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Polynomial functors

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1. *Introduction.* If G is a group, $Z(G)$ its integral group-ring and A_G the augmentation ideal, then we can form the Abelian groups

$$P_n(G) = A_G/A_G^{n+1} \quad \text{and} \quad Q_n(G) = A_G^n/A_G^{n+1}.$$

In (5) we have studied the structure of these Abelian groups which we called polynomial groups. If C denotes the category of Abelian groups, then P_n and Q_n are functors from C into C. We call these functors polynomial functors. The object of this work is to study the nature of these functors. Except for $n = 1$, these functors are non-additive. In fact, in the sense of Eilenberg–Maclane (4) these are functors of degree exactly n (Theorem 2·3). Because of their non-additive nature, their derived functors cannot be calculated in the traditional Cartan–Eilenberg (1) method. We have to make use of the more recent theory of Dold–Puppe (3).

There is a natural epimorphism $\theta_n(M): SP^n(M) \to Q_n(M)$, where $SP^n(M)$ denotes the nth symmetric product of the Abelian group M. We prove (Theorem 4·3) that $\theta_n(M)$ is an isomorphism for free Abelian groups for all $n \geqslant 1$. An immediate consequence of this is that the derived functors of Q_n and SP^n are identical for all $n \geqslant 1$. This enables us to use the results of Dold–Puppe (3) about the homology of symmetric products. In particular, we deduce (Theorem 5·10) that

$$L_q P_n(-, r) \cong L_q P_{n+1}(-, r) \quad \text{for} \quad q < n+1 \quad \text{if} \quad r = 1$$
$$\text{and for} \quad q < 2n+r \quad \text{if} \quad r \geqslant 2.$$

We conclude by showing that $\theta_2(M)$ is an isomorphism for all Abelian groups M.

2. *Degree of polynomial functors.* Let $f: M \to N$ be a homomorphism between two Abelian groups M and N. Then f induces a ring homomorphism $f^*: Z(M) \to Z(N)$, $f^*(\sum_{\substack{n \in Z \\ x \in M}} nx) = \Sigma nf(x)$. f^* maps A_M into A_N. Therefore, f^* maps A_M^n into A_N^n for all $n \geqslant 1$

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and so induces a group homomorphism

$$P_n(f)\colon P_n(M)\to P_n(N), \quad P_n(f)\Big(\sum_{\substack{r\in Z\\ x\in M}} r(x-e_M)+A_M^{n+1}\Big) = \Sigma r(f(x)-e_N)+A_N^{n+1},$$

where e_M is the identity of M and e_N is the identity of N. If $i\colon M\to M$ is the identity homomorphism, then $P_n(i)$ is also the identity homomorphism. Further, if $f\colon M\to N$ and $g\colon N\to L$ are homomorphisms, then $P_n(g\circ f) = P_n(g)\circ P_n(f)$ ('$\circ$' denotes the composition of homomorphisms). Thus P_n is a covariant functor from the category of Abelian groups into the same category. For every homomorphism $f\colon M\to N$, we define $Q_n(f) = P_n(f)|_{Q_n(M)}$. It is easy to see that Q_n is also a functor from the category of Abelian groups into the same category. It is not difficult to check that for $n\geqslant 2$ P_n and Q_n are both non-additive functors.

To study this non-additivity of the polynomial functors we appeal to the theory of deviations and cross-effects as laid out by Eilenberg–Maclane (4). By using a multiplicative notation instead of the additive notation of Eilenberg–Maclane, we can state the definition of the degree of a functor in terms of polynomial maps (5).

2·1. We recall that a map $f\colon G\to T$ from a multiplicative group G into an additive Abelian group T is called a polynomial map of degree $\leqslant n$ if the linear extension of f to $Z(G)$, the integral group-ring of G, vanishes on A_G^{n+1}, where A_G is the augmentation ideal of G.

2·2. *Degree of a functor.* Let $T\colon C\to C$ be a functor such that $T(0) = 0$ where 0 stands for the zero homomorphism. If M and N are Abelian groups, then, by definition, T defines a map $T_{MN}\colon \mathrm{Hom}_Z(M,N)\to \mathrm{Hom}_Z(T(M),T(N))$. Let us write $U = \mathrm{Hom}_Z(M,N)$ multiplicatively and $V = \mathrm{Hom}_Z(T(M),T(N))$ additively. Then $T_{MN}\colon U\to V$ is a map satisfying $T_{MN}(e) = 0$, where e is the identity of U. We say that the functor T is a functor of degree $\leqslant n$ if T_{MN} is a polynomial map of degree $\leqslant n$ for all Abelian groups M, N.

2·3. THEOREM. *P_n and Q_n are functors of degree exactly n, for all $n\geqslant 1$.*

Proof. Let M and N be two Abelian groups and $U = \mathrm{Hom}_Z(M,N)$. Let

$$f_1, f_2, \ldots, f_{n+1}\in U.$$

Consider the homomorphism

$$P_n^*(f_1-e)(f_2-e)\ldots(f_{n+1}-e)\colon P_n(M)\to P_n(N),$$

where $P_n^*\colon Z(U)\to \mathrm{Hom}_Z(P_n(M),P_n(N))$ is the linear extension of P_{nMN} and e is the identity of U. This homomorphism maps $x-e_M+A_M^{n+1}$, $x\in M$, e_M = identity of M, into

$$(f_1(x)-e_N)(f_2(x)-e_N)\ldots(f_{n+1}(x)-e_N)+A_N^{n+1},$$

where e_N is the identity of N. But

$$(f_1(x)-e_N)(f_2(x)-e_N)\ldots(f_{n+1}(x)-e_N)\in A_N^{n+1}.$$

Therefore $P_n^*(f_1-e)(f_2-e)\ldots(f_{n+1}-e)$ maps $x-e_M+A_M^{n+1}$ into zero. Since the elements $x-e_M+A_M^{n+1}$ generate $P_n(M)$, it follows that $P_n(f_1-e)(f_2-e)\ldots(f_{n+1}-e)$

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is the zero homomorphism. Hence P_n is a functor of degree $\leqslant n$. By definition, $Q_n(f) = P_n(f)|_{Q_n(M)}$ for every homomorphism $f: M \to N$. Therefore it follows that

$$Q_n^*(f_1 - e)(f_2 - e) \dots (f_{n+1} - e): Q_n(M) \to Q_n(N)$$

is the zero homomorphism. Hence Q_n is also of degree $\leqslant n$.

To see that the degree of P_n and Q_n is precisely n, observe that if M is an infinite cyclic group and $f: M \to M$ is the identity map, then neither

$$P_n^*(f-e)^n: P_n(M) \to P_n(M) \quad \text{nor} \quad Q_n^*(f-e)^n: Q_n(M) \to Q_n(M)$$

is the zero map.

3. *Symmetric product.* In this section we shall define a natural epimorphism $\theta_n(M): SP^n(M) \to Q_n(M)$, where M is an Abelian group and $SP^n(M)$ is its nth symmetric product. Recall that the nth symmetric product of an Abelian group M is defined as follows:

4·1. Let F be the free Abelian group generated by the symbols $u(m_1, m_2, \dots, m_n)$, $m_i \in M$, $i = 1, 2, \dots, n$. Let R be the subgroup generated by all elements of the type

(*a*) $u(m_1, \dots, m_{i-1}, m_i m_{i+1}, m_{i+2}, \dots, m_{n+1}) - u(m_1, \dots, m_{i-1}, m_i, m_{i+2}, \dots, m_{n+1})$
$- u(m_1, \dots, m_{i-1}, m_{i+1}, m_{i+2}, \dots, m_{n+1}) \quad (i = 1, 2, \dots, n);$

(*b*) $u(m_1, m_2, \dots, m_n) - u(m_{\pi(1)}, m_{\pi(2)}, \dots, m_{\pi(n)})$ where π is a permutation of the integers $1, 2, \dots, n$. $SP^n(M)$ is defined to be the quotient group F/R. If we write $m_1 \hat{\otimes} \dots \hat{\otimes} m_n$ for the coset of $u(m_1, \dots, m_n)$, then a general element of $SP^n(M)$ is a finite sum of the form $\sum_{r \in Z} r m_1 \hat{\otimes} m_2 \hat{\otimes} \dots \hat{\otimes} m_n$.

SP^n is a functor from C into C. The degree of this functor is precisely n (3).

4·2. Proposition. *There is a natural epimorphism* $\theta_n(M): SP^n(M) \to Q_n(M)$ *for all integers* $n \geqslant 1$.

Proof. Consider the homomorphism $\theta: F \to Q_n(M)$ given by

$$\theta(u(m_1, m_2, \dots, m_n)) = (m_1 - e)(m_2 - e) \dots (m_n - e) + A_M^{n+1},$$

where e is the identity of M. It is easy to check that θ maps R into zero and thus induces a homomorphism $\theta_n(M): F/R = SP^n(M) \to Q_n(M)$. Since every element of $Q_n(M)$ is of the form $\Sigma r(m_1 - e)(m_2 - e) \dots (m_n - e) + A_M^{n+1}$, $\theta_n(M)$ is an epimorphism.

If N is another Abelian group, $f: M \to N$ a homomorphism, then it is easy to verify that the following diagram is commutative

$$\begin{array}{ccc} SP^n(M) & \xrightarrow{SP^n(f)} & SP^n(N) \\ \downarrow{\scriptstyle \theta_n(M)} & & \downarrow{\scriptstyle \theta_n(N)} \\ Q_n(M) & \xrightarrow{Q_n(f)} & Q_n(N) \end{array}$$

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4·3. THEOREM. *If F is a free Abelian group, then $\theta_n(F): SP^n(F) \to Q_n(F)$ is an isomorphism for all n.*

Proof. First, let F be a free Abelian group of finite rank. In this case, both $SP^n(F)$ and $Q_n(F)$ are free Abelian groups and their ranks are equal ((2) and (5)). Now let F be an arbitrary free Abelian group, $(v_\lambda)_{\lambda\in\Lambda}$ a set of independent generators of F. Let $x \in SP^n(F)$ be such that $\theta_n(x) = 0$. Since x can be written as

$$\sum_{r\in Z,\, i_j\in\Lambda} r v_{i_1} \hat{\otimes} v_{i_2} \hat{\otimes} \dots \hat{\otimes} v_{i_n},$$

$\theta_n(x) = 0$ implies that the element $\Sigma r(v_{i_1} - e)(v_{i_2} - e) \dots (v_{i_n} - e)$ can be expressed as

$$\sum_{s\in Z,\, f_t\in F} s(f_1 - e)(f_2 - e) \dots (f_{n+1} - e) = J, \quad \text{say.}$$

Let R be the subgroup of F generated by all the independent generators v_λ which appear in J and the expression for x. Then R is a finitely generated free Abelian group. Let $i: R \to F$ be the inclusion homomorphism. Let y be the element of $SP^n(R)$ defined by the element $\Sigma r u(v_{i_1}, v_{i_2}, \dots, v_{i_n})$ (section 4·1). Then $SP^n(i)(y) = x$. The expression J shows that the homomorphism $\theta_n(R): SP^n(R) \to Q_n(R)$ maps y into zero. Since R is finitely generated, we have $y = 0$. Hence $x = 0$. Thus the kernel of

$$\theta_n(F): SP^n(F) \to Q_n(F)$$

consists of the zero element only. This proves Theorem 4·3.

5. *Derived functors of polynomial functors.* We adopt the notation of (3) for the derived functors of a given functor. Thus the left derived functors of a functor T are a doubly indexed system of functors $L_q T(-, r)$.

5·1. THEOREM. *If M is an Abelian group, q, r, n, are integers, $n \geqslant 1$, q and $r \geqslant 0$, then*

$$L_q SP^n(M, r) \cong L_q Q_n(M, r).$$

Proof. Let X be a free FD-resolution of (M, r). Form the FD-complexes $Q_n(X)$ and $SP^n(X)$. In these complexes, $(Q_n X)_q = Q_n(X_q)$ and $(SP^n X)_q = SP^n(X_q)$ and if $\alpha: [m] \to [q]$ is a monotone map, then

$$(Q_n X)_\alpha: (Q_n X)_q \to (Q_n X)_m \quad \text{and} \quad (SP^n X)_\alpha: (SP^n X)_q \to (SP^n X)_m$$

are the homomorphisms

$$Q_n(X_\alpha): Q_n(X_q) \to Q_n(X_m) \quad \text{and} \quad SP^n(X_\alpha): SP^n(X_q) \to SP^n(X_m)$$

respectively. Consider the diagram

$$\begin{array}{ccccccc} \dots \to SP^n(X_{q+1}) & \xrightarrow{d^{(1)}_{q+1}} & SP^n(X_q) & \xrightarrow{d^{(1)}_q} & SP^n(X_{q-1}) \to \dots \\ \downarrow \theta_n(X_{q+1}) & & \downarrow \theta_n(X_q) & & \downarrow \theta_n(X_{q-1}) \\ \dots \to Q_n(X_{q+1}) & \xrightarrow{d^{(2)}_{q+1}} & Q_n(X_q) & \xrightarrow{d^{(2)}_q} & Q_n(X_{q-1}) \to \dots \end{array} \tag{5.2}$$

where θ_n is the natural homomorphism considered in section 4 and $d^{(1)}_q$, $d^{(2)}_q$ are the boundary homomorphisms of the chain complexes $kSP^n X$ and $kQ_n X$ respectively. By definition

$$d^{(1)}_q = \sum_{i=0}^{q} (-1)^i SP^n(X_{\varepsilon^i_q}) \quad \text{and} \quad d^{(2)}_q = \sum_{i=0}^{q} (-1)^i Q_n(X_{\varepsilon^i_q})$$

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where $\epsilon_q^i\colon [q-1]\to[q]$ is the map given by

$$\epsilon_q^i(j) = \begin{cases} j & \text{if } j < i, \\ j+1 & \text{if } j \geqslant i, \end{cases}$$

$$\theta_n(X_{q-1})\,d_q^{(1)} = \theta_n(X_{q-1})\left(\sum_{i=0}^{q}(-1)^i SP^n(X_{\epsilon_q^i})\right) = \sum_{i=0}^{q}(-1)^i\,\theta_n(X_{q-1})\,SP^n(X_{\epsilon_q^i})$$

$$= \sum_{i=0}^{q}(-1)^i Q_n(X_{\epsilon_q^i})\,\theta_n(X_q) = \left(\sum_{i=0}^{q}(-1)^i Q_n(X_{\epsilon_q^i})\right)\theta_n(X_q) = d_q^{(2)}\,\theta_n(X_q).$$

Thus (5·2) is a commutative diagram. Since X_q is free Abelian for all $q \geqslant r$, the homomorphisms θ_n are isomorphisms (4·3). Hence the homology groups of the two complexes kSP^nX and kQ_nX are isomorphic, i.e.

$$L_qSP^n(M,r) \cong L_qQ_n(M,r).$$

Theorem 5·1 enables us to use the results of Dold–Puppe about the homology of symmetric products. In particular, we notice that if X is a free FD-resolution of (M,r), then the hypothesis of theorem 12·1 of (3) is trivially satisfied. Thus we have

$$L_qSP^n(M,r) = H_qSP^nX = 0 \begin{cases} \text{for} & q < n & \text{if} & r = 1, \\ \text{for} & q < 2n+r-2 & \text{if} & r \geqslant 2. \end{cases}$$

Consequently we have the following

5·3. Corollary.

$$L_qQ_n(M,r) = 0 \begin{cases} \textit{for} & q < n & \textit{if} & r = 1 \\ \textit{for} & q < 2n+r-2 & \textit{if} & r \geqslant 2. \end{cases}$$

5·4. *Remark.* The situation in (5·1) is similar to that in Dold–Puppe ((3), (4·13)). $SP^n(M)$ and $Q_n(M)$ are not always isomorphic whereas $LSP^n \cong LQ_n$. That $SP^n(M)$ and $Q_n(M)$ are not always isomorphic is seen by taking $M = Z_p \oplus \ldots \oplus Z_p$, direct sum of, say, m ($\geqslant 2$) copies of the cyclic group of order p, p prime. It has been proved in (5) that in this case the polynomial groups $Q_n(M)$ have a constant structure for large n. This, however, is not the case for $SP^n(M)$ (2).

Concerning functors of finite degree Dold–Puppe (3) have the following theorem:

5·5. *If T has degree $\leqslant k$ and if homological dimension of $A \leqslant r$, then $L_qT(A,n) = 0$ for $q > k(n+r)$.*

(Here $T\colon U \to U$ is a functor between Abelian categories U and U', $A \in U$.) In the category of Abelian groups, every element is of homological dimension $\leqslant 1$. Hence we have the following result about polynomial functors:

5·6. Theorem. (i) $L_qP_n(M,r) = 0$ *for* $q > n(r+1)$,

(ii) $L_qQ_n(M,r) = 0$ *for* $q > n(r+1)$.

Consider the map $A^n\colon C \to C$ given by $A^n(M) = A_M^n$, $M \in C$. If $f\colon M \to N$ is a homomorphism, define $A^n(f)\colon A^n(M) \to A^n(N)$ to be the restriction of $f^*\colon Z(M) \to Z(N)$ to $A^n(M)$, where f^* is the linear extension of f to $Z(M)$.

Then $A^n\colon C \to C$ becomes a covariant functor.

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5·7. COROLLARY.

$$L_q A^n(M,r) \cong L_q A(M,r) \quad \textit{for} \quad q > (n-1)(r+1).$$

Proof. Let X be a free FD-resolution of (M,r). Form the chain-complexes kA^nX, kAX and kP_nX. For any Abelian group N, we have a natural exact sequence

$$0 \to A^n(N) \xrightarrow{i} A(N) \xrightarrow{p} P_{n-1}(N) \to 0,$$

where i is the inclusion map and p is the natural projection. Thus we have an exact sequence of chain complexes,

5·8. $$0 \to kA^nX \to kAX \to kP_{n-1}X \to 0.$$

(5·8) gives us a long exact sequence

$$\ldots \to H_{q+1}kP_{n-1}X \to H_q kA^nX \to H_q kAX \to H_q kP_{n-1}X \to \ldots.$$

By (5·6 (i)) $\quad H_q kP_{n-1}X = 0 = H_{q+1}kP_{n-1}X \quad \text{for} \quad q > (n-1)(r+1).$

Hence $\quad H_q kA^nX \cong H_q kAX \quad \text{for} \quad q > (n-1)(r+1),$

i.e. $\quad L_q A^n(M,r) \cong L_q A(M,r) \quad \text{for} \quad q > (n-1)(r+1).$

5·9. *Remark.* For every Abelian group N, we have a natural split exact sequence $0 \to A(N) \to Z(N) \to Z \to 0$. Therefore, if we consider the group-ring itself as a functor on C, its derived functors are identical with those of A. The derived functors of the group-ring functor have been shown by Dold–Puppe (3) to be the Eilenberg–Maclane functors $H_q(-,r)$. Corollary 5·7, therefore, shows that for fixed q and r and relatively small n, the derived functors $L_q A^n(-,r)$ are also Eilenberg–Maclane functors.

5·10. THEOREM.

$$L_q P_n(M,r) \cong L_q P_{n+1}(M,r) \quad \textit{for} \quad q < n+1 \quad \textit{if} \quad r = 1 \quad \textit{and for} \quad q < 2n+r \quad \textit{if} \quad r \geqslant 2.$$

Proof. For every Abelian group N, we have the natural exact sequence

$$0 \to Q_{n+1}(N) \to P_{n+1}(N) \to P_n(N) \to 0.$$

Let X be a free FD-resolution of (M, r). Form the chain-complexes kP_nX, $kP_{n+1}X$ and $kQ_{n+1}X$. Then we have the following exact sequence of complexes

$$0 \to kQ_{n+1}X \to kP_{n+1}X \to kP_nX \to 0.$$

Consider its homology sequence, i.e. the long exact sequence

$$\ldots \to H_q kQ_{n+1}X \to H_q kP_{n+1}X \to H_q kP_nX \to H_{q-1}kQ_{n+1}X \to \ldots.$$

Suppose that $q < n+1$ or $2n+r$ according as $r = 1$ or $r \geqslant 2$. Then (5·3) implies that $H_q kQ_{n+1}X = H_{q-1}kQ_{n+1}X = 0$. Hence

$$H_q kP_{n+1}X \cong H_q kP_nX, \quad \text{i.e.} \quad L_q P_{n+1}(M,r) \cong L_q P_n(M,r).$$

5·11. COROLLARY.

$$L_r P_n(M,r) \cong M \quad \textit{for all} \quad r,n \geqslant 1.$$

Proof. (5·10) shows that $L_r P_n(M,r) \cong L_r P_{n+1}(M,r)$ for all $r,n \geqslant 1$. Hence $L_r P_n(M,r) \cong L_r P_1(M, r)$. Since P_1 is equivalent to the identity functor (Northcott:

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Introduction to Homological Algebra, p. 230), $L_r P_1(M,r) \cong M$. Hence $L_r P_n(M,r) \cong M$ for all $n, r \geqslant 1$.

5·12. COROLLARY.

$$L_{r+1} P_n(M,r) = 0 \quad \textit{for all} \quad r \geqslant 2 \quad \textit{and} \quad n \geqslant 1.$$

Proof. When $r \geqslant 2$, (5·10) shows that $L_{r+1} P_n(M,r) \cong L_{r+1} P_{n+1}(M,r)$ for all $n \geqslant 1$. Hence $L_{r+1} P_n(M,r) \cong L_{r+1} P_1(M,r)$ for all $n \geqslant 1$. But, as before, P_1 is equivalent to the identity functor. Hence $L_{r+1} P_1(M,r) = 0$, and the corollary is proved.

6. *The homomorphism θ_2.*

6·1. THEOREM. $\theta_2(M)\colon SP^2(M) \to Q_2(M)$ *is an isomorphism for all Abelian groups* M.

Proof. Let N be the kernel of $\theta_2(M)$. Then we have an exact sequence:

$$6\cdot2. \qquad 0 \to N \xrightarrow{i} SP^2(M) \xrightarrow{\theta_2(M)} Q_2(M) \to 0,$$

where i is the inclusion homomorphism. Let T denote the additive group of rationals mod 1. As T is a divisible Abelian group, (6·2) induces an exact sequence

$$6\cdot3. \qquad 0 \to \mathrm{Hom}_Z(Q_2(M), T) \xrightarrow{\theta_2^*} \mathrm{Hom}_Z(SP^2(M), T) \xrightarrow{i^*} \mathrm{Hom}_Z(N, T) \to 0.$$

We assert that θ_2^* is an epimorphism. For, let $f\colon SP^2(M) \to T$ be a homomorphism. Consider the map $\alpha\colon M \times M \to T$ given by $\alpha(x,y) = f(x \hat{\otimes} y)$, $x, y \in M$. Then

$$\begin{aligned}\alpha(xy,z) - \alpha(x,z) - \alpha(y,z) &= f(xy \hat{\otimes} z) - f(x \hat{\otimes} z) - f(y \hat{\otimes} z) \\ &= f(xy \hat{\otimes} z - x \hat{\otimes} z - y \hat{\otimes} z) = f(0) = 0\end{aligned}$$

for all $x, y, z \in M$. Similarly $\alpha(x,yz) - \alpha(x,y) - \alpha(x,z) = 0$. Thus, in particular, α is a (normalized) 2-cocycle. Since $x \hat{\otimes} y = y \hat{\otimes} x$ for all $x, y \in M$, α is a symmetric 2-cocycle, i.e. $\alpha(x,y) = \alpha(y,x)$ for all $x, y \in M$. As T is a divisible Abelian group, it follows that α must be a coboundary. To see this observe that every 2-cocycle $\alpha\colon M \times M \to T$ determines an exact sequence $0 \to T \to G \to M \to 0$ where G is a group containing T in its centre. If α is symmetric, then (since M is Abelian) G must be Abelian. But then, as T is divisible Abelian, the exact sequence must split. This, of course, is equivalent to saying that α is a coboundary. Hence there exists a map $t\colon M \to T$ such that $t(e) = 0$ and $f(x \hat{\otimes} y) = \alpha(x,y) = t(xy) - t(x) - t(y)$ for all $x, y \in M$. t induces a homomorphism $t^*\colon Z(M) \to T$. Since $\alpha(xy,z) - \alpha(x,z) - \alpha(y,z) = 0$, t^* vanishes on A_M^3. Thus t^* induces a homomorphism $t^{**}\colon Z(M)/A_M^3 \to T$. Let $s\colon Q_2(M) \to T$ be the restriction of t^{**} to $Q_2(M)$. Then s is an element of $\mathrm{Hom}_Z(Q_2(M), T)$. By definition,

$$\theta^*(s)(x \hat{\otimes} y) = s((x-e)(y-e) + A_M^3) = t(xy) - t(x) - t(y) = f(x \hat{\otimes} y).$$

Hence $\theta^*(s) = f$, showing that θ^* is an epimorphism. Hence $\mathrm{Hom}_Z(N, T) = 0$. But this implies that $N = 0$.

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