed unit roots in the polynomium such that for $i = 0$ $A(1)$ has reduced rank and we write $A(1) = \alpha(0)\beta(0)^{\mathrm{T}}$. We define $\alpha_0 = \alpha(0)$, $C_0 = \alpha_0^-$ and $\beta_0 = \beta(0)$, $B_0 = \beta_0^-$. To obtain the desired factorization structure, we now pre- and postmultiply by $\tilde{C}_0^{\mathrm{T}}$ and $\tilde{B}_0$, respectively. These matrices are constant matrices and do not change the roots of the polynomial matrix:

$$\begin{aligned}
&\tilde{C}_0^{\mathrm{T}}A(z)\tilde{B}_0 \\
&\quad = \begin{pmatrix} I_{r_0} & 0 \\ 0 & 0 \end{pmatrix} + \tilde{C}_0^{\mathrm{T}}(\mathrm{A}_1(\mathrm{z})\beta_0^-, A_1(z)\tilde{\beta}_1^-)(1-z) \\
&\quad = \left\{ \begin{pmatrix} I_{r_0} & C_0^{\mathrm{T}}F^1(A)(1)\tilde{\beta}_1^- \\ 0 & \tilde{\alpha}_1^{-\mathrm{T}}F^1(A)(1)\tilde{\beta}_1^- \end{pmatrix} + \tilde{C}_0^{\mathrm{T}}(H_0(A)(z), F^1(A)_1(z)\tilde{\beta}_1^-)(1-z) \right\} \\
&\qquad \times \begin{pmatrix} I_{r_0} & 0 \\ 0 & I_{p-r_0}(1-z) \end{pmatrix} \\
&\quad \equiv G_0(z)T_0(z).
\end{aligned}$$

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Postmultiplication of this equation by $\tilde{B}_0^{*\mathrm{T}}$ gives the factorization of the theorem, and if $\tilde{\alpha}_1^{-\mathrm{T}} F^1(A)(1)\tilde{\beta}_1^-$ is of full rank the result is proved for $i = 1$. For $i = 1$ some of the basic features of the factorization structure become clear, but much of the complexity encountered for $i > 1$ is still missing. Therefore, we now assume that $\tilde{\alpha}_1^{-\mathrm{T}} F^1(A)(1)\tilde{\beta}_1^-$ is of reduced rank such that there exist $\alpha(1)$ and $\beta(1)$ for which $\tilde{\alpha}_1^{-\mathrm{T}} F^1(A)(1)\tilde{\beta}_1^- = \alpha(1)\beta(1)^{\mathrm{T}}$ and we give the factorization structure for $i = 2$. First postmultiply by $\tilde{K}_1(\beta)$ and premultiply by $\tilde{K}_1(\alpha)^{\mathrm{T}}$, such that the last row and column blocks of the preceding expression are split into two row/column blocks:

$$\tilde{C}_1^{\mathrm{T}} A(z)\tilde{B}_1 = \left\{ \begin{pmatrix} I_{r_0} & C_0^{\mathrm{T}} F^1(A)(1)\beta_1^- & C_0^{\mathrm{T}} F^1(A)(1)\tilde{\beta}_2^- \\ 0 & I_{r_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tilde{C}_1^{\mathrm{T}} \left(H_1(A)(z), F^1(A)_1(z)\tilde{\beta}_2^-\right)(1-z) \right\} \begin{pmatrix} I_{r_0} & 0 \\ 0 & I_{p-r_0}(1-z) \end{pmatrix}.$$

We then simultaneously postmultiply by a triangular matrix the first factor of the preceding right-hand-side expression and premultiply the second factor by its inverse. This triangular matrix has the following form:

$$\begin{pmatrix} I_{r_0} & 0 & -C_0^{\mathrm{T}} F^1(A)(1)\tilde{\beta}_2^- \\ 0 & I_{r_1} & 0 \\ 0 & 0 & I_{p-R_1} \end{pmatrix}.$$

This operation gives

$$\begin{aligned} &\tilde{C}_1^{\mathrm{T}} A(z)\tilde{B}_1 \\ &\quad = \left\{ \begin{pmatrix} I_{r_0} & C_0^{\mathrm{T}} F^1(A)(1)\beta_1^- & 0 \\ 0 & I_{r_1} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \tilde{C}_1^{\mathrm{T}} (H_1(A)(z), F^2(A)(z)\tilde{\beta}_2^-)(1-z) \right\} \\ &\qquad \times \begin{pmatrix} I_{r_0} & 0 & C_0^{\mathrm{T}} F^1(A)(1)\tilde{\beta}_2^-(1-z) \\ 0 & I_{r_1}(1-z) & 0 \\ 0 & 0 & I_{p-R_1}(1-z) \end{pmatrix} \\ &\quad = \left\{ \begin{pmatrix} U_1(A) & C_1^{\mathrm{T}} F^2(A)(1)\tilde{\beta}_2^- \\ 0 & \tilde{\alpha}_2^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_2^- \end{pmatrix} + \tilde{C}_1^{\mathrm{T}} \left(H_1(A)(z), F^2(A)_1(z)\tilde{\beta}_2^-\right)(1-z) \right\} \\ &\qquad \times \begin{pmatrix} I_{r_0} & 0 & C_0^{\mathrm{T}} F^1(A)(1)\tilde{\beta}_2^-(1-z) \\ 0 & I_{r_1}(1-z) & 0 \\ 0 & 0 & I_{p-R_1}(1-z)^2 \end{pmatrix} \\ &\quad \equiv G_1(z)T_1(z). \end{aligned}$$

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The determinant of the first factor evaluated at $z = 1$ equals the determinant of $\tilde{\alpha}_2^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_2^-$ such that when this latter matrix is of full rank all unit roots have been removed from the first factor. Postmultiplication of the previous equation $\tilde{B}_1^{*\mathrm{T}}$ gives the factorization of the theorem, which is then proved for $i = 2$.

Finally, we consider a general step in the induction. Assume that for $i = m$ the matrices $\tilde{C}_j^{\mathrm{T}} A(z)\tilde{B}_j\tilde{B}_j^*$ obey the general factorization structure and that $\tilde{\alpha}_{j+1}^{-\mathrm{T}} F^{j+1}(A)(1)\tilde{\beta}_{j+1}^-$ has reduced rank for $j = 0, 1, \ldots, m-1$. The important part of the factorization structure assumed here is

$$\tilde{C}_j^{\mathrm{T}} A(z)\tilde{B}_j = \left\{ \begin{pmatrix} U_j(A) & C_j^{\mathrm{T}} F^{j+1}(A)(1)\tilde{\beta}_{j+1}^- \\ 0 & \tilde{\alpha}_{j+1}^{-\mathrm{T}} F^{j+1}(A)(1)\tilde{\beta}_{j+1}^- \end{pmatrix} + \tilde{C}_j^{\mathrm{T}}(H_j(A)(z), F^{j+1}(A)_1(z)\tilde{\beta}_{j+1}^-)(1-z) \right\} T_j(z), \tag{2}$$

where $T_j(z)$ is a block upper triangular matrix whose block dimensions are $(r_u \times r_v)$ with $(r_u, r_v) \in \{r_0, \ldots, r_j, p - R_j\}^2$ and $(u,v) \in \{0, \ldots, j, j+1\}^2$, the diagonal submatrices are equal to $(1-z)^u I_{r_u}$ *or* $(1-z)^{j+1} I_{p-R_j}$, the second-upper block diagonal is equal to zero, and the other submatrices indexed by (u,v) (the last index $j+1$ is associated to the dimension $p - R_j$) are polynomials without a constant term and degree $(v-1)$.

We now consider the mth factorization (at this stage, just ignore the last $\tilde{B}_{m-1}^{*\mathrm{T}}$ factor):

$$\tilde{C}_{m-1}^{\mathrm{T}} A(z)\tilde{B}_{m-1} = \left\{ \begin{pmatrix} U_{m-1}(A) & C_{m-1}^{\mathrm{T}} F^m(A)(1)\tilde{\beta}_m^- \\ 0 & \tilde{\alpha}_m^{-\mathrm{T}} F^m(A)(1)\tilde{\beta}_m^- \end{pmatrix} + \tilde{C}_{m-1}^{\mathrm{T}}(H_{m-1}(A)(z), F^m(A)_1(z)\tilde{\beta}_m^-)(1-z) \right\} T_{m-1}(z).$$

We now postmultiply by $\tilde{K}_m(\beta)$ and premultiply by $\tilde{K}_m(\alpha)^{\mathrm{T}}$.

From the definitions of $\tilde{K}$ and K we see that $\tilde{K}_m(\alpha)^{\mathrm{T}}\tilde{C}_{m-1}^{\mathrm{T}} A(z)\tilde{B}_{m-1}\tilde{K}_m(\beta) = \tilde{C}_m^{\mathrm{T}} A(z)\tilde{B}_m$, i.e., the last block of $\tilde{C}_{m-1}$ and $\tilde{B}_{m-1}$ is split into two blocks. Next, observe that

$$\begin{aligned} &K(C_{m-1}, \alpha(m))^{\mathrm{T}}\tilde{\alpha}_m^{-\mathrm{T}} F^m(A)(1)\tilde{\beta}_m^- K(B_{m-1}, \beta(m)) \\ &\quad = K(C_{m-1}, \alpha(m))^{\mathrm{T}}\alpha(m)\beta(m)^{\mathrm{T}} K(B_{m-1}, \beta(m)) \\ &\quad = K(C_{m-1}, \alpha(m))^{\mathrm{T}}\tilde{\alpha}_m^{-\mathrm{T}}\tilde{\alpha}_m\alpha(m)\beta(m)^{\mathrm{T}}\tilde{\beta}_m^{\mathrm{T}}\tilde{\beta}_m^- K(B_{m-1}, \beta(m)) \\ &\quad = (\alpha_m^-, \tilde{\alpha}_{m+1}^-)^{\mathrm{T}}\alpha_m\beta_m^{\mathrm{T}}(\beta_m^-, \tilde{\beta}_{m+1}^-) \\ &\quad = \begin{pmatrix} I_{r_m} \\ 0 \end{pmatrix}(I_{r_m} \quad 0) = \begin{pmatrix} I_{r_m} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

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such that

$$\begin{aligned}
\tilde{C}_m^{\mathrm{T}} A(z) \tilde{B}_m &= \left\{ \begin{pmatrix} U_{m-1}(A) & C_{m-1}^{\mathrm{T}} F^m(A)(1) \beta_m^- & C_{m-1}^{\mathrm{T}} F^m(A)(1) \tilde{\beta}_{m+1}^- \\ 0 & I_{r_m} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\
&\quad + \tilde{C}_m^{\mathrm{T}} (H_{m-1}(A)(z), F^m(A)_1(z) \tilde{\beta}_m^-)(1 - z) \\
&\quad \left. \times \begin{pmatrix} I_{R_{m-1}} & 0 \\ 0 & K(B_{m-1}, \beta(m)) \end{pmatrix} \right\} \tilde{K}_m^{-1} T_{m-1}(z) \tilde{K}_m \\
&= \left\{ \begin{pmatrix} \text{first term} \\ \text{unchanged} \end{pmatrix} + \tilde{C}_m^{\mathrm{T}} (H_{m-1}(A)(z), F^m(A)_1(z) \beta_m^-, \right. \\
&\qquad \left. F^m(A)_1(z) \tilde{\beta}_{m+1}^-)(1 - z) \right\} \tilde{K}_m^{-1} T_{m-1}(z) \tilde{K}_m \\
&= \left\{ \begin{pmatrix} \text{first term} \\ \text{unchanged} \end{pmatrix} + \tilde{C}_m^{\mathrm{T}} (H_m(A)(z), F^m(A)_1(z) \tilde{\beta}_{m+1}^-)(1 - z) \right\} \\
&\quad \times \tilde{K}_m^{-1} T_{m-1}(z) \tilde{K}_m \\
&= \{(\ ) + \tilde{C}_m^{\mathrm{T}} (H_m(A)(z), F^m(A)_1(z) \tilde{\beta}_{m+1}^-)(1 - z)\} \\
&\quad \times \begin{pmatrix} I_{R_{m-1}} & 0 & -U_{m-1}(A)^{-1} C_{m-1}^{\mathrm{T}} F^m(A)(1) \tilde{\beta}_{m+1}^- \\ 0 & I_{r_m} & 0 \\ 0 & 0 & I_{p-R_m} \end{pmatrix} \\
&\quad \times \begin{pmatrix} I_{R_{m-1}} & 0 & U_{m-1}(A)^{-1} C_{m-1}^{\mathrm{T}} F^m(A)(1) \tilde{\beta}_{m+1}^- \\ 0 & I_{r_m} & 0 \\ 0 & 0 & I_{p-R_m} \end{pmatrix} \\
&\quad \times \tilde{K}_m^{-1} T_{m-1}(z) \tilde{K}_m \\
&= \left\{ \begin{pmatrix} U_{m-1}(A) & C_{m-1}^{\mathrm{T}} F^m(A)(1) \beta_m^- & 0 \\ 0 & I_{r_m} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right. \\
&\quad \left. + \tilde{C}_m^{\mathrm{T}} (H_m(A)(z), F^{m+1}(A)(z) \tilde{\beta}_{m+1}^-)(1 - z) \right\} \times (4 \text{ factors}) \\
&= \left\{ \begin{pmatrix} U_m(A) & 0 \\ 0 & 0 \end{pmatrix} + \tilde{C}_m^{\mathrm{T}} (H_m(A)(z), F^{m+1}(A)(z) \tilde{\beta}_{m+1}^-)(1 - z) \right\} \\
&\quad \times (4 \text{ factors})
\end{aligned}$$

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$$= \left\{ \begin{pmatrix} U_m(A) & 0 \\ 0 & 0 \end{pmatrix} + \tilde{C}_m^{\mathrm{T}}(H_m(A)(z), F^{m+1}(A)(1)\tilde{\beta}_{m+1}^{-} + F^{m+1}(A)_1(z)\tilde{\beta}_{m+1}^{-}(1-z))(1-z) \right\}$$

$$\times \text{(4 factors)}$$

$$= \left\{ \begin{pmatrix} U_m(A) & C_m^{\mathrm{T}} F^{m+1}(A)(1)\tilde{\beta}_{m+1}^{-} \\ 0 & \tilde{\alpha}_{m+1}^{\mathrm{T}} F^{m+1}(A)(1)\tilde{\beta}_{m+1}^{-} \end{pmatrix} + \tilde{C}_m^{\mathrm{T}}(H_m(A)(z), F^{m+1}(A)_1(z)\tilde{\beta}_{m+1}^{-})(1-z) \right\}$$

$$\times \begin{pmatrix} I_{R_m} & 0 \\ 0 & I_{p-R_m}(1-z) \end{pmatrix} \times \text{(4 factors)}.$$

Now we consider the last 5 factors of the preceding equation:

$$\begin{pmatrix} I_{R_m} & 0 \\ 0 & I_{p-R_m}(1-z) \end{pmatrix} \begin{pmatrix} I_{R_{m-1}} & 0 & U_{m-1}(A)^{-1} C_{m-1}^{\mathrm{T}} F^m(A)(1)\tilde{\beta}_{m+1}^{-} \\ 0 & I_{r_m} & 0 \\ 0 & 0 & I_{p-R_m} \end{pmatrix}$$

$$\times \tilde{K}_m^{-1} T_{m-1}(z) \tilde{K}_m$$

$$= \begin{pmatrix} I_{R_{m-1}} & 0 & U_{m-1}(A)^{-1} C_{m-1}^{\mathrm{T}} F^m(A)(1)\tilde{\beta}_{m+1}^{-} \\ 0 & I_{r_m} & 0 \\ 0 & 0 & I_{p-R_m}(1-z) \end{pmatrix}$$

$$\times \begin{pmatrix} I_{R_{m-1}} & 0 \\ 0 & K(B_{m-1}, \beta(m))^{-1} \end{pmatrix}$$

$$\times \begin{pmatrix} T_{m-2}^*(z) & t_0 \\ 0 & I_{p-R_{m-1}}(1-z)^m \end{pmatrix} \begin{pmatrix} I_{R_{m-1}} & 0 \\ 0 & K(B_{m-1}, \beta(m)) \end{pmatrix}$$

$$= \begin{pmatrix} I_{R_{m-1}} & 0 & U_{m-1}(A)^{-1} C_{m-1}^{\mathrm{T}} F^m(A)(1)\tilde{\beta}_{m+1}^{-} \\ 0 & I_{r_m} & 0 \\ 0 & 0 & I_{p-R_m}(1-z) \end{pmatrix}$$

$$\times \begin{pmatrix} T_{m-2}^*(z) & t_1 & t_2 \\ 0 & I_{r_m}(1-z)^m & 0 \\ 0 & 0 & I_{p-R_m}(1-z)^m \end{pmatrix}$$

$$= \begin{pmatrix} T_{m-2}^*(z) & t_1 & t_2 + U_{m-1}(A)^{-1} C_{m-1}^{\mathrm{T}} F^m(A)(1)\tilde{\beta}_{m+1}^{-}(1-z)^m \\ 0 & I_{r_m}(1-z)^m & 0 \\ 0 & 0 & I_{p-R_m}(1-z)^{m+1} \end{pmatrix}$$

$$= \begin{pmatrix} T_{m-1}^*(z) & t_3 \\ 0 & I_{p-R_m}(1-z)^{m+1} \end{pmatrix} \equiv T_m(z),$$

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with $T_j^*(z)$ having the same structure as $T_j(z)$, but with the last column/row of dimensions corresponding to β_{j+1}^- instead of $\tilde{\beta}_{j+1}^-$. The t_0 block is simply the upper right block of $T_{m-1}(z)$ of row dimension R_{m-1}. The t_1 and t_2 blocks are simply t_0 split into two columns, i.e., with $\tilde{\beta}_m^-$ turned into $(\beta_m^-, \tilde{\beta}_{m+1}^-)$. The block t_3 is simply defined as the two upper right blocks in the previous matrix.

Now evaluate the determinant of $\tilde{C}_m^{\mathrm{T}} A(z) \tilde{B}_m$ at $z = 1$. For $\tilde{\alpha}_{m+1}^{-\mathrm{T}} F^{m+1}(A)(1) \tilde{\beta}_{m+1}^-$ of full rank postmultiplication of the equation by $\tilde{B}_m^{*\mathrm{T}}$ means that the factorization theorem is proved. Otherwise, only an iteration of the factorization procedure (from $i = m$ to $i = m + 1$) has been described, but we know that one or more iterations similar in structure will eventually lead to the desired factorization and the theorem is proved. ■

4. CHARACTERIZATION OF INTEGRATED VAR PROCESSES

It is a well-known fact that for the kind of processes considered here the condition for stationarity is $|A(1)| \neq 0$. Now with unit roots present in the process, some other characterization theorems are needed to describe the integration, cointegration, and polynomial cointegration properties of the process. Let us start by recalling some well-known results—written, however, with the notation of the present paper.

THEOREM 2. *Parametric characterization of an $I(1)$ process. Under the basic assumption, the necessary and sufficient conditions for the process Z_t to be $I(1)$ are that there exist $\alpha(0)$ and $\beta(0)$ matrices such that*

$A(1) = \alpha(0)\beta(0)^{\mathrm{T}}$,
$\tilde{\alpha}_1^{-\mathrm{T}} F^1(A)(1) \tilde{\beta}_1^-$ *has full rank.*

By a comparison of the present theorem with that of the $I(1)$ representation given in Johansen (1992) or in la Cour (1992), it is easily seen that apart from pre- and postmultiplication by $(p \times p)$ full-rank matrices that leave the essential rank condition unaffected, only notational differences in the characterization part of the theorem are present. Hence, the proof from one of these sources will apply and is not repeated here. Note that in case of an $I(1)$ series both ΔZ_t and $\beta_0^{\mathrm{T}} Z_t$ are stationary, a property that is easily derived from $T_0(z) \tilde{B}_0^{*\mathrm{T}}$ from the polynomial factorization applied to the $I(1)$ case.

Now the theorem for the parametric characterization of $I(2)$ processes is given.

THEOREM 3. *Parametric characterization of an $I(2)$ process. Under the basic assumption, the necessary and sufficient conditions for the process Z_t to be $I(2)$ are that there exist $\alpha(0)$, $\beta(0)$ matrices and $\alpha(1)$, $\beta(1)$ matrices such that*

$$F^0(A)(1) \equiv A(1) = \alpha(0)\beta(0)^{\mathrm{T}},$$
$$\tilde{\alpha}_1^{-\mathrm{T}} F^1(A)(1) \tilde{\beta}_1^- = \alpha(1)\beta(1)^{\mathrm{T}},$$
$\tilde{\alpha}_2^{-\mathrm{T}} F^2(A)(1) \tilde{\beta}_2^-$ *has full rank.*

Again by comparisons of notation the theorems are essentially identical to those found in either Johansen (1992) or in la Cour (1992), and the proof

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will not be repeated here. Note that $\tilde{\beta}_2^{\mathrm{T}}\Delta^2 Z_t$, $\beta_1^{\mathrm{T}}\Delta Z_t$ and $(\beta_0^{\mathrm{T}} Z_t + C_0^{\mathrm{T}} F^1(A)(1)\tilde{\beta}_2^{-}\tilde{\beta}_2^{\mathrm{T}}\Delta Z_t)$ are stationary, a property that is easily found from $T_1(z)\tilde{B}_1^{*\mathrm{T}}$ of the polynomial factorization theorem for an $I(2)$ process.

Now a general parametric characterization theorem for a process integrated of order d is given:

THEOREM 4. *Parametric characterization of an $I(d)$ process. Under the basic assumption, the necessary and sufficient conditions for the process Z_t to be $I(d)$ are that there exist $\alpha(0)$, $\beta(0)$ matrices and $\alpha(j)$, $\beta(j)$ matrices $j = 1, 2, \ldots, (d-1)$ such that*

$$F^0(A)(1) \equiv A(1) = \alpha(0)\beta(0)^{\mathrm{T}},$$

$$\tilde{\alpha}_j^{-\mathrm{T}} F^j(A)(1)\tilde{\beta}_j^{-} = \alpha(j)\beta(j)^{\mathrm{T}},\ j = 1, \ldots, (d-1),$$

$\tilde{\alpha}_d^{-\mathrm{T}} F^d(A)(1)\tilde{\beta}_d^{-}$ *has full rank.*

Proof of Theorem 4. To prove sufficiency we use the polynomial factorization theorem. Starting from $A(z)Z_t = \epsilon_t$ we pre- and postmultiply $A(z)$ such that the full-rank condition of Theorem 4 ensures the absence of unit roots in $G_{d-1}(z)$, i.e., in the first factor of the right-hand-side expression in (2) for $j = (d-1)$:

$$\tilde{C}_{d-1}^{\mathrm{T}} A(z)\tilde{B}_{d-1}\tilde{B}_{d-1}^{*\mathrm{T}} Z_t = G_{d-1}(z)T_{d-1}(z)\tilde{B}_{d-1}^{*\mathrm{T}} Z_t = \tilde{C}_{d-1}^{\mathrm{T}}\epsilon_t.$$

From all the reduced rank conditions of Theorem 4 we can stepwise build up the partition of the coefficient matrix until the preceding equation that satisfies the polynomial factorization theorem is reached. From $T_{d-1}(z)$ we see that in some directions it is necessary to take differences of order d to make the process stationary, implying that the process is integrated of order d such that the sufficiency part of the proof is completed.

As for the necessity part of the proof, we apply the polynomial factorization theorem once more. Knowing that the process is integrated of order d implies that it cannot be integrated of a lower order such that the full-rank conditions of processes integrated of a lower order will be violated. This forces us to perform one more iteration from the polynomial factorization that is valid for an $I(d-1)$ process. This leads to a $T_j(z)$ factor that allows for differences of order d of the process, i.e., $T_{d-1}(z)$. For an $I(d)$ process we know that it is sufficient to take differences of order d to make the process stationary such that we do not need another iteration to find the polynomial factorization of an $I(d)$ process. But for the factorization including $T_{d-1}(z)$ we see that it is the full rank of $\tilde{\alpha}_d^{-\mathrm{T}} F^d(A)(1)\tilde{\beta}_d^{-}$ that makes the factorization valid; hence it proves that the conditions of the parametric characterization theorem are indeed necessary conditions. ■

To give a complete description of the polynomial cointegration properties of an $I(d)$ process a close examination of $T_{d-1}(z)\tilde{B}_{d-1}^{*\mathrm{T}}$ will suffice. Each row block, of which there are $(d+1)$ when solved backward, corresponds to a different direction in the β-space, i.e., the space spanned by $\beta_0, \beta_1, \ldots, \beta_{d-1}, \tilde{\beta}_d$. In the directions spanned by $\tilde{\beta}_d$ there is no cointegration, and only dth order differences of the process will be stationary. In the directions spanned by β_{d-1} the process

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cointegrates from $I(d)$ to $I(d-1)$, and $(d-1)$th order differences will suffice to make the process stationary. For all the other directions polynomial cointegration patterns exist, patterns that become increasingly more complex each time we consider a new β with a lower subscript. Without actually writing down each row of $T_{d-1}(z)\tilde{B}_{d-1}^{*\mathrm{T}}$ a more detailed discussion and interpretation of these properties is difficult. What we will do here is to exemplify the discussion by the polynomial cointegration properties of an $I(3)$ process that already gives a good indication of the interpretation in the general case.

4.1. An Example: The Characterization of an $I(3)$ Process

First the parametric representation theorem of an $I(3)$ process is given (this is just an application of Theorem 4 with $d=3$).

THEOREM 5. *Parametric characterization of an $I(3)$ process. Under the basic assumption, the necessary and sufficient conditions for the process Z_t to be $I(3)$ are that there exist $\alpha(0), \beta(0)$ matrices, $\alpha(1), \beta(1)$ matrices, and $\alpha(2), \beta(2)$ matrices such that*

$$
\begin{aligned}
F^0(A)(1) \equiv A(1) &= \alpha(0)\beta(0)^{\mathrm{T}},\\
\tilde{\alpha}_1^{-\mathrm{T}} F^1(A)(1)\tilde{\beta}_1^{-} &= \alpha(1)\beta(1)^{\mathrm{T}},\\
\tilde{\alpha}_2^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_2^{-} &= \alpha(2)\beta(2)^{\mathrm{T}},\\
\tilde{\alpha}_3^{-\mathrm{T}} F^3(A)(1)\tilde{\beta}_3^{-} &\text{ has full rank.}
\end{aligned}
$$

Now, by taking a close look at $T_2(z)$ from the polynomial factorization theorem, which in this case is the relevant one, we find the following structure (to save space we write $(1-z)$ as Δ):

$$
T_2(z) = \begin{pmatrix}
I_{r_0} & 0 & \alpha_0^{-\mathrm{T}} F^1(A)(1)\beta_2^{-}\Delta & \alpha_0^{-\mathrm{T}} F^1(A)(1)\tilde{\beta}_3^{-}\Delta + (\alpha_0^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_3^{-} - \alpha_0^{-\mathrm{T}} F^1(A)(1)\beta_1^{-}\alpha_1^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_3^{-})\Delta^2 \\
0 & I_{r_1}\Delta & 0 & \alpha_1^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_3^{-}\Delta^2 \\
0 & 0 & I_{r_2}\Delta^2 & 0 \\
0 & 0 & 0 & I_{p-R_2}\Delta^3
\end{pmatrix}.
$$

The interpretation of the cointegration and polynomial cointegration properties of an $I(3)$ process is based on this $T_2(z)$ postmultiplied by $\tilde{B}_2^{*\mathrm{T}} = (\beta_0, \beta_1, \beta_2, \tilde{\beta}_3)^{\mathrm{T}}$. Transformed by this matrix product, the process is stationary. We start from the bottom. At the first level in the directions of $\tilde{\beta}_3$ there is no cointegration and the difference operator must be applied three times. At the second level, in the subspace spanned by β_2, cointegration is present, reducing the order of integration from 3 to 2, making Δ^2 the sufficient operator to transform the process to stationarity. At the third level, observe that normally by taking β_1 combinations of the process the order of integration will be reduced from 3 to 2, implying that the use of Δ will lead to an $I(1)$ process. But polynomial cointegration properties are present, and these $I(1)$ series cointegrate in turn with certain other $I(1)$ combinations of the process, $\tilde{\beta}_3^{\mathrm{T}}\Delta^2 Z_t$ with coefficient matrices $(I_{r_1}, \alpha_1^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_3^{-})$.

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The most complex case appears at the fourth level in the β_0 direction. Here $\beta_0^{\mathrm{T}} Z_t$ is generally $I(2)$ but cointegrates with $\tilde{\beta}_3^{\mathrm{T}} \Delta Z_t$, which is also $I(2)$, to an $I(1)$ process with coefficient matrix $(I_{r_0}, \alpha_0^{-\mathrm{T}} F^1(A)(1)\tilde{\beta}_3^-)$. This $I(1)$ process cointegrates with other $I(1)$ processes, $\beta_2^{\mathrm{T}} \Delta Z_t$ and $\tilde{\beta}_3^{\mathrm{T}} \Delta^2 Z_t$, and the coefficient matrices of these two terms are to be $\alpha_0^{-\mathrm{T}} F^1(A)(1)\beta_2^-$ and $\alpha_0^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_3^- - \alpha_0^{-\mathrm{T}} F^1(A)(1)\beta_1^- \alpha_1^{-\mathrm{T}} F^2(A)(1)\tilde{\beta}_3^-$, respectively, which can be directly derived from $T_2(z)$. The last of these coefficient matrices consists of two terms where the last term in a sense "corrects for" the polynomial cointegration that took place at the previous level.

This type of "correcting for" interpretation is general in the sense that for an $I(d)$ process the more complex coefficient matrices of the polynomial coefficient matrices contain similar kinds of terms that always contain the structure of the coefficient matrices that works at the levels already considered. Based on the $I(3)$ example it is suggested that the way to proceed with the discussion and interpretation in the general case of an $I(d)$ process, $d > 3$, is again to start by looking at the bottom row, the row of dimension $((p - R_{d-1}) \times p)$, of the relevant matrix. By moving up the matrix block-row–wise for each of the dimensions $(r_j \times p), j = (d - 1), (d - 2), \ldots, 1, 0$, the full structure will be revealed.

5. CONCLUSION AND DISCUSSION

This paper sets up a polynomial factorization theorem that makes it easy to formulate a parametric characterization theorem for an $I(d)$ process, thereby extending to the general case the parametric characterization theorems of $I(1)$ and $I(2)$ processes provided in Johansen (1992). The factorization theorem also provides a basis for the discussion and interpretation of the complicated cointegration and polynomial cointegration properties of processes integrated of orders higher than 1. The discussion of the relation between the polynomial cointegration coefficient matrices and the coefficient matrices of the basic VAR model is exemplified in the study of an $I(3)$ process.

The paper does not include the corresponding moving average representation (or Granger representation) of an $I(d)$ process. Such a representation is in principle even in the general case easy to find but difficult to write up analytically, because it for one thing needs inversion of $(d + 1)$ matrices. Also notice that to compare the parametric characterization conditions of a finite VAR process as studied in the present paper with other articles starting from the moving average representation of an integrated process (see, e.g., Gregoir and Laroque, 1993), one has to consider conditions under which the factorization theorem works even for infinite order VAR's.

Even though the paper does not contain any statements or propositions concerning estimation and inference in $I(d)$ processes, it is of interest to time series econometricians because the characterization theorems are necessary first steps in the ongoing process of developing the statistical analysis of integrated time series.

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NOTES

1. Such a factorization has certain similarities with the Smith–McMillan decomposition (see, e.g., Engle and Yoo, 1991; Kailath, 1980).

2. I thank an anonymous referee for suggesting this notation as an alternative to the one used in a previous version of the paper. It is hoped that the present notation will make the reading of the proof considerably easier.

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