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On ($\mathscr{D}$, h(n)) summability methods

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Abstract. In 1967 Segal introduced the Dirichlet convolution $(\mathscr{D}, h(n))$, generalizing a method of Ingham developed in studies on the Prime Number Theorem. In this paper we establish necessary and sufficient conditions on the sequence $h(n)$ in order that the convolution method $(\mathscr{D}, h(n))$ be conservative. Further conditions are established for the method to be absolutely conservative.

1. In his studies on the prime number theorem, Ingham [3] in 1945 introduced a novel summability method. This was generalized in 1967 by Segal [5, 6]: he defined the notion of $(\mathscr{D}, h(n))$ summability, where $h\colon \mathbb{N} \to \mathbb{R}$ denotes a function with $h(1) = 1$: the series Σa_n is said to be $(\mathscr{D}, h(n))$ summable to L if $t_n \to L$, where t_n is the $(\mathscr{D}, h(n))$ transform given by

$$t_n = \frac{1}{n}\sum_{v=1}^{n} v \sum_{d|v} a_d h(v/d) = \frac{1}{n}\sum_{d=1}^{n} d a_d \sum_{rd\leqslant n} rh(r).$$

We observe that, on writing $\alpha_{n,d} = (d/n)\sum_{rd\leqslant n} rh(r)$, we have

$$t_n = \sum_{d=1}^{n} \alpha_{n,d} a_d. \tag{1}$$

Ingham's method corresponds to the case $h(n) = 1/n$. Let us also introduce the notion of *absolute* $(\mathscr{D}, h(n))$ *summability*: this occurs when $\sum_{n=2}^{\infty} |t_n - t_{n-1}| < \infty$.

Now let T denote any summability method. T is said to be *regular* if it sums every convergent series to its ordinary sum, whilst T is said to be *conservative* if it merely sums every convergent series (see Hardy [2]). Furthermore, T is *absolutely conservative*

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if it transforms each absolutely convergent series to another such (see [4]), and *absolutely regular* if it absolutely sums every absolutely convergent series to its ordinary sum.

Hardy [2] proved that Ingham's method is not regular, whilst Segal [5] generalized this, and showed that indeed $(\mathscr{D}, h(n))$ is never regular. It is thus of interest to study when $(\mathscr{D}, h(n))$ is conservative and even absolutely conservative. This is the object of the present paper: we establish in Theorems 1 and 2 necessary and sufficient conditions on the sequence $h(n)$ in order that each property obtains for $(\mathscr{D}, h(n))$. These theorems admit several interesting corollaries; in particular, Segal's result on the non-regularity of $(\mathscr{D}, h(n))$ is an immediate consequence of Theorem 1. We shall prove the following theorems.

THEOREM 1. *$(\mathscr{D}, h(n))$ is conservative if and only if $\sum_{r=1}^{\infty} |h(r)| < \infty$. Furthermore, if this condition is satisfied then any convergent series is $(\mathscr{D}, h(n))$ summable to zero.*

THEOREM 2. *$(\mathscr{D}, h(n))$ is absolutely conservative if and only if it is conservative.*

2. In our proofs the following lemmas are employed.

LEMMA 1. *If $\alpha_{n,1} \to c$ as $n \to \infty$, then, for each d, $\alpha_{n,d} \to c$ as $n \to \infty$.*

Proof. Let us write $[n/d] = m$ and $\{n/d\} = \theta$. Then from the definition of $\alpha_{n,d}$ we obtain

$$\alpha_{n,d} = \alpha_{m,1} m/(m+\theta),$$

from which the result follows.

The next result is both elementary to prove and well known.

LEMMA 2. *If $\sum_{n=1}^{\infty} a_n$ converges then $(1/n)\sum_{r=1}^{n} r a_r \to 0$ as $n \to \infty$.*

Transformations of the form (1) have been well studied (see [1], p. 68): they are regular if and only if

$$\sum_{d=1}^{n} |\alpha_{n,d} - \alpha_{n,d+1}| \quad \text{is bounded as} \quad n \to \infty \tag{2}$$

and

$$\text{for each } d,\ \alpha_{n,d} \to 1 \quad \text{as} \quad n \to \infty. \tag{3}$$

In view of Lemma 1, in our case it would be sufficient that (3) hold for $d = 1$ only, in other words that $nh(n)$ be Cesaro summable to 1. The transformation (1) is conservative if and only if (2) holds and (3) is weakened to

$$\text{for each } d,\ \alpha_{n,d} \quad \text{converges as} \quad n \to \infty. \tag{4}$$

As with (3), in our case it suffices only that $\alpha_{n,1}$ converges.

We now state and prove our key lemma.

LEMMA 3. *Suppose that $(1/n)\sum_{r=1}^{n} r\,h(r)$ is bounded as $n \to \infty$, but that $\sum_{r=1}^{\infty} |h(r)|$ diverges. Then (2) fails.*

Proof. For convenience we write $\Delta_d \alpha_{n,d} = \alpha_{n,d} - \alpha_{n,d+1}$. Now if $d \geqslant \sqrt{n}$, then $n/d - n/(d+1) < 1$, and so there can be at most one integer s satisfying

$$n/d \geqslant s > n/(d+1).$$

Now

$$\sum_{d=1}^{n} |\Delta_d \alpha_{n,d}| \geqslant \sum_{(n)} |\Delta_d \alpha_{n,d}|,$$

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where the latter sum extends over those d satisfying $\sqrt{n} \leqslant d \leqslant n$ for which an integer s exists as above. Plainly it suffices for the proof to demonstrate that the second sum is unbounded as $n \to \infty$. We have

$$\Delta_d \alpha_{n,d} = (1/n)\{(d+1)\,\Delta_d(\sum_{rd \leqslant n} rh(r)) - \sum_{rd \leqslant n} rh(r)\},$$

so it in fact suffices to prove that

$$(1/n)\sum_{(n)} |\sum_{rd \leqslant n} rh(r)| \quad \text{is bounded as} \quad n \to \infty \tag{5}$$

and

$$(1/n)\sum_{(n)} (d+1)\,|\Delta_d \sum_{rd \leqslant n} rh(r)| \quad \text{is unbounded.} \tag{6}$$

Let us consider the assertion (5) first. By assumption that there is a constant A such that $|\sum_{r=1}^{n} rh(r)| \leqslant An$ for each n. Hence $|\sum_{rd \leqslant n} rh(r)|$

$$\leqslant A[n/d] = As,$$

and so the sum in (5) is at most

$$(A/n) \sum_{s \leqslant \sqrt{n}} s,$$

which is certainly bounded as $n \to \infty$.

Now let us consider the assertion (6). We observe that $\Delta_d \sum_{rd \leqslant n} rh(r) = sh(s)$, where s is the integer specified above. Hence if m denotes the smallest integer with $m \geqslant \sqrt{n}$, then the sum in (6) is at least

$$(1/n) \sum_{s \leqslant n/m} (d+1)\, s|h(s)| \geqslant \sum_{s \leqslant n/m} |h(s)|,$$

which diverges by assumption. Thus the proof of the lemma is complete.

3. We shall now prove Theorem 1: let us first suppose that

$$\sum_{r=1}^{\infty} |h(r)| < \infty.$$

For convenience we shall write $y_n = (1/n)\sum_{v=1}^{n} va_v$. In order to establish that $(\mathscr{D}, h(n))$ is conservative we shall show that $t_n \to 0$ whenever $y_n \to 0$: by Lemma 2 this is indeed sufficient. Now from the definitions we obtain

$$t_n = (1/n) \sum_{r=1}^{n} rh(r)\,[n/r]\, y_{[n/r]}.$$

For each s, $1 \leqslant s \leqslant n$, $[n/r] = s$ if and only if

$$n/(s+1) < r \leqslant n/s,$$

so we may write

$$t_n = \sum_{s=1}^{n} \beta_{n,s}\, y_s, \tag{7}$$

where

$$\beta_{n,s} = (s/n) \sum rh(r),$$

and the latter sum extends over those r with $n/(s+1) < r \leqslant n/s$.

It is well known that the transformation (7) sends null sequences to null sequences if and only if

$$\sum_{s=1}^{n} |\beta_{n,s}| \quad \text{is bounded as} \quad n \to \infty,$$

and

$$\text{for each } s,\ \beta_{n,s} \to 0 \quad \text{as} \quad n \to \infty.$$

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However, since $$|\beta_{n,s}| \leqslant \sum_{n/(s+1)<r\leqslant n/s} |h(r)|,$$

both conditions are immediately verified: thus $(\mathscr{D}, h(n))$ is conservative as required.

Finally we note that if $(\mathscr{D}, h(n))$ is conservative then $\alpha_{n,1}$ converges as $n \to \infty$, and thus is bounded. Hence $\sum_{r=1}^{\infty} |h(r)| < \infty$, for otherwise Lemma 3 yields an immediate contradiction. The proof of Theorem 1 is now complete.

Theorem 1 admits several immediate corollaries.

COROLLARY 1. *If $h(n) = 0$ for n sufficiently large, then $(\mathscr{D}, h(n))$ is conservative.*

COROLLARY 2. *If $(\mathscr{D}, h(n))$ is conservative, then $\sum_{r=1}^{n} rh(r) = O(n)$.*

Proof. From Lemma 2 and Theorem 1 we in fact obtain $\sum_{r=1}^{n} rh(r) = o(n)$.

COROLLARY 3. (Segal [5]) *$(\mathscr{D}, h(n))$ is never regular.*

Proof. If $(\mathscr{D}, h(n))$ is regular, then it is conservative, and so by Theorem 1 it sums to zero any convergent sum. We thus obtain a contradiction by considering any convergent sum whose limit is non-zero.

We remark that our proof differs from that of Segal: we consider sums taken over those values of d for which there is an integer in the range $(n/(d+1), n/d]$, whereas he considers those values of d for which no such integer exists.

COROLLARY 4. *It is not possible for $(\mathscr{D}, h(n))$ and its inverse to be both conservative.*

Proof. Let us first choose a sequence t_n so that t_n converges to some non-zero limit, for example $t_n = 1 - 1/n$: the corresponding sequence a_d may then be chosen recursively from (1). Now, if the inverse of $(\mathscr{D}, h(n))$ is conservative, Σa_d converges, and so by Theorem 1 if $(\mathscr{D}, h(n))$ is also conservative then $t_n \to 0$, a contradiction.

4. Before embarking on the proof of Theorem 2 we require the following lemma.

LEMMA 4. *If $\sum_{r=1}^{\infty} |h(r)|$ converges then so also does $\sum_{m=1}^{\infty} m^{-2} |\sum_{r=1}^{m} rh(r)|$.*

Proof. For
$$\sum_{m=1}^{\infty} m^{-2} \left| \sum_{r=1}^{m} rh(r) \right| \leqslant \sum_{r=1}^{\infty} r|h(r)| \sum_{m=1}^{\infty} m^{-2} \leqslant 2 \sum_{r=1}^{\infty} |h(r)|.$$

In order to prove Theorem 2, we will show that $(\mathscr{D}, h(n))$ is absolutely conservative if and only if $\sum_{r=1}^{\infty} |h(r)| < \infty$. Now the necessary and sufficient condition for the transformation (1) to be absolutely conservative is that (see [1])
$$\sup_{d} \sum_{n=1}^{\infty} |\alpha_{n,d} - \alpha_{n-1,d}| < \infty,$$
where we write $\alpha_{0,d} = 0$ for each d. In our case this condition becomes
$$\sup_{d} d \sum_{n=d}^{\infty} \left| n^{-1} \sum_{rd\leqslant n} rh(r) - (n-1)^{-1} \sum_{rd<n} rh(r) \right| < \infty, \tag{8}$$
where the second factor inside the moduli bars is interpreted as 0 when $n = 1$. We shall split the sum in (8) into two parts, according as d divides n or not, and estimate the size of the two parts separately. Consider first those terms in which d does not divide n, so $d \geqslant 2$. If we write $[n/d] = m$, then the term inside the moduli bars in (8) is just
$$(n^{-1} - (n-1)^{-1}) \sum_{r=1}^{m} rh(r),$$

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and so the sum of the moduli of such terms over n satisfying $[n/d] = m$ is

$$((md)^{-1} - (md+d-1)^{-1}) \left| \sum_{r=1}^{m} rh(r) \right|.$$

Since $d \geqslant 2$, we have

$$(\tfrac{1}{4})\, m^{-2} d^{-1} \leqslant (md)^{-1} - (md+d-1)^{-1} \leqslant m^{-2} d^{-1},$$

and thus

$$\left(\frac{1}{4}\right) \sum_{m=1}^{\infty} m^{-2} \left| \sum_{r=1}^{m} rh(r) \right| \leqslant \sum_{d \mid n=1}^{\infty} |\alpha_{n,d} - \alpha_{n-1,d}| \leqslant \sum_{m=1}^{\infty} m^{-2} \left| \sum_{r=1}^{m} rh(r) \right|. \tag{9}$$

Let us now examine the contribution to the sum in (8) from those n divisible by d. In this case the term inside the moduli bars in (8) is

$$((md)^{-1} - (md-1)^{-1}) \sum_{r<m} rh(r) + h(m)/d, \tag{10}$$

where we have written $m = n/d$.

However, for $m \geqslant 2$ we have

$$(md-1)^{-1} - (md)^{-1} \leqslant (m-1)^{-2} d^{-2},$$

so the first summand in (10) contributes at most

$$d^{-1} \sum_{p=1}^{\infty} p^{-2} \left| \sum_{r=1}^{p} rh(r) \right|,$$

where we have replaced $m-1$ by p, whilst the second summand in (10) contributes at most

$$\sum_{m=1}^{\infty} |h(m)| \tag{11}$$

to (8). Hence if $\sum_{m=1}^{\infty} |h(m)| < \infty$, we deduce from Lemma 4 and the above that $(\mathscr{D}, h(n))$ is absolutely conservative. Conversely, if $(\mathscr{D}, h(n))$ is absolutely conservative then from (9) $\sum_{m=1}^{\infty} m^{-2} \left| \sum_{r=1}^{m} rh(r) \right| < \infty$, and so (11) is finite as required. This completes the proof of Theorem 2.

COROLLARY 5. *$(\mathscr{D}, h(n))$ is never absolutely regular.*

Proof. As with Corollary 3, we merely have to consider a sequence $t_n \not\to 0$ with $\sum_{n=2}^{\infty} |t_n - t_{n-1}| < \infty$ to obtain the desired results.

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