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Entropy and r equivalence

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Abstract. In this paper, the structure of r equivalence, which was introduced by Vershik and which classifies group actions of the group $G = \sum_{n=1}^{\infty} \mathbb{Z}/r_n\mathbb{Z}$, $r_n \in \mathbb{N} \setminus \{1\}$, is examined. This is an equivalence relation that naturally arises from looking at certain sequences of σ-algebras. Vershik proved that if a sequence $r = (r_1, r_2, \ldots)$ does not satisfy a super-rapid growth rate, then entropy is an invariant for r equivalence. In this paper, a strong converse of this is proven: for any r which does satisfy this super-rapid growth rate, we can find a zero entropy action in every r equivalence class.

1. *Introduction*

In this paper the restricted orbit equivalence theory developed by Kammeyer and Rudolph is used to determine the role that entropy plays in classifying G actions up to r equivalence, where $G = \sum_{n=1}^{\infty} \mathbb{Z}/r_n\mathbb{Z}$, and $r = (r_1, r_2, \ldots)$. First r equivalence is defined, and is tied to isomorphism of decreasing sequences of σ-algebras. Then in §1.2 some basic notions of restricted orbit equivalence (abbreviated r.o.e.) theory are introduced. §2 of the paper is devoted to proving that r.o.e. theory applies to classifying these systems. In other words, it is shown that r equivalence is a restricted orbit equivalence. Vershik proved a sufficient condition on the growth rate of r for entropy to be an invariant for r equivalence. In §3, r.o.e. theory is used to prove that if the converse of this condition holds, then r is entropy free, as well as reproving sufficiency.

1.1. *r equivalence.*

Let $(X, \mathcal{F}, \mu)$ be a non-atomic Lebesgue probability space, and let $\{T_g\}_{g \in G}$ be a free measure-preserving ergodic group action on the space, where $G = \sum_{n=1}^{\infty} \mathbb{Z}/r_n\mathbb{Z}$ and $r_n \in \mathbb{N} \setminus \{1\}$. Set $r = (r_1, r_2, \ldots)$. Notice that G is countable. We take the discrete topology on G, and since G is abelian, it is an amenable group. Let $G_i = \sum_{n=1}^{i} \mathbb{Z}/r_n\mathbb{Z}$. Set $q_i = \#G_i = \prod_{n=1}^{i} r_n$. Notice that $G = \cup G_i$. The central example of a G action, which we call the Kronecker action, is the following.

Example. Let $X = \prod_{n=1}^{\infty} \mathbb{Z}/r_n\mathbb{Z}$, μ = Haar measure, $\mathcal{F}$ = Borel σ-algebra and $(T_g x)_n = (x_n + g_n) \bmod r_n$. Since the action is rotation on a compact abelian group it has discrete spectrum and entropy zero.

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Another example is the natural generalization of the Bernoulli action for this group.

Example. Let $X = \{0, \dots, n-1\}^G$, μ be the Bernoulli measure on X given by a probability vector $(p_0, \dots, p_{n-1})$ on the n coordinates, let $\mathcal{F}$ be the Borel σ-algebra, and T be the shift. Notice that the partition P_0 into n sets at the identity is a generator for this action, so

$$h(T) = h(T, P) = -\sum_{i=0}^{n-1} p_i \log p_i.$$

There is a natural equivalence relation, which we call r equivalence, that arises when considering group actions of this group.

Definition. Suppose T and S are two $\sum \mathbb{Z}/r_n\mathbb{Z}$ actions. We say T is r equivalent to S if there is a one-to-one measurable measure-preserving map $\phi : X \to Y$ such that $\{\phi(T_g x)\}_{g \in G_i} = \{T_g \phi x\}_{g \in G_i}$ for every i. In this case, we write $T \underset{r}{\sim} S$.

Dye's theorem says that any two ergodic G actions are orbit equivalent, since G is a countable discrete amenable group. However, this equivalence relation asks for more than just orbit equivalence, since it asks to preserve each G_i orbit. This equivalence relation arises naturally from classifying certain decreasing sequences of σ-algebras up to isomorphism, which is an area first introduced by Vershik. The basic setup is as follows: a sequence of non-atomic σ-algebras $\mathcal{F} = \mathcal{F}_0 \supset \mathcal{F}_1 \cdots$ is called r-adic if the fibers of $\mathcal{F}_n$ over $\mathcal{F}_{n+1}$ all contain r_n points of equal mass. Two of these sequences are isomorphic if there is a one-to-one measure-preserving map between the two spaces that takes the nth σ-algebras to each other. If for each n we choose measurable sections of $\mathcal{F}_n | \mathcal{F}_{n+1}$ this gives an action of G. Likewise, given any G action such a sequence can be defined by looking at the σ-algebras of G_n invariant sets. From the definitions it follows that isomorphism between r-adic σ-algebras is the same as the notion of r equivalence between two G actions. A sequence is standard, which is terminology due to Vershik, if it is isomorphic to the sequence of σ-algebras generated by the G_i invariant sets of the Kronecker action. Vershik proves in [**V1**] that a certain entropy which he defines for sequences of σ-algebras is an invariant for isomorphism of dyadic sequences, which is the case when $r_n = 2\ \forall n$. He also shows that for the case of Bernoulli dyadic actions, the entropy that he defines coincides with the regular entropy. In the same volume of the same journal Stepin proves that entropy is an invariant for dyadic equivalence of dyadic actions and that Vershik's entropy corresponds to the entropy of an arbitrary G action [**S1**]. Later, Vershik shows in [**V2**] that in the case when r does not grow too fast, entropy is an invariant for the corresponding r equivalence. A proof of this can also be found in [**S2**]. In particular, this condition on r encompasses the previous case of dyadic equivalence. On the other hand, Vershik proved that there exist sequences r such that entropy is not an invariant for r equivalence [**V3**]. This naturally leads to the problem of determining when entropy is an invariant.

1.2. *Restricted orbit equivalence.* This section provides a brief overview of some basic definitions of restricted orbit equivalence theory. Let $(X, \mathcal{F}, \mu)$ be a non-atomic Lebesgue probability space and let $G = \sum_{n=1}^{\infty} \mathbb{Z}/r_n\mathbb{Z}$. Let $\mathcal{O} \subset X \times X$ be the orbit

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relation associated with some free, ergodic measure-preserving G action $\{T_g\}_{g\in G}$. In other words, $\mathcal{O} = \{(x, T_g x) : x \in X, g \in G\}$.

Definition. A *G arrangement* $\alpha : \mathcal{O} \to G$ is any map that satisfies the following:
(1) For a.e. x and every $g \in G$ there is an x' such that $\alpha(x, x') = g$. We write $T_g^\alpha x = x'$.
(2) For all $A \in \mathcal{F}$ and $g \in G$, both $T_g^\alpha A \in \mathcal{F}$ and $\mu(T_g^\alpha A) = \mu(A)$.
(3) α satisfies the cocycle equation $\alpha(x_2, x_3)\alpha(x_1, x_2) = \alpha(x_1, x_3)$.

Notice that α is a G arrangement iff there is a measure-preserving free ergodic G action $\{T_g^\alpha\}$ whose orbit relation is $\mathcal{O}$ and $\alpha(x, T_g^\alpha x) = g \ \forall (x, T_g^\alpha x) \in \mathcal{O}$.

Definition. $FG(\mathcal{O}) = \{\phi : X \to X \mid \phi$ is one-to-one, measurable, and for μ a.e. $x, (x, \phi x) \in \mathcal{O}\}$. This is the full group of $\mathcal{O}$.

Definition. A *G rearrangement* of $\mathcal{O}$ is a pair (α, ϕ) where α is a G arrangement and $\phi \in FG(\mathcal{O})$. We write $\alpha\phi(x, x') = \alpha(\phi x, \phi x')$.

Notice that $T_g^{\alpha\phi} x = \phi^{-1} T_g^\alpha \phi x$. Define a metric on $FG(\mathcal{O})$, called the *strong* L_1 *distance*, given by $\|\phi_1, \phi_2\|_s = \mu\{x : \phi_1 x \neq \phi_2 x\}$. Set $m^1(\alpha, \phi) = \|\phi, \mathrm{id}\|_s$. Since this does not depend on α, it can be written simply as $m^1(\phi)$.

Now define a metric on the invertible elements of G^G as follows. Place an ordering on the elements of G which respects the subgroup structure of the group. Let

$$d(f, h) = \sum_{g_i \in G} \frac{d_0(f(g_i), h(g_i)) + d_0(f^{-1}(g_i), h^{-1}(g_i))}{2^{i+1}},$$

where d_0 is the 0, 1 valued metric on G. Define $h_x^{\alpha,\beta}(g) = \alpha(x, T_g^\beta x)$ and notice that this function is invertible. Let

$$\|\alpha, \beta\| = \int d(h_x^{\alpha,\beta}, \mathrm{id})\, d\mu$$

and

$$\|\phi, \psi\|_w^\alpha = \int d(h_x^{\alpha,\alpha\phi}, h_x^{\alpha,\alpha\psi})\, d\mu$$

be the *weak* L_1 *distance* on arrangements and on $FG(\mathcal{O})$, repectively. Set $m^0(\alpha, \phi) = \|\phi, \mathrm{id}\|_w^\alpha$. This topology was introduced in Halmos' *Ergodic Theory* [**H**].

For the last notion of distance, let $f_x^{\alpha,\phi} \in G^G$ be defined by $f_x^{\alpha,\phi}(g) = \alpha(T_g^\alpha x, \phi(T_g^\alpha x))$. This gives rise to a partition $f^{\alpha,\phi}$ of X with functions in G^G as the labels. Let D be a metric on $\mathcal{M}(G^G)$ which induces the weak * topology. Define the *distribution distance* between two rearrangements to be

$$\|(\alpha, \phi), (\beta, \psi)\|_* = D((f^{\alpha,\phi})^*\mu, (f^{\beta,\psi})^*\mu).$$

Remark. In the definition of distribution distance we can choose D so that if $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta$ and $i = [1/\delta]$, then there is a set B, $\mu(B) > 1-\delta$, and a measure-preserving map $\pi : B \to \pi B$ such that if $x \in B$ and $g \in G_i$ then $f_x^{\alpha,\phi}(g) = f_{\pi x}^{\beta,\psi}(g)$. Later it is shown that π can be chosen so that it is G_i equivariant on a G_i invariant set.

Definition. A *size* is a function $m : \{(\alpha, \phi)\} \to \mathbb{R}^+$ such that if we write $m_\alpha(\phi, \psi) \underset{\text{def}}{=} m(\alpha\phi, \phi^{-1}\psi)$ then m satisfies the following three axioms.

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AXIOM 1. *m_α is a pseudometric on $FG(\mathcal{O})$.*

AXIOM 2. *m lies between m^0 and m^1 in the following sense:*
(1) *$\forall \epsilon > 0$, $\exists \delta > 0$ such that if $m^1(\alpha, \phi) < \delta$ then $m(\alpha, \phi) < \epsilon$;*
(2) *$\forall \epsilon > 0$, $\exists \delta > 0$ such that if $m(\alpha, \phi) < \delta$, then $m^0(\alpha, \phi) < \epsilon$.*

AXIOM 3. *For every $\epsilon > 0$ there is a $\delta = \delta(\epsilon, \alpha, \phi) > 0$ such that if (β, ψ) is another rearrangement and $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta$, then $m(\beta, \psi) < m(\alpha, \phi) + \epsilon$.*

Definition. α is *m-equivalent* to β if there exists a sequence $\{\phi_i\} \subset FG(\mathcal{O})$ such that the following are true:
(1) $\{\phi_i\}$ is m_α cauchy;
(2) $\{\phi_i^{-1}\}$ is m_β cauchy;
(3) $\|\alpha\phi_i, \beta\| \to 0$.
In this case, we write $\alpha \underset{m}{\sim} \beta$.

Remark. m^0 and m^1 are sizes which give rise to orbit equivalence and isomorphism, respectively [**KR**].

Define

$$h_m(T^\alpha) = \inf_{\alpha \underset{m}{\sim} \beta} h(T^\beta).$$

THEOREM. ([**KR**]) *Given any size m, either*
(1) *$h_m(T^\alpha) = 0$ for all T or*
(2) *$h_m(T^\alpha) = h(T^\alpha)$ for all T.*

In the first case m is called entropy free, and in the second case m is called entropy preserving. We also have the following result.

THEOREM. ([**KR**]) *When m is entropy free, the zero entropy actions are a dense G_δ set in every m equivalence class.*

The immediate goal is to show that there exists a size that corresponds to r equivalence, since then r.o.e. theory can be applied to determine whether a given sequence is entropy free or entropy preserving. §2.1 is devoted to proving that a certain function on G rearrangements is a size. Then in §2.2 the link between r equivalence and the equivalence relation generated by this size is established.

2. *r equivalence is a restricted orbit equivalence*

2.1. *Defining a size.*

Fix $r = (r_1, r_2, \ldots)$. Recall $G_i = \sum_{n=1}^{i} \mathbb{Z}/r_n\mathbb{Z}$.

Definition. We say (α, ϕ) is $\{G_i\}_{i=1}^\infty$ compatible on $B \subset X$ if whenever $(x, x') \in \mathcal{O}$ and $x, x' \in B$, then $\alpha(x, x') \in G_i \Leftrightarrow \alpha\phi(x, x') \in G_i$ for all i.

Unless otherwise indicated $\{G_i\}$ will mean $\{G_i\}_{i=1}^\infty$.

Definition. $c_r(\alpha, \phi) = 1 - \sup\{\mu(B) \mid (\alpha, \phi)$ is $\{G_i\}$ compatible on $B\}$.

Set $m^r(\alpha, \phi) = c_r(\alpha, \phi) + m^0(\alpha, \phi)$.

THEOREM. *m^r is a size.*

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For convenience, once r is fixed we can drop the subscript. Let $m(\alpha, \phi) = m^r(\alpha, \phi)$, and $c(\alpha, \phi) = c_r(\alpha, \phi)$. Define $c_\alpha(\phi, \psi) = c(\alpha\phi, \phi^{-1}\psi)$. Let

$$m_\alpha(\phi, \psi) = c_\alpha(\phi, \psi) + \|\phi, \psi\|_w^\alpha.$$

AXIOM 1. *m_α is a psuedometric on $FG(\mathcal{O})$.*

Proof. (1) $m_\alpha(\phi, \psi) \geq 0$ and if $\phi = \psi$ then $m_\alpha(\phi, \psi) = 0$. This is true by definition.

(2) $m_\alpha(\phi, \psi) = m_\alpha(\psi, \phi)$. First notice that $c_\alpha(\phi, \psi) = c_\alpha(\psi, \phi)$ since $(\alpha\phi, \phi^{-1}\psi)$ is $\{G_i\}$ compatible on a set B iff $(\alpha\psi, \psi^{-1}\phi)$ is $\{G_i\}$ compatible on B. Furthermore, $\|\phi, \psi\|_\alpha^w = \|\psi, \phi\|_\alpha^w$ since m^0 is a size.

(3) $m_\alpha(\phi, \psi) \leq m_\alpha(\phi, \theta) + m_\alpha(\theta, \psi)$. Since m^0 is a size, it suffices to show that $c_\alpha(\phi, \psi) \leq c_\alpha(\phi, \theta) + c_\alpha(\theta, \psi)$. Fix $\delta > 0$. Let A be a set for which $(\alpha\phi, \phi^{-1}\theta)$ is $\{G_i\}$ compatible, with $1 - \mu A < c_\alpha(\phi, \theta) + \delta$. Let B be a set on which $(\alpha\theta, \theta^{-1}\psi)$ is $\{G_i\}$ compatible, with $1 - \mu B < c_\alpha(\theta, \psi) + \delta$. Then $(\alpha\phi, \phi^{-1}\psi)$ is $\{G_i\}$ compatible on $A \cap B$. Hence

$$\begin{aligned} c_\alpha(\phi, \psi) &\leq 1 - \mu(A \cap B) = \mu(A^c \cup B^c) \\ &\leq 1 - \mu A + 1 - \mu B \leq c_\alpha(\phi, \theta) + c_\alpha(\theta, \psi) + 2\delta. \end{aligned}$$

Since δ was arbitrary, the result follows. □

AXIOM 2.

(1) $\forall \epsilon > 0$, $\exists \delta > 0$ *such that if* $m^1(\alpha, \phi) < \delta$ *then* $m(\alpha, \phi) < \epsilon$.

(2) $\forall \epsilon > 0$, $\exists \delta > 0$ *such that if* $m(\alpha, \phi) < \delta$ *then* $m^0(\alpha, \phi) < \epsilon$.

Proof. (1) m^0 is weaker than m^1, so it suffices to show that $m^1(\alpha, \phi) < \delta \Rightarrow c(\alpha, \phi) < \delta$. Notice that (α, ϕ) is $\{G_i\}$ compatible on $\{x \mid \phi x = x\}$. Hence $c(\alpha, \phi) < \delta$.

(2) By definition, $m^0(\alpha, \phi) \leqslant m(\alpha, \phi) < \delta$.

AXIOM 3. *$\forall \epsilon > 0$ and (α, ϕ), $\exists \delta > 0$ such that if (β, ψ) is a rearrangement on $\mathcal{O}'$ and $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta$, then $m(\beta, \psi) < m(\alpha, \phi) + \epsilon$.*

Since $m^0(\alpha, \phi)$ is a size, it suffices to show that given $\epsilon > 0$ and (α, ϕ), $\exists \delta$ so that if $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta$, then $c(\beta, \psi) < c(\alpha, \phi) + \epsilon$. In order to do this some lemmas are established.

Remark. If $f_x^{\alpha,\phi} = f_y^{\beta,\psi}$ on G_i then $f_{x'}^{\alpha,\phi} = f_{y'}^{\beta,\psi}$ on G_i, where $\alpha(x, x') = \beta(y, y') \in G_i$.

Remark. Given any i and α there exists a set F such that $X = \cup_{g \in G_i} T_g^\alpha F$ and $\{T_g^\alpha F\}_{g \in G_i}$ are mutually disjoint. This follows from the fact that we can pick a measurable section of the factor map associated to the σ-algebra of G_i invariant sets.

Definition. Fix arrangements (α, ϕ) and (β, ψ) on X. We say that a map $\pi : X \to X$ is *suitable* with respect to G_i on a set D if π is one-to-one, measure preserving, $\forall x \in D$, $f_x^{\alpha,\phi} = f_{\pi x}^{\beta,\psi}$ on G_i, D is G_i invariant, and $\forall x \in D$ and $\forall g \in G_i$, $\pi T_g^\alpha x = T_g^\beta \pi x$.

LEMMA. *Fix (α, ϕ) and (β, ψ) on X. If $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta < 1$ and $i = [1/\delta]$, then there exists a map π which is suitable with respect to G_i on a set D of positive measure.*

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Proof. Let $X = \cup_{g\in G_i} T_g^\alpha F_x'$, where $\{T_g^\alpha F_x'\}$ are mutually disjoint, and $X = \cup_{g\in G_i} T_g^\beta F_y'$, where $\{T_g^\beta F_y'\}$ are mutually disjoint. Since the distribution distance is less than δ, this guarantees that there are sets $F_x, F_y \subset X$ such that:
(1) $\mu F_x = \mu F_y > 0$;
(2) F_x and F_y are contained in a single level of their respective towers;
(3) $\forall x \in F_x$, $\forall y \in F_y$, and $\forall g \in G_i$, $f_x^{\alpha,\alpha\phi}(g) = f_y^{\beta,\beta\psi}(g)$.
Define $\pi : F_x \to F_y$ to be any one-to-one measure-preserving map. Now define π on $D = \cup_{g\in G_i} T_g^\alpha(F_x)$ by $\pi T_g^\alpha x = T_g^\beta \pi y$ $\forall g \in G_i$. With this definition, we see that $\mu D > 0$, D is G_i invariant and π is G_i equivariant on D. To see that $f_x^{\alpha,\phi} = f_{\pi x}^{\beta,\psi}$ on G_i, notice that for $x \in F_x$ this is true by definition. Now apply the previous remark to get $\forall g \in G_i$,

$$f_{T_g^\alpha x}^{\alpha,\phi}(g) = f_{T_g^\beta \pi x}^{\beta,\psi}(g).$$

Let $\pi|_{D^c}$ be any one-to-one measure-preserving map. □

LEMMA. *Suppose* (α, ϕ) *and* (β, ψ) *are rearrangements on* X. *If* $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta < 1$ *and* $i = [1/\delta]$, *then there is a map* π *which is suitable with respect to* G_i *on a set* D, *and* $\mu D > 1 - \delta$.

Proof. Consider $\Pi_i = \{\hat{\pi} : \hat{\pi}$ is suitable with respect to G_i on a set D of positive measure$\}$. This set is non-empty by the previous lemma. We can put a partial ordering on this set as follows: we say that $\hat{\pi}_1 < \hat{\pi}_2$ if $D_1 \subset D_2$ and $\hat{\pi}_2|_{D_1} = \hat{\pi}_1|_{D_1}$. Let π with domain D be a maximal element of this collection. Suppose $\mu D \le 1 - \delta$. Then $\mu D^c \ge \delta$ and D^c is G_i invariant. By the same reasoning as in the previous lemma, we can find a set of positive measure in D^c on which π is suitable with respect to G_i. This extends π, which contradicts maximality. Hence $\mu D > 1 - \delta$. □

Proof (Axiom 3). We show that given (α, ϕ) and $\epsilon > 0$, $\exists \delta > 0$ such that $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta \Rightarrow c(\beta, \psi) < c(\alpha, \phi) + \epsilon$.

Pick I such that $\mu\{x \mid \alpha(x, \phi x) \in G_I\} > 1 - \epsilon/8$. Let $\delta = \min(\epsilon/8, 1/(I + 1))$, and set $E = \{y \mid \beta(y, \psi y) \in G_I\}$. Notice that $\mu E > 1 - \epsilon/4$, and if $y, y' \in E$ then $\beta(y, y') \in G_i \Leftrightarrow \beta\psi(y, y') \in G_i$ for $i \ge I$. Let $\tilde{B}_x$ be a set such that (α, ϕ) is $\{G_i\}$ compatible on $\tilde{B}_x$, with $1 - \mu\tilde{B}_x < c(\alpha, \phi) + \delta$. Since $\|(\alpha, \phi), (\beta, \psi)\|_* < \delta$, apply the previous lemma to get a map π which is suitable with respect to G_I on a set D_x, and $\mu D_x > 1 - \delta$. Set $B_y = E \cap \pi(D_x \cap \tilde{B}_x)$. First show that if $y, y' \in B_y$, and $\beta(y, y') \in G_i$, then $\beta\psi(y, y') \in G_i$ for $i \le I$. Set $\pi x = y$ and $\pi x' = y'$. Notice that $\alpha(x, \phi x) = \beta(y, \psi y)$ and $\alpha(x', \phi x') = \beta(y', \psi y')$ since $x, x' \in D_x$. By the equivariance of π we have $\alpha(x, x') = \beta(y, y')$. Applying the cocycle equation yields $\alpha\phi(x, x') = \beta\psi(y, y')$. Since $x, x' \in B_x$ and $\alpha(x, x') \in G_i$, this implies that $\beta\psi(y, y') = \alpha\phi(x, x') \in G_i$.

Now it suffices to show that if $y, y' \in B_y$, and $\beta\psi(y, y') \in G_i$ then $\beta(y, y') \in G_i$ for all $i \le I$. The problem in this direction is that π does not commute with $T^{\beta\psi}$. But if $y' = T_g^{\beta\psi} y$ for some $g \in G_i$, then $y' = T_{g_0}^\beta y$ for some $g_0 \in G_I$:

$$\beta(y, y') = \beta(\psi y', y')\beta(\psi y, \psi y')\beta(y, \psi y).$$

All of the terms on the right-hand side are in G_I since $y, y' \in E$. Hence $\beta(y, y') = g_0 \in G_I$. By the equivariance of π on D_x we have that $\alpha(x, x') = g_0$, where $\pi x = y$ and

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$\pi x' = y'$. Again we have that $\alpha(x, \phi x) = \beta(y, \psi y)$ and $\alpha(x', \phi x') = \beta(y', \psi y')$, so we apply the cocycle equation to obtain that $\alpha\phi(x, x') = \beta\psi(y, y') \in G_i$. Now $x, x' \in \tilde{B}_x$, and $\alpha\phi(x, x') \in G_i \Rightarrow \alpha(x, x') \in G_i$. Hence $\beta(y, y') \in G_i$, which shows that (β, ψ) is $\{G_i\}$ compatible on B_y. Furthermore,

$$\begin{aligned} c(\beta, \psi) &\leq 1 - \mu B_y \leq 1 - \mu(E \cap \pi(D_x \cap \tilde{B}_x)) \\ &< 1 - \mu\tilde{B}_x + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \frac{\epsilon}{8} < c(\alpha, \phi) + \epsilon. \end{aligned}$$ □

2.2. *Relating m and r equivalence.* The first step in exhibiting the relation between m^r equivalence and r equivalence is the flattening lemma. This says that an arrangement (α, ϕ) can be altered by an element ψ of $FG(\mathcal{O})$ with $m^1(\psi) \leq c(\alpha, \phi)$, so that $c(\alpha, \phi\psi) = 0$. This will help prove a theorem that gives a characterization of m^r equivalence. The link between m^r and r equivalence will arise as a corollary of this theorem.

Choose B to be a set so that (α, ϕ) is $\{G_i\}$ compatible on B, $\mu B > 0$. Define $N_i = \{x' \in B^c \mid \exists x \in B$ such that $\alpha(x, x') \in G_i$ and $\nexists y \in B$ such that $\alpha(y, x') \in G_{i-1}\}$. Notice that:

(1) $\{N_i\}$, B are mutually disjoint;
(2) $B_i = B \cup_{j=1}^{i} N_j$ is G_i invariant;
(3) $\mu(X \setminus \cup_{i=1}^{\infty} B_i) = 0$ (by ergodicity).

LEMMA. *There exists a measurable function $\rho : X \to B$ with the property that if $x', y' \in N_i$ and $\alpha(x', y') \in G_i$, then $\rho(x') = \rho(y')$.*

Proof. Let $\rho(x) = x$ if $x \in B$. Define $\rho_i : N_i \to B$ as follows. Consider $\mathcal{I}_i = \sigma$-algebra of G_i invariant sets and let $\pi_i : X \to Y_i$ be the associated factor map. Take a measurable section $\hat{\rho}_i$ such that $\hat{\rho}_i \circ \pi_i/_{N_i} \in B$. Let $\rho_i = \hat{\rho}_i \circ \pi_i/_{N_i}$. Define $\rho(x) = \rho_i(x)$ if $x \in N_i$. Now ρ is defined on a.a. of X with the desired property. □

FLATTENING LEMMA. *If $c(\alpha, \phi) < \delta < 1$, then $\exists\psi \in FG(\mathcal{O})$ such that $m^1(\psi) < \delta$ and $c(\alpha, \phi\psi) = 0$.*

Proof. Since $c(\alpha, \phi) > 1 - \delta$, we can choose B with $\mu(B) > 1 - \delta$ and (α, ϕ) is $\{G_i\}$ compatible on B. Let $\psi(x) = x$ $\forall x \in B$. On B^c, partition each N_i into sets as follows. Let ρ be the function from the previous lemma, fix x' and set $\rho x' = x$. Let

$$H_i(x') = G_i \setminus \{\alpha\phi\psi(x, y) : y \in \mathcal{O}_{G_i}(x), y \in B_{i-1}\}$$

and let

$$H_i'(x') = \{\alpha(x, y) : y \in N_i, y \in \mathcal{O}_{G_i}(x)\}.$$

Notice that $\#H_i(x') = \#H_i'(x')$ and this defines a measurable partition $\mathcal{P}$ of N_i based on the H_i, H_i' name of x'. Define $\Phi_{x'}^i : H_i'(x') \to H_i(x')$ to be a map with the following properties:

(1) $\Phi_{x'}^i$ is one-to-one;
(2) $\Phi_{x'}^i$ is constant on elements of $\mathcal{P}$ (hence $\rho(x') = \rho(y) \Rightarrow \Phi_{x'}^i = \Phi_y^i$);
(3) $\Phi_{x'}^i$ only changes the ith coordinate of any group element;

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(4) if g, h are in the same G_{i-1} coset then so are $\Phi^i_{x'}(g)$, $\Phi^i_{x'}(h)$.

Now for $x' \in p \in \mathcal{P}$, if $\alpha(x, x') = g$ define

$$\phi\psi(x') = T^{\alpha}_{\alpha(x,\phi x)} T^{\alpha}_{\Phi^i_{x'}(g)} x = T^{\alpha}_{\Phi^i_{x'}(g)} \phi x.$$

This is measurable since we have a composition of measurable functions on measurable sets. The proof that this is one-to-one and that $c(\alpha, \phi\psi) = 0$ is by induction on the sets B_i. The base case for our induction is $B_0 = B$. $\phi\psi$ is one-to-one on B, and $(\alpha, \phi\psi)$ is $\{G_i\}$ compatible on B since it was chosen that way. Now assume that $\phi\psi$ is one-to-one on B_j and $(\alpha, \phi\psi)$ is $\{G_i\}$ compatible on B_j for all $j < i$. It is necessary to show these properties for $\phi\psi$ on B_i. The (one-to-one)-ness proof breaks into four cases. Fix $x' \in N_i$.

(1) $y \in N_i$ and $\alpha(y, x') \in G_{i-1}$. Since $\rho x' = \rho y = x$ we have that $\Phi^i_{x'} = \Phi^i_y$. But $y \neq x' \Rightarrow \Phi^i_{x'}(\alpha(x, y)) \neq \Phi^i_{x'}(\alpha(x, x'))$ so $\phi\psi(x') \neq \phi\psi(y)$.

(2) $y \in N_i$ and $\alpha(y, x') \in G_i \setminus G_{i-1}$. Since $\rho x' = \rho y = x$, we have that

$$T^{\alpha}_{\Phi^i_{x'}(\alpha(x,y))} \phi x = T^{\alpha}_{\Phi^i_{x'}(\alpha(x,x'))} \phi x \Leftrightarrow \alpha(x, y) = \alpha(x, x').$$

Hence $\phi\psi(x') \neq \phi\psi(y)$.

(3) $y \in B_{i-1}$ and $\alpha(x', y) \in G_i \setminus G_{i-1}$. Notice that $\alpha(x', y) \in G_i \Rightarrow \alpha(x, y) \in G_i \Rightarrow \alpha\phi\psi(x, y) \in G_i \setminus H_i(x')$. But $\alpha\phi\psi(x, x') = \Phi^i_{x'}(\alpha(x, x')) \in H_i(x')$. Hence $\alpha\phi\psi(x, y) \neq \alpha\phi\psi(x, x')$.

(4) $y \in B_i$ and $\alpha(x', y') \in G_j \setminus G_i$ for some j. Let $x = \rho x'$ and $y = \rho y'$. By the cocycle equation we have $\alpha(x, y) \in G_j \setminus G_i$. But $x, y \in B \Rightarrow \alpha\phi\psi(x, y) \in G_j \setminus G_i$. Hence $\alpha\phi\psi(x', y') \in G_j \setminus G_i$.

Thus ψ is one-to-one on B_i.

Also, $(\alpha, \phi\psi)$ is $\{G_i\}$ compatible on B_i. It suffices to show that if $y, x' \in B_i$ and $x' \in \mathcal{O}(y)$ then $\alpha(y, x') \in G_j \setminus G_{j-1} \Rightarrow \alpha\phi\psi(y, x') \in G_j \setminus G_{j-1}$. Fix $x' \in N_i$, $\rho(x') = x$, $\rho(y') = y$ and consider four cases.

(1) $y' \in N_i$ and $\alpha(y', x') \in G_{i-1}$. Pick k so that $\alpha(y', x') \in G_k \setminus G_{k-1}$.

$$\alpha\phi\psi(y, x') = \Phi^i_{x'}(\alpha(x, x'))\Phi^i_{x'}(\alpha(x, y'))^{-1} \in G_k.$$

Let $g'_k = k$th coordinate of $\alpha(x, y')$, and $g_k = k$th coordinate of $\alpha(x, x')$. Since $\Phi^i_{x'}$ acts as the identity on these coordinates and by hypothesis $g'_k g_k^{-1} \neq \mathrm{id}$, we have $\alpha\phi\psi(x', y') \notin G_k$.

(2) $y' \in N_i$ and $\alpha(y', x') \in G_i \setminus G_{i-1}$. Since $\alpha(x, x'), \alpha(x, y')$ are in different G_{i-1} cosets, $\Phi^i_{x'}$ maps them to different cosets, hence $\alpha\phi\psi(y', x') \in G_i \setminus G_{i-1}$.

(3) $y' \in B_{i-1}$ and $\alpha(y', x') \in G_i$.

$$\alpha\phi\psi(y', x') = \alpha\phi\psi(x, x')\alpha\phi\psi(y', x) \in G_i.$$

Also, $\alpha\phi\psi(x, x') = \Phi^i_{x'}(\alpha(x, x')) \in G_{i-1}(x')$ and $\alpha\phi\psi(x, y') \in G_i \setminus G_{i-1}(x')$. Hence $\alpha\phi\psi(y', x') \notin G_{i-1}$.

(4) $y' \in B_i$ and $\alpha(y', x') \in G_j \setminus G_i$. Pick k such that $\alpha(y', x') \in G_k \setminus G_{k-1}$. Notice that

$$\alpha(y', x') \in G_k \setminus G_{k-1} \Rightarrow \alpha(y, x) \in G_k \setminus G_{k-1} \Rightarrow$$

$$\alpha\phi\psi(y, x) \in G_k \setminus G_{k-1} \Rightarrow \alpha\phi\psi(y', x') \in G_k \setminus G_{k-1}. \qquad \square$$

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Definition. We say $\alpha \overset{r}{\sim} \beta$ if whenever $x \in \mathcal{O}(x')$, $\alpha(x, x') \in G_i \Leftrightarrow \beta(x, x') \in G_i$ for every i.

THEOREM. $\alpha \underset{m^r}{\sim} \beta \Leftrightarrow \exists \phi \in FG(\mathcal{O})$ *such that* $\alpha\phi \overset{r}{\sim} \beta$.

Proof. First it is shown that if $\alpha \underset{m^r}{\sim} \beta$ then $\exists \phi \in FG(\mathcal{O})$ such that $\alpha\phi \overset{r}{\sim} \beta$. Fix r and let $m = m^r$. Since α is m-equivalent to β, there exists a sequence $\{\phi_i\} \subset FG(\mathcal{O})$ such that $\{\phi_i\}$ is m-cauchy and $\|\alpha\phi_i, \beta\| \to 0$. Choose ϕ_i so that $m(\alpha\phi_i, \phi_i^{-1}\phi_{i+1}) = m_\alpha(\phi_i, \phi_{i+1}) < 1/2^{i+1}$. By induction it is shown that a sequence of reorderings $\{\psi_i\}$ can be found such that

$$\alpha\phi_1\psi_1 \overset{r}{\sim} \alpha\phi_2\psi_2 \overset{r}{\sim} \cdots \overset{r}{\sim} \alpha\phi_n\psi_n \overset{r}{\sim} \cdots$$

with $m^1(\psi_n, \psi_{n+1}) < 1/2^{n+1}$. For the base case of the induction, set $\psi_1 = \mathrm{id}$ and find ψ_2 so that $\alpha\phi_1 \overset{r}{\sim} \alpha\phi_2\psi_2$ and $m^1(\psi_2) < \frac{1}{4}$. Since $c(\alpha\phi_1, \phi_1^{-1}\phi_2) < \frac{1}{4}$, the flattening lemma gives a $\hat{\psi}_2 \in FG(\mathcal{O})$ with $m^1(\hat{\psi}_2) < \frac{1}{4}$ and $\alpha\phi_1 \overset{r}{\sim} \alpha\phi_2\hat{\psi}_2$. Set $\psi_2 = \hat{\psi}_2$, which finishes the base case. Now assume that there exists ψ_n with these properties for $n < k$, and show there exists ψ_k with the desired properties. In order to apply the previous lemma to this case notice that

$$m_\alpha(\phi_{k-1}\psi_{k-1}, \phi_k\psi_{k-1}) = m_\alpha(\phi_{k-1}, \phi_k) < \frac{1}{2^k}.$$

Hence $\exists \hat{\psi}_k \in FG(\mathcal{O})$ such that $m^1(\hat{\psi}_k) < 1/2^k$ and

$$\alpha\phi_{k-1}\psi_{k-1} \overset{r}{\sim} \alpha\phi_k\psi_{k-1}\hat{\psi}_k.$$

Setting $\psi_k = \psi_{k-1}\hat{\psi}_k$ yields a map with the desired properties.

CLAIM. $\exists \psi \in FG(\mathcal{O})$ *such that* $m^1(\psi_i, \psi) \to 0$.

Proof. Let $A_i = \{x : \hat{\psi}_i \neq \mathrm{id}\}$. Then $\sum_{i=1}^{\infty} \mu A_i < \frac{1}{2} < \infty$. By the Borel–Cantelli lemma $\mu\{x : x \in A_i \text{ for infinitely many } i\text{'s}\} = 0$. Set $\psi(x) = \psi_n(x)$, where $n = n(x) = \max\{i : x \in A_i\}$. This defines ψ on almost all of X. ψ is one-to-one since $\psi(x) = \psi(y) \Rightarrow \psi_n(x) = \psi_n(y)$ for some n, which cannot happen since each ψ_n is one-to-one. Furthermore, ψ is onto because it is measure preserving. Thus $\psi \in FG(\mathcal{O})$. Furthermore, $\mu\{x \mid \psi_i \neq \psi\} \to 0$. □

The previous claim implies that $\|\beta\psi_n, \beta\psi\| \to 0$. It is also clear that $\|\alpha\phi_n\psi_n, \beta\psi_n\| = \|\alpha\phi_n, \beta\| \to 0$. Hence $\|\alpha\phi_n\psi_n, \beta\psi\| \to 0$. This fact combined with the fact that $\alpha\phi_n\psi_n \overset{r}{\sim} \alpha\phi_m\psi_m$ for all m, n shows that $\alpha\phi_n\psi_n \overset{r}{\sim} \beta\psi$ for all n. Setting $\phi = \phi_1\psi^{-1}$ implies that $\alpha\phi \overset{r}{\sim} \beta$.

To show that if there is $\phi \in FG(\mathcal{O})$ such that if $\alpha\phi \overset{r}{\sim} \beta$, then $\alpha \underset{m^r}{\sim} \beta$, it suffices to show that if $\alpha \overset{r}{\sim} \beta$ then $\alpha \underset{m^r}{\sim} \beta$, since α is m^r equivalent to $\alpha\phi$. Find a set F such that $X = \cup_{g \in G_i} T_g^\alpha F$ and the $T_g^\alpha F$ are mutually disjoint. Define $\phi_i(x) = x$ if $x \in F$ and $\phi_i(T_g^\alpha x) = T^\alpha_{\beta(x, T_g^\alpha x)} x$ and verify that the properties in the definition of m equivalence hold for this sequence.

(1) $\|\alpha\phi_i, \beta\| \to 0$. It suffices to show that for every pair x, x' such that $x' \in \mathcal{O}_{G_i}(x)$ and $x \in F$, $\alpha\phi_i(x, x') = \beta(x, x')$. But

$$\alpha\phi_i(x, x') = \alpha(x, \phi_i x') = \alpha(x, T^\alpha_{\beta(x,x')} x) = \beta(x, x').$$

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(2) $c_\alpha(\phi_i, \phi_j) = 0\ \forall i, j$. It is clear by construction that for all i, $c_\alpha(\mathrm{id}, \phi_i) = 0$.
(3) $c_\beta(\phi_i^{-1}, \phi_j^{-1}) = 0\ \forall i, j$. $\phi_i^{-1}(x) = x$ if $x \in F$ and $\phi_i^{-1}(T_h^\beta x) = T^\beta_{\alpha(x, T_h^\beta x)} x$ for $h \in G_i$. Again we have $\forall i$, $\beta\phi_i^{-1} \overset{r}{\sim} \beta$. □

Let α be an arrangement on X associated with T^α and let β be an arrangement on Y associated with S^β. Define $OE(\mathcal{O}, \mathcal{O}') = \{\phi : X \to Y \mid \phi$ is one-to-one, measurable, and $\phi\mathcal{O} = \mathcal{O}'\}$. We obtain as a corollary from the previous theorem that r equivalence is the same as m equivalence for this m.

COROLLARY. *$\alpha \underset{m^r}{\sim} \beta\phi$ for some $\phi \in OE(\mathcal{O}, \mathcal{O}') \Leftrightarrow T^\alpha \underset{r}{\sim} S^\beta$.*

Proof. It suffices to show that $T^\alpha \underset{r}{\sim} S^\beta$ iff there is a $\phi \in OE(\mathcal{O}, \mathcal{O}')$ such that $\alpha \overset{r}{\sim} \beta\phi$.

($\Leftarrow$) If $\exists\phi \in OE(\mathcal{O}, \mathcal{O}')$ such that $\alpha \overset{r}{\sim} \beta\phi$, then $\alpha(x, x') \in G_i \Leftrightarrow \beta(\phi x, \phi x') \in G_i$. This implies that

$$\{\phi T_g^\alpha x\}_{g \in G_i} = \{S_g^\beta \phi x\}_{g \in G_i}$$

which tells us that $T^\alpha \underset{r}{\sim} S^\beta$.

($\Rightarrow$) If $T^\alpha \underset{r}{\sim} S^\beta$ then $\exists\phi \in OE(\mathcal{O}, \mathcal{O}')$ such that $\{\phi T_g^\alpha x\}_{g \in G_i} = \{S_g^\beta \phi x\}_{g \in G_i}$ for all i. But this implies that $\alpha(x, x') \in G_i \Leftrightarrow \beta(\phi x, \phi x') \in G_i$ for all i, which implies $\alpha \overset{r}{\sim} \beta\phi$. □

For a fixed sequence r define

$$h_r(T^\beta) = \inf_{S^\beta \underset{r}{\sim} T^\alpha} h(S^\beta).$$

A consequence of the previous theorem is the following.

COROLLARY. $h_r(T^\alpha) = h_{m^r}(T^\alpha)$.

In the case that m^r is entropy free or entropy preserving, we just say r is entropy free or entropy preserving, respectively.

3. *Entropy*

3.1. *Introduction.* This section gives a necessary and sufficient condition on the growth rate of r for entropy to be an invariant for r equivalence. More specifically, it is shown that

$$h_r(T) = h(T)\ \forall T \Leftrightarrow \sum_{k=1}^{\infty} \frac{\log r_{k+1}}{H_k} < \infty.$$

Vershik demonstrated that if $\sum_{k=1}^{\infty} \log(r_k!)/H_k < \infty$, then r is entropy preserving. The equivalence lemma proves that this condition is equivalent to the convergence of the previous growth rate condition, which is the one that Stepin used. §3.2 is devoted to reproving sufficiency, while §3.3 is devoted to proving necessity.

EQUIVALENCE LEMMA.

$$\sum_{k=1}^{\infty} \frac{\log r_k}{q_{k-1}} < \infty \quad \textit{iff} \quad \sum_{k=1}^{\infty} \frac{\log(r_k!)}{q_k} < \infty.$$

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Proof. ($\Rightarrow$) Notice that $r_k! < r_k^{r_k}$ so

$$\sum_{k=1}^{\infty} \frac{\log(r_k!)}{q_k} < \sum_{k=1}^{\infty} \frac{\log r_k^{r_k}}{q_k} = \sum_{k=1}^{\infty} \frac{\log r_k}{q_{k-1}} < \infty.$$

($\Leftarrow$) By Stirling's formula we have that $r_k! > (r_k)^{r_k - 1/2} e^{-r_k}$ so

$$\sum_{k=1}^{\infty} \left(\frac{\log r_k}{q_{k-1}} - \frac{\log r_k}{2q_k} - \frac{\log e}{q_{k-1}} \right) \leq \sum_{k=1}^{\infty} \frac{\log(r_k!)}{q_k} < \infty.$$

But the second two terms in the first sum are summable, which tells us that the first term in the sum must also be summable. □

3.2. *Reproving sufficiency of Vershik's entropy condition.* This section focuses on reproving the following theorem using r.o.e. theory.

THEOREM. ([**V3**]) *If* $\sum_{k=1}^{\infty} \log r_{k+1}/q_k < \infty$, *then* $h_r(T^\alpha) = h(T^\alpha)$ $\forall \alpha$.

The proof is by contradiction using a theorem from r.o.e. theory. Let $m = m^r$.

THEOREM. ([**KR**]) *Let* $c > 0$ *and* β *be any ordering such that* $h(T^\beta) > 0$. *Suppose* $h_r(T^\beta) < h(T^\beta) - c$. *Let* (T^α, P) *be any Bernoulli process with* $h(T^\alpha, P) = h(T^\beta)$. *For any* $\epsilon > 0$ $\exists \phi \in FG(\mathcal{O})$ *such that* $m(\alpha, \phi) < \epsilon$ *and* $h(T^{\alpha\phi}, P) < h(T^\alpha, P) - c$.

This theorem is one of the building blocks used in proving that r is either entropy free or entropy preserving. This theorem says that, in particular, if the r entropy can be lowered for some arrangement β, then the regular entropy can be lowered relative to a partition, by an element of $FG(\mathcal{O})$, on any Bernoulli process with equal entropy. Pick the following Bernoulli process. Let $h = h(T^\beta)$. Pick m such that $h < \log m$. By the continuity of the function $-\sum_{i=0}^{m-1} p_i \log p_i$, there is a probability vector $(p_0, \ldots, p_{m-1})$ on $\{0, \ldots, m-1\}$ such that $-\sum_{i=0}^{m-1} p_i \log p_i = h$. Also assume that each $p_i > 0$ by dropping to a smaller number of symbols if necessary. Let $X = \{0, \ldots, m-1\}^G$, T^α = shift map, μ = Bernoulli measure with probability vector $(p_0, \ldots, p_{m-1})$, and P = partition at the identity into m sets. Notice that $h(T^\alpha, P) = h$. The contradiction hypothesis is that $\exists \{\phi_i\} \subset FG(\mathcal{O})$ such that $m(\alpha, \phi_i) \to 0$ and $h(T^{\alpha\phi_i}, P) < h(T^\alpha, P) - c$. A further reduction can be made by only considering a specific type of element of the full group. The following lemma is needed to accomplish this.

LEMMA. *Suppose that* $\delta < q_i^{-1}$, $m^0(\alpha, \phi) < \delta^2$ *and* $c(\alpha, \phi) = 0$. *Then* $\exists \psi \in FG(\mathcal{O})$ *such that* $m^1(\psi) < \delta$, $c(\alpha, \phi\psi) = 0$, *and if* $\alpha(x, x') \in G_i$, *then* $\alpha(x, x') = \alpha\phi\psi(x, x')$.

Proof. If $d(h_x^{\alpha,\alpha\phi}, \mathrm{id}) < \delta$ then $h_x^{\alpha,\alpha\phi} = \mathrm{id}$ on G_i. Let $B = \{x \in X : h_x^{\alpha,\alpha\phi} = \mathrm{id} \text{ on } G_i\}$. By Chebychev's inequality, $\mu B > 1 - \delta$. Furthermore, by applying the cocycle equation, we can see that B is G_i invariant. Let $\psi = \mathrm{id}$ on B. Define ψ on B^c as follows. Since B^c is G_i invariant, build an $\alpha\phi, G_i$ tower with base F to partition B^c, and define ψ to be the identity on F. On the rest of the tower, if $x \in F$ and $\alpha\phi(x, x') \in G_i$ let $\alpha\phi\psi(x, x') = \alpha(x, x')$. This map has the desired properties. □

Remark. Notice that $h(T^{\alpha\phi}, P) = h(T^\alpha, \phi P)$.

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Remark. If $\epsilon < \frac{1}{2}$ and $m^1(\alpha, \phi) < \epsilon^2$, then $m^0(\alpha, \phi) < \epsilon$.

Proof. Pick $j = \max\{i \mid 2\epsilon < q_i^{-1}\}$. Let $E = \{x \mid \phi T_g^\alpha x = T_g^\alpha x \ \forall g \in G_j\}$. Notice $\mu E > 1 - \epsilon/2$ and $\forall x \in E$

$$d(h_x^{\alpha,\alpha\phi}, \mathrm{id}) < 2^{-q_j} < \frac{\epsilon}{2}.$$

Hence $m^0(\alpha, \phi) < \epsilon$. □

The following proposition proves that it suffices to consider a sequence $\{\phi_i\}$ such that $c(\alpha, \phi_i) = 0$ and if $\alpha(x, x') \in G_i$ then $\alpha\phi_i(x, x') = \alpha(x, x')$.

PROPOSITION. *Suppose* $\exists\{\phi_i\} \subset FG(\mathcal{O})$ *such that* $m(\alpha, \phi_i) \to 0$ *and* $h(T^{\alpha\phi_i}, P) < h(T^\alpha, P) - c$. *Then given any* $0 < \epsilon < \frac{1}{2}$, *there exists* $\hat{\phi} \in FG(\mathcal{O})$ *such that* $m(\alpha, \hat{\phi}) < \epsilon$, $c(\alpha, \hat{\phi}) = 0$, $h(T^{\alpha\hat{\phi}}, P) < h(T^\alpha, P) - c + \epsilon$, *and if* $\epsilon < q_j^{-1}$ *and* $\alpha(x, x') \in G_j$, *then* $\alpha(x, x') = \alpha\hat{\phi}(x, x')$.

Proof. Pick δ depending on ϵ and $m = \#P$ such that:
(1) if Q, Q' are two m-set partitions with $\mu(Q \triangle Q') < 2\delta$, then

$$h(T^\alpha, Q) < h(T^\alpha, Q') + \epsilon;$$

(2) $\delta < (\epsilon/2)^2$.
Pick $\delta_0 < (\delta/2)^2$, and $\phi \in \{\phi_i\}$ such that $m(\alpha, \phi) < \delta_0$. By the flattening lemma, $\exists\psi \in FG(\mathcal{O})$ such that $m^1(\psi) < \delta_0$ and $c(\alpha, \phi\psi) = 0$. Also,

$$\begin{aligned} \|\alpha, \alpha\phi\psi\| &\leq \|\alpha, \alpha\phi\| + \|\alpha\phi, \alpha\phi\psi\| \\ &\leq \delta_0 + \frac{\delta}{2} < \delta. \end{aligned}$$

Apply the previous lemma to $(\alpha, \phi\psi)$ to produce $\psi' \in FG(\mathcal{O})$ such that $m^1(\psi') < \delta$, $c(\alpha, \phi\psi\psi') = 0$, and if $\epsilon < q_j^{-1}$ and $\alpha(x, x') \in G_j$, then $\alpha(x, x') = \alpha\phi\psi\psi'(x, x')$. Set $\hat{\phi} = \phi\psi\psi'$ and notice that

$$\begin{aligned} m(\alpha, \hat{\phi}) &\leq \|\alpha, \alpha\phi\| + \|\alpha\phi, \alpha\phi\psi\| + \|\alpha\phi\psi, \alpha\phi\psi\psi'\| \\ &\leq \delta_0 + \frac{\delta}{2} + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

Also notice that $\mu(\hat{\phi}P \triangle \phi P) < \delta_0 + \delta < 2\delta$. Hence by the first remark, $h(T^{\alpha\hat{\phi}}, P) < h(T^\alpha, P) - c + \epsilon$. □

THEOREM. *If* $\sum_{k=1}^\infty \log r_{k+1}/q_k < \infty$ *then* $h_r(T^\alpha) = h(T^\alpha)\ \forall\alpha$.

Proof. Assume $\exists c$, $0 < c < 1$, and $\{\phi_i\} \subset FG(\mathcal{O})$ such that $c(\alpha, \phi_i) = 0$, $h(T^{\alpha\phi_i}, P) < h(T^\alpha, P) - c$, and if $\alpha(x, x') \in G_i$, then for all $i \geq I$, $\alpha\phi_i(x, x') = \alpha(x, x')$. Pick i such that:
(1) $\sum_{k=i+1}^\infty \log r_k!/q_k < c/8$ (by the equivalence lemma);
(2) $\log q_i/q_i < c/8$.
Set $\epsilon = c/8$, $\phi = \phi_i$, $h = h(T^\alpha, P)$, $(\alpha, P^n) = \bigvee_{g\in G_n} T_g^\alpha P$ and $(\alpha\phi, P^n) = \bigvee_{g\in G_n} T_g^{\alpha\phi} P$. The collection of names of one of these partitions is denoted with curly

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brackets, i.e. $\{(\alpha, P^n)\}$. If p is an element of some partition, let $\bar{p}$ be the name of that element. The Shannon–McMillan theorem [**OW**] says that $\exists N$ such that $n \geq N$ implies that for all but $c/8$ of $p_n^\phi \in (\alpha\phi, P^n)$,

$$\mu p_n^\phi \geq 2^{-G_n(h-7c/8)}.$$

On the other hand, for all but $c/8$ of $p_n^\phi \in (\alpha\phi, P^n)$,

$$\mu p_n^\phi \leq 2^{-G_n(h-3c/8)}$$

which will give the desired contradiction.

Create partitions inductively, where the 0th stage corresponds to a partition

$$\bar{\Gamma}_i : \{(\alpha, P^i)\} \to \Gamma_i,$$

which is fundamentally different from the partitions at the following stages. Let $\rho_i = \#\Gamma_i$ and $\bar{\Gamma}_i(\bar{w}_i) = \bar{\Gamma}_i(\bar{v}_i)$ if $\bar{v}_i = T_g^\alpha \bar{w}_i$ for some $g \in G_i$. Set $W_i =$ maximum number of words in any partition element of $\bar{\Gamma}_i$ and notice that $W_i \leq q_i$. The first step is to create a partition

$$\bar{\Gamma}_{i+1} : \{(\alpha, P^{i+1})\} \to \Gamma_{i+1}, \quad \rho_{i+1} = \#\Gamma_{i+1},$$
$$\bar{\Gamma}_{i+1}(\bar{w}_{i+1}) = (\gamma_1^{i+1}, \ldots, \gamma_{\rho_i}^{i+1}),$$

where $\gamma_j^{i+1} =$ # of occurrences of any (α, P^i) word belonging to the jth partition element of $\bar{\Gamma}_i$ as the name of a G_i coset in $\bar{w}_{i+1}$. Let $W_{i+1} =$ the maximum number of words in any partition element. Notice that:

(1) $\sum_{j=1}^{\rho_i} \gamma_j^{i+1} = r_{i+1}$;

(2) $W_{i+1} \leq (q_i)^{r_{i+1}}(r_{i+1}!)$, since inside any G_i coset in $\bar{w}_{i+1}$ there are at most G_i different arrangements of that coset in the same element of $\bar{\Gamma}_i$, with r_{i+1} different places to make those choices, and at most $r_{i+1}!$ different ways of arranging the G_i cosets in $\bar{w}_{i+1}$.

Assume that the partitions $\bar{\Gamma}_k$ on $\{(\alpha, P^k)\}$ have been created for $k < n$ with $\rho_k = \#\Gamma_k$, $\sum_{j=1}^{\rho_{k-1}} \gamma_j^k = r_k$ and the maximum number of words in any partition element is W_k. Let

$$\bar{\Gamma}_n : \{(\alpha, P^n)\} \to \Gamma_n, \quad \rho_n = \#\Gamma_n,$$
$$\bar{\Gamma}_n(\bar{w}_n) = (\gamma_1^n, \ldots, \gamma_{\rho_{n-1}}^n),$$

where $\gamma_j^n =$ # of occurrences of any (α, P^{n-1}) word that belongs to the jth element of $\bar{\Gamma}_{n-1}$ as the name of a G_{n-1} coset in $\bar{w}_n$. Again we have that $\sum_{j=1}^{\rho_{n-1}} \gamma_j^n = r_n$. Notice that $W_n \leq (W_{n-1})^{r_n}(r_n!)$, by the same reasoning as in step 1. Hence

$$\begin{aligned} W_n &\leq (q_i)^{r_{i+1}\cdots r_n}(r_{i+1}!)^{r_{i+2}\cdots r_n} \cdots r_n! \\ &\leq 2^{q_n(\log q_i/q_i + \sum_{k=i+1}^n \log r_k!/q_k)} \\ &\leq 2^{q_n(c/4)}. \end{aligned}$$

What does an atom of $(\alpha\phi, P^n)$ look like? By the previous reduction, inside any G_i coset of an (α, P^n) word, ϕ can only rotate the (α, P^i) name, since if $\alpha(x, x') \in G_i$, then $\alpha\phi(x, x') = \alpha(x, x')$. Furthermore, ϕ can only rearrange G_k cosets inside of G_{k+1}

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cosets since $c(\alpha, \phi) = 0$. Hence any atom of $(\alpha\phi, P^n)$ is a union of pieces of atoms of (α, P^n), all of whose names must belong to the same element of $\bar{\Gamma}_n$. To each p_n^ϕ we can assign a representative $p_n \in (\alpha, P^n)$ whose name belongs to this partition element such that

$$\mu p_n^\phi \leq (\mu p_n) W_n.$$

(This assignment is independent of which atom we choose from the partition element. If $\bar{\Gamma}_n(\bar{p}_n) = \bar{\Gamma}_n(\bar{q}_n)$ then $\mu p_n = \mu q_n$ since T^α is Bernoulli.)

By applying the Shannon–McMillan theorem to T^α with $\epsilon = c/8$ $\exists N$ such that for $n \geq N$ and for all but $c/8$ of $p_n \in \{(\alpha, P^n)\}$,

$$\mu p_n \leq 2^{-q_n(h-c/8)}.$$

This implies that for all but $c/8$ of $p_n^\phi \in \{(\alpha\phi, P^n)\}$,

$$\mu p_n^\phi \leq 2^{-q_n(h-c/8-c/4)} = 2^{-q_n(h-3c/8)}$$

which gives the desired contradiction. □

3.3. *Necessity of Vershik's entropy condition.* The goal of this section is to prove the following theorem.

THEOREM. *If* $\sum_{k=1}^{\infty} \log r_{k+1}/q_k$ *diverges, then* r *is entropy free.*

Some preliminary lemmas are established in order to prove this. Let Λ be the number of n-tuples of non-negative integers that sum to r.

COMBINATORIAL LEMMA. $\Lambda = \binom{r+n-1}{n-1}$

Proof. First count the number of n-tuples of positive integers that add up to r. We have r objects and we want to place $n-1$ partition lines. The number of spaces for partition lines is $r-1$ and the number of partition lines is $n-1$, so the number of ways to choose this is $\binom{r-1}{n-1}$.

Now suppose we let j positions be 0. There are $\binom{n}{j}$ ways this can occur, and there are $n-j$ positions left which must sum to r. Hence we get $\binom{n}{j}\binom{r-1}{n-j-1}$ ways this can happen. The total number of vectors that can occur is given by

$$\Lambda = \sum_{j=0}^{n-1} \binom{n}{j}\binom{r-1}{n-j-1} = \binom{n+r-1}{n-1}.$$ □

TECHNICAL LEMMA. *Suppose* $\exists n, r \in \mathbb{N}$ *such that* $r < n$ *and* $n \geq 2^{2^{j+2}}$. *Then*

$$\left[\frac{n}{r} + 1\right]^{r+n} \leq n^{r/2^{j+1}} \left[\frac{n}{r}\right]^{r+n}.$$

Proof. It suffices to show that

$$\log\left(1 + \frac{r}{n}\right) \leq \frac{r \log n}{2^{j+1}(r+n)} = \frac{r}{n} \frac{\log n}{2^{j+1}(1 + r/n)}$$

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which comes down to showing that

$$\frac{\log(1+r/n)}{r/n} \leq \frac{\log n}{2^{j+2}}.$$

Now $\log(1+x)/x \leq 1$ on $[0,\infty)$, so it suffices to show that

$$1 \leq \frac{\log n}{2^{j+2}}.$$

But by assumption, $\log n \geq 2^{j+2}$. □

The following theorem provides a tool for proving the entropy result.

THEOREM. ([**KR**]) *Let $c > 0$. Suppose for some arrangement α and some finite partition P there is a sequence $\{\phi_i\}$ with $\phi_i \in FG(\mathcal{O})$, $m^r(\alpha,\phi) \to 0$ and $h(T^{\alpha\phi_i}, P) < h(T^\alpha, P) - c$. Then $h_r(T^\beta) = 0$ for all arrangements β.*

This theorem states that if the regular entropy of a partition can be lowered by some fixed positive amount along a sequence of full group elements whose m^r size is tending to 0, then r is entropy free.

THEOREM. *If $\sum_{k=1}^{\infty} \log r_{k+1}/H_k$ diverges, then $\{r_k\}$ is entropy free.*

Proof. In order to apply the previous theorem, choose the Bernoulli process mentioned in §1.2 for α, where $X = \{0,1\}^G$, T^α = shift, $\mathcal{F}$ = Borel σ-algebra, P = the partition at the identity, and μ is i.i.d. measure $(\frac{1}{2},\frac{1}{2})$. Notice that $h(T^\alpha, P) = 1$. As usual, $m = m^r$. We construct a sequence $\{\phi_i\} \subset FG(\mathcal{O})$ with the following properties:
(1) if $\alpha(x,x') \in G_j$, for $j \leq i$, then $\alpha(x,x') = \alpha\phi_i(x,x')$;
(2) $c(\alpha,\phi_i) = 0$;
(3) $h(T^{\alpha\phi_i}, P) < h(T^\alpha, P) - c = 1 - c$.
The first two properties imply that $m(\alpha,\phi_i) \to 0$. Consider three cases:
(1) $\exists\{k_j\}$ and a such that $0 < a < \log r_{k_j}/q_{k_j-1} < 1$;
(2) $\exists\{k_j\}$ such that $\log r_{k_j}/q_{k_j-1} \geq 1$;
(3) $\log r_{k+1}/q_k \to 0$.
The construction of ϕ_I is outlined for the first case. The construction does not change in the second case. However, the third case requires an inductive argument to construct ϕ_i which will use the basic construction in the first case as the base case.

Case 1. $\exists\{k_j\}$ and a such that $0 < a < \log r_{k_j}/q_{k_j-1} < 1$.

Set $c = a/2$. We construct ϕ_I by only rearranging one level of a tower. Let $i+1 = \min\{k_j \mid k_j > I\}$. Notice that $2^{aG_i} < r_{i+1} < 2^{G_i}$. Start constructing ϕ_I far enough out in the sequence so that the following conditions are met:
(1) $q_i > 2^{i+2}$;
(2) $1/2^{i+1} < a/4$;
(3) $\log q_{i+1}/q_{i+1} < a/4$.

Find a set F so that $X = \bigcup_{g\in q_{i+1}} T_g^\alpha F$ and the $T_g^\alpha F$ are mutually disjoint. Again, let $(\alpha, P^j) = \bigvee_{g\in G_j} T_g^\alpha P$, and $\{(\alpha, P^j)\}$ = collection of (α, P^j) words. If each point $x \in F$ is considered as the identity element in G_{i+1}, then the G_{i+1} orbit of x is really a copy of the group G_{i+1}. The map will switch the G_i cosets around in G_{i+1} depending

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on the (α, P^{i+1}) name of x, but inside the G_i cosets it will not move points around. This yields the first two properties listed. How does ϕ_I move the cosets around? Any (α, P^{i+1}) word is a string of length r_{i+1} of (α, P^i) words. Choose an ordering on the (α, P^i) words, $\{\bar{w}_j^i\}_{j=1}^{2^{q_i}}$, where each $\bar{w}_j^i \in \{(\alpha, P^i)\}$. Assign to each point $x \in F$ a vector $\Lambda_{i+1}(x) = (\lambda_1(x), \ldots, \lambda_{2^{q_i}}(x))$, where $\lambda_j(x) =$ number of times $\bar{w}_j^i$ occurs as the name of a G_i coset in the (α, P^{i+1}) name of x. The name of the mth G_i coset of x means the map $\bar{P}^i : G_i \to \{0, 1\}$, where $(0, 0, 0, \ldots)$ is identified with $T_{(0,\ldots,0,m)}(x)$. Now ϕ_I moves the G_i cosets around in the G_{i+1} orbit of x as follows. Set $j(m, x) = \min\{j : m + 1 \leq \sum_{i=1}^{j} \lambda_i(x)\}$. Now define the $(\alpha\phi_I, P^{i+1})$ name of x, which is a map

$$\bar{P}_x^{i+1} : G_i \times \mathbb{Z}/r_{i+1}\mathbb{Z} \to \{0, 1\}$$

with the following formula:

$$\bar{P}_x^{i+1}(h, m) = \bar{w}_{j(m,x)}^i(h)$$

since $\bar{w}_{j(m,x)}^i : G_i \to \{0, 1\}$. Notice the only important property that this assignment has is that any two points in the base with the same Λ_{i+1} vector will now have the same $(\alpha\phi_I, P^{i+1})$ name. Any similar assignment with the same property will also work. If $\eta_j =$ number of $(\alpha\phi_I, P^j)$ words, then

$$h(T^{\alpha\phi_I}, P) \leq \lim_{j\to\infty} \frac{\log \eta_j}{q_j} \leq \frac{\log \eta_{i+1}}{q_{i+1}}.$$

In order to obtain property 3 it suffices to show that $\log \eta_{i+1}/q_{i+1} \leq 1 - c$. Notice that $\eta_{i+1} \leq$ (number of different $\Lambda_{i+1}(x)$ that can occur)q_{i+1}. Apply the combinatorial lemma with $r = r_{i+1}$, $n = \eta_i = 2^{G_i}$ and $\Lambda =$ number of different $\Lambda_{i+1}(x)$ that can occur, to obtain

$$\eta_{i+1} \leq \binom{r_{i+1} + \eta_i - 1}{\eta_i - 1} q_{i+1}.$$

In order to put an estimate on η_{i+1} apply the technical lemma with $r = r_{i+1}$ and $n = \eta_i = 2^{q_i}$ to obtain

$$\left[1 + \frac{\eta_i}{r_{i+1}}\right]^{\eta_i + r_{i+1}} \leq \eta_i^{r_{i+1}/2^{i+1}} \left(\frac{\eta_i}{r_{i+1}}\right)^{\eta_i + r_{i+1}}.$$

Hence

$$\begin{aligned}
\eta_{i+1} &\leq \binom{r_{i+1} + \eta_i - 1}{\eta_i - 1} q_{i+1} \\
&\qquad \text{(by combinatorial lemma)} \\
&\leq \binom{r_{i+1} + \eta_i}{r_{i+1}} q_{i+1} = \binom{r_{i+1}[1 + \eta_i/r_{i+1}]}{r_{i+1}} q_{i+1} \\
&\leq \frac{r_{i+1}^{\eta_i}[1 + \eta_i/r_{i+1}]^{\eta_i + r_{i+1}}}{\eta_i^{\eta_i}} q_{i+1} \qquad \text{(by Stirling's formula)} \\
&\leq \left[\frac{r_{i+1}}{\eta_i}\right]^{\eta_i} \left[\frac{\eta_i}{r_{i+1}}\right]^{\eta_i + r_{i+1}} (\eta_i)^{r_{i+1}/2^{i+1}} q_{i+1} \\
&\qquad \text{(by technical lemma)} \\
&\leq \left[\frac{\eta_i}{r_{i+1}}\right]^{r_{i+1}} (\eta_i)^{r_{i+1}/2^{i+1}} q_{i+1}.
\end{aligned}$$

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This estimate yields the desired entropy calculation since

$$\begin{aligned} h(T^{\alpha\phi_I}, P) &\leq \frac{\log \eta_{i+1}}{q_{i+1}} \\ &\leq \frac{\log([\eta_i/r_{i+1}]^{r_{i+1}}(\eta_i)^{r_{i+1}/2^{i+1}}q_{i+1})}{q_{i+1}} \\ &\leq 1 - \frac{\log r_{i+1}}{G_i} + \frac{1}{2^{i+1}} + \frac{\log q_{i+1}}{q_{i+1}} \\ &\leq 1 - a + \frac{a}{4} + \frac{a}{4} = 1 - c. \end{aligned}$$

Case 2. $\exists\{k_j\}$ such that $\log r_{k_j}/q_{k_j-1} \geq 1$.

Set $c = \frac{1}{4}$. Pick I so that $2/q_I < \frac{1}{8}$ and $\log q_{I+1}/q_{I+1} < \frac{1}{8}$, and set $i+1 = \min\{k_j \mid k_j > I\}$. Construct ϕ_I by altering the G_i cosets in G_{i+1} in the same way as in case 1. As the construction of ϕ_I does not differ, it is only necessary to check the entropy property. Apply the combinatorial lemma to $n = \eta_i = 2^{q_i}$ and $r = r_{i+1}$ to get an estimate on η_{i+1} = number of $(\alpha\phi_I, P^{i+1})$ words. In this case, notice that $r_{i+1} > \eta_i$. Thus

$$\begin{aligned} \eta_{i+1} &\leq \binom{r_{i+1} + \eta_i - 1}{\eta_i - 1} q_{i+1} \leq \binom{r_{i+1} + \eta_i}{r_{i+1}} q_{i+1} \\ &\leq \binom{2r_{i+1}}{r_{i+1}} q_{i+1} \leq 2^{2r_{i+1}} q_{i+1} \end{aligned}$$

by using Stirlings formula for the last inequality. Hence

$$h(T^{\alpha\phi_I}, P) \leq \frac{\log \eta_{i+1}}{q_{i+1}} \leq \frac{2}{q_i} + \frac{\log q_{i+1}}{q_{i+1}} < \frac{1}{8} + \frac{1}{8} < 1 - c.$$

Case 3. $\log r_{k+1}/q_k \to 0$.

Let $0 < 3\epsilon < c < \frac{1}{2}$ and pick i with:
(1) $\prod_{k=i+1}^{\infty}(1 + 2^{-k}) < 1 + \epsilon$;
(2) $q_i > 2^{i+3}$;
(3) $\sum_{k=i+1}^{\infty} \log q_k/q_k < \epsilon$. (This is possible since $\log q_k/q_k < k/2^k$ which is summable.)
Pick j depending on i such that:
(1) $c + 3\epsilon < \sum_{k=i+1}^{j} \log r_{k+1}/q_k$ (since the sum diverges);
(2) $\eta_k > 2^{q_k/2}$ $\forall k$, $i < k \leq j$. (If this were not true then $h(T^{\alpha\phi_i}, P) \leq \log \eta_k/q_k < \frac{1}{2}$, which proves this case.)

The idea is to construct ϕ_i by rearranging $j - i$ successive towers instead of just one as in the previous cases. Create ϕ_i inductively. The first step has already been outlined in case 1, i.e. over each point in the base of a G_{i+1} tower the map moves G_i cosets around so that every point with the same Λ_{i+1} vector has the same $(\alpha\phi_i, P^{i+1})$ name. Assume k steps have been completed in the construction of ϕ_i for $i < k < j$. Now we need to complete the $k+1$ step in the construction of ϕ_i. Build a T^{α} tower of height q_{k+1} that covers the whole space. Apply ϕ_i to this tower. This will only move points around inside each G_k coset. Call the base of this altered tower F_{k+1}. To any point $x \in F_{k+1}$, assign a vector Λ_{k+1} based on the $(\alpha\phi_i, P^k)$ names that are seen in the r_{k+1} different G_k cosets sitting over x in the altered tower. Let $\Lambda_{k+1}(x) = (\lambda_1^{k+1}(x), \ldots, \lambda_{\eta_k}^{k+1}(x))$, where

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$\eta_k = \#P^k$ words of $\alpha\phi_i$, $\lambda_n^{k+1}(x) =$ number of times the nth $(\alpha\phi_i, P^k)$ word appeared as the name of a G_k coset in the altered tower over x, and $\sum_{n=1}^{\eta_k} \lambda_n^{k+1}(x) = r_{k+1}$. Move G_k cosets around in the copy of G_{k+1} sitting over each point x in the base, so that any two points with the same Λ_{k+1} vector have the same $(\alpha\phi_i, P^{k+1})$ name. This construction again gives $m(\alpha, \phi_i) \to 0$. It suffices to show that $\log \eta_j/q_j < 1 - c$, since this implies property 3. To do this, we need an estimate on η_j. Notice that $\eta_{k+1} \leq$ (number of different Λ_{k+1} vectors that can occur)q_{k+1}. Applying the combinatorial lemma with $\eta_k = n$ and $r_{k+1} = r$ yields

$$\eta_{k+1} \leq \binom{r_{k+1} + \eta_k - 1}{r_{k+1}} q_{k+1} \quad \forall i \leq k < j.$$

Furthermore, since $r_{k+1} < \eta_k$ and $\eta_k > 2^{q_k/2} > 2^{2^{k+2}}$, applying the technical lemma with $n = \eta_k$ and $r = r_{k+1}$ gives

$$\left[\frac{\eta_k}{r_{k+1}} + 1\right]^{r_{k+1}+\eta_k} \leq \eta_k^{r_{k+1}/2^{k+1}} \left[\frac{\eta_k}{r_{k+1}}\right]^{r_{k+1}+\eta_k} \quad \forall k, \quad i \leq k < j.$$

We know that whenever $i \leq k < j$,

$$\begin{aligned}
\eta_{k+1} &\leq \binom{r_{k+1}[\eta_k/r_{k+1} + 1]}{r_{k+1}} q_{k+1} \\
&\leq \left(\frac{r_{k+1}}{\eta_k}\right)^{\eta_k} \left(\frac{\eta_k}{r_{k+1}} + 1\right)^{r_{k+1}+\eta_k} q_{k+1} \\
&\qquad \text{(by Stirling's formula)} \\
&\leq \left(\frac{\eta_k}{r_{k+1}}\right)^{r_{k+1}} \eta_k^{r_{k+1}/2^{k+1}} q_{k+1} \\
&\qquad \text{(by Technical Lemma)} \\
&\leq \frac{\eta_k^{r_{k+1}(1+1/2^{k+1})} q_{k+1}}{r_{k+1}^{r_{k+1}}}.
\end{aligned}$$

Thus we obtain an estimate for η_j in terms of $\eta_i = 2^{q_i}$. Let $R_{k,j} = \prod_{t=k+1}^{j} r_t(1 + 2^{-t})$.

$$\begin{aligned}
\eta_j &\leq \frac{\eta_i^{R_{i,j}} (\prod_{k=i+1}^{j-1} q_k^{R_{k,j}} q_j)}{\prod_{k=i+1}^{j} r_k^{\prod_{t=k}^{j} r_t}} \\
&\leq \frac{2^{q_j \prod_{k=i+1}^{j}(1+2^{-k})} (\prod_{k=i+1}^{j-1} q_k^{R_{k,j}} q_j)}{\prod_{k=i+1}^{j} r_k^{\prod_{t=k}^{j} r_t}}.
\end{aligned}$$

If we take logs and divide by q_j then

$$\begin{aligned}
\frac{\log(\prod_{k=i+1}^{j-1} q_k^{R_{k,j}} q_j)}{q_j} &= \sum_{k=i+1}^{j-1} \frac{\prod_{t=k+1}^{j}(1 + 2^{-t}) \log q_k}{q_k} + \frac{\log q_j}{q_j} \\
&\leq (1 + \epsilon) \sum_{k=i+1}^{j} \frac{\log q_k}{q_k} < 2\epsilon.
\end{aligned}$$

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Hence

$$\begin{aligned} h(T^{\alpha\phi_i}, P) &\leq \frac{\log \eta_j}{q_j} \leq 1 + \epsilon + 2\epsilon - \sum_{k=i+1}^{j} \frac{\log r_k}{q_{k-1}} \\ &< 1 + 3\epsilon - (c + 3\epsilon) = 1 - c. \end{aligned}$$

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