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Intersection multiplicities and tangent cones

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0. *Introduction*

The following result has at least the appeal of intuitive plausibility. Let U and V be subvarieties of an algebraic variety X; let $x \in X$ be an isolated point of the intersection of U and V, and let $I(X, U.V, x)$ denote the intersection multiplicity (in some sense to be made precise) of U and V at x. Now let T_X be the tangent space to X at x, within which lie the tangent cones C_U, C_V to U and V at x. Then

$$I(X, U.V, x) = I(T_X, C_U.C_V, x).$$

In this paper we give suitable conditions for this result to hold, and prove it; indeed, we give the generalization to the scheme-theoretic context, and state and prove the corresponding result for the multiplicities at a proper component of the intersection of two subschemes, involving normal cones in place of tangent cones.

We work throughout in the theory of schemes (3) and adopt the definition of intersection multiplicities developed in (12), which, as stated in (7), p. 445, is at present the most popular way to assign multiplicities to proper components of intersection. In (12) it is proved that this intersection multiplicity agrees with that of Weil and Samuel whenever the latter is defined, so the result applies in this context too.

The notions of tangent and normal cones are developed in Section 1, since the only suitable reference seems to be (4), and this does not contain all the necessary details (but see also (9) and (5)).

1. *Tangent cones and normal cones*

(1·1). Let S be a scheme (not necessarily separated, in accordance with the terminology of (3), I), with structure sheaf $\mathcal{O}_S$. From a quasi-coherent $\mathcal{O}_S$-algebra A, one constructs, by glueing spectra of rings of sections, an S-scheme $\mathrm{Spec}_S A$. In this way, indeed, one obtains a contravariant equivalence of categories ((3), I, § 9)

$$\mathrm{Spec}_S\text{: (quasi-coherent } \mathcal{O}_S\text{-algebras)} \longrightarrow \text{(affine } S\text{-schemes)}$$

with the properties:

(α) If $A \to B$ is a surjective morphism of $\mathcal{O}_S$-algebras (surjective on each stalk, that is), then the induced morphism $\mathrm{Spec}_S B \to \mathrm{Spec}_S A$ is a closed immersion of S-schemes.

(β) If $f: S' \to S$ is a base change, $(\mathrm{Spec}_S A) \times_S S'$ is naturally isomorphic to

$$\mathrm{Spec}_{S'}(f^*A).$$

(γ) If $f: X \to S$ is any S-scheme, there is a natural bijection

$$(\mathrm{Sch}/S)(X, \mathrm{Spec}_S A) \simeq (\mathcal{O}_S\text{-alg})(A, f_* \mathcal{O}_X)$$

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(where Sch/S denotes the category of S-schemes, $\mathcal{O}_S$-alg the category of $\mathcal{O}_S$-algebras, and, here as below, for objects A, B of a category C, $C(A, B)$ denotes the set of C-morphisms from A to B).

In particular, a quasi-coherent $\mathcal{O}_S$-module M gives rise to an $\mathcal{O}_S$-algebra $\mathrm{Sym}_{\mathcal{O}_S} M$ by glueing symmetric algebras of modules of sections; and thus to an affine S-scheme $\mathbb{V}_S(M) = \mathrm{Spec}_S(\mathrm{Sym}_{\mathcal{O}_S} M)$.

If M is locally free and of finite type, then $\mathbb{V}_S(M)$ is a *vectorbundlescheme* over S; that is, $\mathbb{V}_S(M)$ is Zariski-locally on S of the form $\mathbb{A}^n_S$ with $n \in \mathbb{N}$. In this way we get a contravariant equivalence of categories

$$\mathbb{V}_S\text{: (locally free } \mathcal{O}_S\text{-modules of f.t.)} \longrightarrow \text{(vectorbundleschemes/}S\text{)}.$$

(Compare (9), pp. 293–299, where a different convention is used.)

The functor $\mathbb{V}_S$ has the properties:

(*a*) If $M \to N$ is a surjection (on the stalks) of quasi-coherent $\mathcal{O}_S$-modules, then $\mathbb{V}_S(N) \to \mathbb{V}_S(M)$ is a closed immersion.

(*b*) If $f: S' \to S$ is a base change, then $\mathbb{V}_S(M) \times_S S'$ is naturally isomorphic to $\mathbb{V}_{S'}(f^*M)$.

(*c*) If $f: X \to S$ is an S-scheme, then the X-valued points of $\mathbb{V}_S(M)$ are

$$\mathbb{V}_S(M)(X) = (\mathrm{Sch}/S)(X, \mathbb{V}_S(M)) \simeq (\mathcal{O}_S\text{-mod})(M, f_*\mathcal{O}_X).$$

(*d*) In particular,

$$\mathbb{V}_S(M)(S) \simeq (\mathcal{O}_S\text{-mod})(M, \mathcal{O}_S);$$

indeed, the sheaf of S-sections of the morphism $\mathbb{V}_S(M) \to S$ is naturally isomorphic to the dual $M^\vee = \mathrm{Hom}_{\mathcal{O}_S}(M, \mathcal{O}_S)$ of M.

(1·2). As an example of the applications of (1·1), we can define the tangent bundle of a scheme X over a field k; more generally, we discuss the relative tangent bundle of an S-scheme X, and specialize later to $S = \mathrm{Spec}\, k$.

Let $\Omega^1_{X/S}$ be the $\mathcal{O}_X$-module of relative differentials, constructed locally from derivations ((1), VI-1·21, (9), ch. 3, § 1, (3), IV$_4$ (16·3·1)). Set $T_S(X) = \mathbb{V}_X(\Omega^1_{X/S})$; this may also be denoted by $T(X)$ or $T_{X/S}$. $T_S(X)$ is an X-scheme, the *tangent bundle*, or *tangent-bundle-scheme* of X over S; if $\Omega^1_{X/S}$ is locally free – for example if X is smooth over S ((3), IV$_4$ (17·2·3), (1), VII-(5·1)), then as in (1·1) $T_S(X)$ is a vectorbundlescheme over X.

The fibres of the morphism $\pi: T_S(X) \to X$ are given as follows. Let x be a point of X, with local ring $\mathcal{O}_{X,x}$; let $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}_{X,x}$ and $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}$ its residue field; let $s = f(x)$, where f is the structure morphism $X \to S$. Applying (1·1)(*b*) to the natural morphism $\mathrm{Spec}\,(\kappa(x)) = \{x\} \to X$, we have

$$\begin{aligned} \pi^{-1}(x) &= \mathbb{V}_X(\Omega^1_{X/S}) \otimes_{\mathcal{O}_{X,x}} \kappa(x) \\ &= \mathbb{V}_{\kappa(x)}((\Omega^1_{X/S})_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)). \end{aligned}$$

As in (3), IV$_4$ (16·4·16·2),

$$(\Omega^1_{X/S})_x = \Omega^1_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}} = D, \quad \text{say},$$

and the $\kappa(x)$-module is $D/\mathfrak{m}D$. If x is rational over $\kappa(s)$, this is, by (3), IV$_4$ (16·4·13),

$$(\Omega^1_{X/S})_x \otimes_{\mathcal{O}_{S,s}} \kappa(s) \simeq \mathfrak{n}/\mathfrak{n}^2,$$

where $\mathfrak{n}$ is the maximal ideal of the local ring $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$ of x in its fibre over s ($\mathfrak{m}_s$ being the maximal ideal of $\mathcal{O}_{S,s}$).

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Thus if x is rational over $\kappa(s)$, the fibre $\pi^{-1}(x)$ of $T_S(X)$ over x is the $\kappa(x)$-scheme $\mathbb{V}_{\kappa(x)}(\mathfrak{n}/\mathfrak{n}^2)$; its $\kappa(x)$-rational points are given, by (1·1)(*d*), by

$$\pi^{-1}(x)(\kappa(x)) = (\mathfrak{n}/\mathfrak{n}^2)^\vee, \text{ the dual of the } \kappa(x)\text{-vectorspace } \mathfrak{n}/\mathfrak{n}^2.$$

For example, if $S = \operatorname{Spec}\kappa(s)$ then $\mathfrak{n} = \mathfrak{m}$, and for a $k = \kappa(s)$-rational point x of X,

$$\pi^{-1}(x)(k) = (\mathfrak{m}/\mathfrak{m}^2)^\vee$$

is the Zariski tangent space to the k-scheme X at x. Thus, in this case,

$$\pi^{-1}(x) = \mathbb{V}_k(\mathfrak{m}/\mathfrak{m}^2)$$

should be thought of as the *tangent-space-scheme* to X at x.

One reason why $T_S(X)$ is to be thought of as the tangent bundle rather than the cotangent bundle is the fact that it is covariant in X. For a morphism $g: X \to Y$ of S-schemes gives rise to a morphism of $\mathcal{O}_X$-modules $g^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ ((1), VI-1·21), and so to a commutative diagram

$$\begin{array}{ccccc} T_S(X) = \mathbb{V}_X(\Omega^1_{X/S}) & \longrightarrow & \mathbb{V}_X(g^*\Omega^1_{Y/S}) = \mathbb{V}_Y(\Omega^1_{Y/S}) \times_Y X & \longrightarrow & \mathbb{V}_Y(\Omega^1_{Y/S}) = T_S(Y) \\ & \searrow & \downarrow & & \downarrow \\ & & X & \xrightarrow{g} & Y \end{array}$$

defining $T_S(g): T_S(X) \to T_S(Y)$. In particular, if $g: X \to Y$ is an immersion, then

$$g^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$$

is surjective ((1), VI-1·21) and so $T_S(g): T_S(X) \to T_S(Y)$ is a closed immersion.

We note that this approach differs from, and is more natural than, that of (9), ch. 3, §4; compare too (3), IV$_4$ (16·5).

(1·3). Now suppose that W is a closed subscheme of a scheme X, defined by a quasi-coherent sheaf of ideals I in $\mathcal{O}_X$ (so that $\mathcal{O}_W$ is $\mathcal{O}_X/I$, restricted to W). Then I/I^2 is naturally an $\mathcal{O}_W$-module, and

$$gr_W(X) = \oplus_{n \in \mathbb{N}} I^n/I^{n+1} = \mathcal{O}_W \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$$

an $\mathcal{O}_W$-algebra. We define

$$C_{W,X} = \operatorname{Spec}_W(gr_W(X)),$$

the *normal cone* to W in X,

and

$$N_{W,X} = \mathbb{V}_W(I/I^2),$$

the *normal space* (or *normal bundle*) to W in X. These are each W-schemes, and the surjection $\operatorname{Sym}_W(I/I^2) \to \operatorname{gr}_W(X)$ (arising from the fact that the graded $\mathcal{O}_W$-algebra $\operatorname{gr}_W(X)$ is generated by its piece I/I^2 in degree 1) gives rise to a closed immersion $C_{W,X} \to N_{W,X}$ (by (1·1)(α)).

In the special case where k is a field, X a k-scheme, and $W = \{x\}$ is a k-rational point of X, we have, using (1·1)(*b*),

$$N_{\{x\},X} = \mathbb{V}_{\operatorname{Spec} k}(\mathfrak{m}/\mathfrak{m}^2),$$

where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,x}$; hence, in this case, $N_{\{x\},X}$ is the tangent-space-scheme to X at x, whose k-points make up the Zariski tangent space to X at x (compare (1·2)). Also in this case, $C_{\{x\},X}$ is called the *tangent cone* to X at x.

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Returning to the general case, $N_{W,X}$ is, in a sense easily made precise, the 'linear space' spanned by $C_{W,X}$. Note that $C_{W,X}$ need not be a reduced scheme, even if X and W are, and $N_{W,X}$ need not be spanned by $(C_{W,X})_{\mathrm{red}}$; this occurs for example if W is a cuspidal point of a plane curve X (compare (11), p. 68).

Since, by (1·1) (γ),

$$C_{W,X}(W) = (\mathrm{Sch}/W)(W, \mathrm{Spec}_W(\mathrm{gr}_W X)) \simeq \mathcal{O}_W\text{-alg}(\mathrm{gr}_W X, \mathcal{O}_W),$$

we see that $C_{W,X}$ has a canonical W-valued point (that is, section of $C_{W,X} \to W$), corresponding to the augmentation $\mathrm{gr}_W X \to \mathcal{O}_W$; since this last morphism is surjective, the morphism $W \to C_{W,X}$ is in fact a closed immersion. Composing with $C_{W,X} \to N_{W,X}$, we find a closed immersion $W \to N_{W,X}$, which is the 'zero section' of the normal space, corresponding under (1·1) (d) to the zero map in $(I/I^2)^\vee$. In the case where $W = \{x\}$ is a k-point of a k-scheme, this is the distinguished point (0) of the tangent space to X at x.

Thus we may say that each of $C_{W,X}$ and $N_{W,X}$ contains a canonical copy of W: $W \hookrightarrow C_{W,X} \hookrightarrow N_{W,X}$.

(1·4). We now consider base changes, and the form of these notions in the affine case.

LEMMA. *Suppose that W is a closed subscheme of a scheme X, $X' \to X$ is a base change (any morphism), $W' = W \times_X X'$. Then*

$$C_{W',X'} = C_{W,X} \times_W W'$$

and

$$N_{W',X'} = N_{W,X} \times_W W'$$

(each up to canonical isomorphism).

Proof. Let I be the sheaf of ideals in $\mathcal{O}_X$ defining W, and the morphisms be named as shown.

$$\begin{array}{ccc} W' & \xrightarrow{f} & W \\ \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle i} \\ X' & \xrightarrow{g} & X \end{array}$$

Defining $\mathrm{pow}_I \mathcal{O}_X = \oplus_{n \in \mathbb{N}} I^n$ with the obvious $\mathcal{O}_X$-algebra structure, we see that

$$\mathrm{gr}_W X = \mathrm{pow}_I \mathcal{O}_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X/I) = i^* \mathrm{pow}_I \mathcal{O}_X.$$

By (1·1) (β),

$$C_{W,X} \times_W W' = \mathrm{Spec}_{W'} f^* \mathrm{gr}_W X, \quad \text{and} \quad f^* \mathrm{gr}_W X = f^* i^* \mathrm{pow}_I \mathcal{O}_X = i'^* g^* \mathrm{pow}_I \mathcal{O}_X.$$

But g^*I defines W' in X', and the result for $C_{W',X'}$ follows. Hence $f^*(I/I^2) = I'/I'^2$ (where $I' = g^*I$) and thus the result for $N_{W',X'}$ follows too. ∎

Recall that associated to an ideal $\mathfrak{a}$ of a ring A, and an A-module M, we have a graded ring

$$\mathrm{gr}_{\mathfrak{a}} A = \oplus_{n \in \mathbb{N}} \mathfrak{a}^n/\mathfrak{a}^{n+1}$$

and over it the graded module

$$\mathrm{gr}_{\mathfrak{a}} M = \oplus_{n \in \mathbb{N}} \mathfrak{a}^n M/\mathfrak{a}^{n+1} M.$$

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COROLLARY 1. *Let W be a closed subscheme of a scheme X, $U = \operatorname{Spec} A$ an open affine of X, and suppose that $W \cap U$ is defined by the ideal $\mathfrak{a}$ of A. Then*

$$C_{W,X}|\,U = \operatorname{Spec} \operatorname{gr}_{\mathfrak{a}} A$$

and

$$N_{W,X}|\,U = \operatorname{Spec} \operatorname{Sym}_{A/\mathfrak{a}}(\mathfrak{a}/\mathfrak{a}^2).$$

Proof. Apply the Lemma to $X' = U$.

COROLLARY 2. *Let $X = \operatorname{Spec} A$ be an affine scheme, W the closed subscheme defined by an ideal $\mathfrak{a}$ of A, w a point of W corresponding to a prime ideal $\mathfrak{p}$ of A (with $\mathfrak{p} \supseteq \mathfrak{a}$). Then the fibre of $C_{W,X}$ over the point w is the spectrum of*

$$\operatorname{gr}_{\mathfrak{a}} A \otimes_{A/\mathfrak{a}} \kappa(\mathfrak{p}) \simeq \operatorname{gr}_{\mathfrak{a}'} R \otimes_{R/\mathfrak{a}'} (R/\mathfrak{m}),$$

where R is the localization $A_{\mathfrak{p}}$ of A at $\mathfrak{p}$, with maximal ideal $\mathfrak{m} = \mathfrak{p}A_{\mathfrak{p}}$, and $\mathfrak{a}' = \mathfrak{a}R$. The fibre of $N_{W,X}$ over w is the spectrum of

$$(\operatorname{Sym}_{A/\mathfrak{a}} \mathfrak{a}/\mathfrak{a}^2) \otimes_{A/\mathfrak{a}} \kappa(\mathfrak{p}) \simeq (\operatorname{Sym}_{R/\mathfrak{a}'} \mathfrak{a}'/\mathfrak{a}'^2) \otimes_{R/\mathfrak{a}'} (R/\mathfrak{m}).$$

Proof. Apply the Lemma to the base changes $\operatorname{Spec} \kappa(\mathfrak{p}) \to \operatorname{Spec} R \to X$. (This corollary can also be proved directly algebraically by computing

$$\operatorname{pow}_{\mathfrak{a}} A \otimes_A A/\mathfrak{a} \otimes_{A/\mathfrak{a}} \kappa(\mathfrak{p})$$

in two ways.)

COROLLARY 3. *If W is a reduced, irreducible closed subscheme of a scheme X, let $\mathcal{O}_{X|W}$ be the local ring of W in X (the stalk of $\mathcal{O}_X$ at the generic point w of W), and $\mathfrak{m}$ be its maximal ideal. Then the fibre of $C_{W,X} \to W$ at the generic point w of W is the $\kappa(w) = k(W)$-scheme*

$$\operatorname{Spec} (\operatorname{gr}_{\mathfrak{m}} \mathcal{O}_{X|W})$$

and the fibre of $N_{W,X}$ at w is

$$\operatorname{Spec} (\operatorname{Sym}_{\kappa(w)} \mathfrak{m}/\mathfrak{m}^2).$$

Proof. Replace X by an open affine neighbourhood of w and apply Corollary 2 (in which then $\mathfrak{a}' = \mathfrak{m}$).

PROPOSITION. *Let W be a reduced, irreducible closed subscheme of a scheme X, with generic point w and local ring $\mathcal{O}_{X|W} = R$ with maximal ideal $\mathfrak{m}$. Let $\mathfrak{p}$ (respectively $\mathfrak{q}$) denote the augmentation ideal of $\operatorname{Sym}_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$ (respectively $\operatorname{gr}_{\mathfrak{m}} R$) (generated in each case by the homogeneous elements of degree $\geqslant 1$). Then the local ring in $N_{W,X}$ of the copy of W therein (1·3) is (up to natural isomorphism)*

$$\mathcal{O}_{N_{W,X}|W} = (\operatorname{Sym}_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2)_{\mathfrak{p}},$$

and the local ring in $C_{W,X}$ of the copy of W therein (1·3) is

$$\mathcal{O}_{C_{W,X}|W} = (\operatorname{gr}_{\mathfrak{m}} R)_{\mathfrak{q}},$$

where in each case the subscript ($\mathfrak{p}$ or $\mathfrak{q}$) denotes localization.

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Proof. By Corollary 1 we may assume X (and so W) affine. Let A be the affine ring of W (so that A is integral), $S = A\backslash\{0\}$. Denoting by B either of the affine rings of $N_{W,X}$ and $C_{W,X}$, we get from Corollary 3 a commutative diagram

$$\begin{array}{ccc} S^{-1}A & \xleftarrow{f} & A \\ {\scriptstyle i'}\uparrow\downarrow & & {\scriptstyle i}\uparrow\downarrow \\ R' = S^{-1}B & \longleftarrow & B \end{array}$$

with f injective, where i is the ring morphism expressing W as a closed subscheme of $N_{W,X}$ (or $C_{W,X}$), and where R' is either $\mathrm{Sym}_{R/\mathfrak{m}}\,\mathfrak{m}/\mathfrak{m}^2$ or $\mathrm{gr}_{\mathfrak{m}}R$.

We now wish to show that $T^{-1}S^{-1}B \simeq T_1^{-1}B$, where $T_1 = B\backslash\ker(i)$ and

$$T = R\backslash\ker(i').$$

(Note that $\ker(i')$ is either $\mathfrak{p}$ or $\mathfrak{q}$.) Consider the morphism $B \to T^{-1}S^{-1}B$; it is easy to check that the $b \in B$ which are inverted under this morphism are exactly those $b \in B$ which belong to T_1. Thus the result is proved.

(1·5). LEMMA. *Suppose that W is a closed subscheme of a locally noetherian scheme X, defined by an $\mathcal{O}_X$-ideal I. Then the following conditions are equivalent:*

(*a*) *$C_{W,X} = N_{W,X}$ (that is, the immersion $C_{W,X} \to N_{W,X}$ of (1·3) is an isomorphism) and I/I^2 is a locally free $\mathcal{O}_W = (\mathcal{O}_X/I)$-module;*

(*b*) *At each point w of W, I is generated by a regular sequence in $\mathcal{O}_{X,w}$ (that is, W is a local complete intersection in X);*

(*c*) *$C_{W,X} \to W$ is a vectorbundlescheme.*

Furthermore, if these conditions hold and W is reduced and irreducible, then the local ring $\mathcal{O}_{X/W}$ of W in X is regular. Conversely, if A is a regular local ring, $X = \mathrm{Spec}\,A$ and W is the closed point of X, then the conditions hold.

Proof. In the terminology of (3), IV$_4$ (16·9), (*b*) asserts, by (3), IV$_4$ (16·9·10), that the immersion $W \to X$ is regular. Thus (*a*) $\Leftrightarrow$ (*b*) follows from (3), IV$_4$ (16·9·4).

(*a*) $\Rightarrow$ (*c*) is clear. To see that (*c*) $\Rightarrow$ (*a*), note that the grading on $\mathrm{gr}_W X$ shows that if $\mathrm{gr}_W X \simeq \mathrm{Sym}_{\mathcal{O}_W} M$ for some $\mathcal{O}_W$-module M, then we must have $M \simeq I/I^2$; and for M to define a vectorbundlescheme it must be locally free.

If now W is reduced and irreducible, then setting $A = \mathcal{O}_{X/W}$ and letting $\mathfrak{m}$ be its maximal ideal, by Corollary 3 of (1·4), (*a*) shows (taking fibres over the generic point of W) that

$$\mathrm{Sym}_{A/\mathfrak{m}}\,\mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\sim} \mathrm{gr}_{\mathfrak{m}}A$$

so that A is regular by (3), 0_{IV} (17·1·1). Conversely, this last criterion for regularity shows that $W = \mathrm{Spec}\,(A/\mathfrak{m}) \hookrightarrow X = \mathrm{Spec}\,A$ satisfies the conditions.

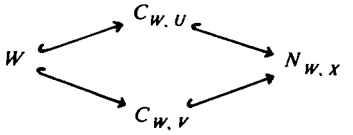
(1·6). Now suppose that W, U are each closed subschemes of a scheme X, with quasi-coherent ideal sheaves I, J respectively; and suppose that $W \subseteq U$, so that $I \supseteq J$. Let $\bar{I} = I/J$ denote the image of I in $\mathcal{O}_U = \mathcal{O}_X/J$. Then the surjection $I \to \bar{I}$ induces surjections $I^n/I^{n+1} \to \bar{I}^n/\bar{I}^{n+1}$ and so (by (1·1)(α)) commutative diagrams

$$\begin{array}{ccc} gr_W U & \longleftarrow & gr_W X \\ \uparrow & & \uparrow \\ \mathrm{Sym}_W \bar{I}/\bar{I}^2 & \longleftarrow & \mathrm{Sym}_W I/I^2 \end{array} \qquad \begin{array}{ccc} C_{W,U} & \longrightarrow & C_{W,X} \\ \downarrow & & \downarrow \\ N_{W,U} & \longrightarrow & N_{W,X} \end{array}$$

in the second of which all morphisms are closed immersions.

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In particular, if U, V, W are closed subschemes of X with $W \subseteq U \cap V$, we obtain a diagram of closed immersions (using (1·3))



which will interest us below.

2. *Intersection multiplicities*

(2·1). (12), ch. v, provides a theory of intersection multiplicities appropriate to schemes satisfying certain properties.

Namely, let X be a locally noetherian scheme. Let U, V be two reduced, irreducible closed subschemes of X, and W an irreducible component of $U \cap V$; give W its reduced subscheme structure. Let $A = \mathcal{O}_{X,W}$ be the local ring of W in X (the stalk of $\mathcal{O}_X$ at the generic point of W). We suppose that A is a *regular* local ring; that is, that (the generic point of) W is a regular point of X.

The subschemes U and V correspond 'at W' to two prime ideals $\mathfrak{p}_U, \mathfrak{p}_V$ of A, and W corresponds to the maximal ideal $\mathfrak{m}$ of A. We set

$$I(X, U.V, W) = \chi^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V),$$

which is shorthand for

$$\sum_{i=0}^{\dim A} (-1)^i l_A(\mathrm{Tor}_i^A(A/\mathfrak{p}_U, A/\mathfrak{p}_V)) \qquad (*)$$

to be the *intersection multiplicity* ($\in \mathbb{N}$) of U and V at W.

We note in passing that for this definition to have the most desirable properties, A should satisfy the conjecture of (12), VB, §4; see also (6). It is conjectured that any regular local ring satisfies the conjecture. However, these concerns will not bother us here.

(12), VC, shows that for the case where X is an algebraic variety (reduced scheme of finite type) over an algebraically closed field, the quantity (*) coincides with the intersection multiplicity in the sense of Weil and Samuel (11).

By its nature, the definition (*) extends naturally to define the intersection multiplicity of two coherent sheaves at an irreducible component W of their supports (provided this latter is regular on X); namely, they correspond 'at W' to two modules M, N over the local ring $A = \mathcal{O}_{X,W}$, and one sets the multiplicity to be equal to

$$\chi^A(M, N) = \sum_{i=0}^{\dim A} (-1)^i l_A(\mathrm{Tor}_i^A(M, N)). \qquad (**)$$

In particular, this shows how to cope with (*) when U, V need no longer be reduced nor irreducible; for they then correspond to ideals $\mathfrak{a}_U, \mathfrak{a}_V$ of A, and we set $M = A/\mathfrak{a}_U$, $N = A/\mathfrak{a}_V$ in (**).

However, let M, N be any two modules of finite type over the regular local ring A; then by ((2), IV, §1·4, theorem 1) each of M, N has a composition series whose successive quotients are A-modules of the form $A/\mathfrak{p}$ ($\mathfrak{p} \in \mathrm{Spec}\, A$). Since the exactness of

$$0 \to M' \to M \to M'' \to 0$$

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implies easily that $\chi^A(M,N) = \chi^A(M',N)+\chi^A(M'',N)$, this reduces the quantity $\chi^A(M,N)$ by additivity to a sum of quantities of the type (*).

(2·2). We now turn to the problem of relating the intersection multiplicity of two subschemes to that of their normal cones.

As in (2·1), let X be a locally noetherian scheme, U, V two closed subschemes, W an irreducible component of $U \cap V$, and $A = \mathcal{O}_{X/W}$ the local ring of W in X; and suppose that A is a regular local ring.

As in (1·6), we have the normal cones $C_{W,U}$ and $C_{W,V}$ to W in U and V realized as closed subschemes of the normal space $N_{W,X}$ to W in X. We note that, by (1·5) and Corollary 3 of (1·4), since A is regular, $N_{W,X}$ and $C_{W,X}$ have the same fibre over the generic point of W.

We also have a copy of W lying inside $C_{W,U} \cap C_{W,V}$. It need not be true that W is an irreducible component of $C_{W,U} \cap C_{W,V}$: for example, let $X = \mathbb{A}^2$ with coordinates x, y, $W = \{(0,0)\}$, $U: y = x^2$, $V: y = 2x^2$; then $C_{W,U} = C_{W,V}$, defined in $N_{W,X} \simeq X$ by $y = 0$.

Thus we must assume a small amount of transversality; namely, we suppose that

$$\operatorname{supp} C_{W,U} \cap \operatorname{supp} C_{W,V} = \operatorname{supp} W,$$

all supports being taken in $N_{W,X}$.

(2·2·1) THEOREM. *Under the above hypotheses,*

$$I(X, U.V, W) = I(N_{W,X}, C_{W,U}.C_{W,V}, W)$$

Below, we first examine the translation of this into local algebra, and then prove it.

(2·3) For the algebraic version of (2·2), we let $\mathfrak{m}$ denote the maximal ideal of A, and suppose that U, V are defined in A by ideals $\mathfrak{a}_U$, $\mathfrak{a}_V$; we set $M = A/\mathfrak{a}_U$, $N = A/\mathfrak{a}_V$, so that the left hand side of (2·2·1) is $\chi^A(M,N)$.

By the Proposition of (1·4), the local ring of W in $N_{W,X}$ is $B_{\mathfrak{q}}$, where $B = \operatorname{gr}_{\mathfrak{m}} A$ and $\mathfrak{q}$ is its augmentation ideal; for by (1·5) since A is regular, $B \simeq \operatorname{Sym}_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. By (1·6), the realizations of $C_{W,U}$ and $C_{W,V}$ inside $N_{W,X}$ correspond to the B-modules $M' = \operatorname{gr}_{\mathfrak{m}} M$ and $N' = \operatorname{gr}_{\mathfrak{m}} N$, and the right-hand side of (2·2·1) is

$$\chi^B_{\mathfrak{q}}(M', N') \quad \text{(notation for } \chi^{B_{\mathfrak{q}}}(M'_{\mathfrak{q}}, N'_{\mathfrak{q}})).$$

Our hypotheses become: $M \otimes_A N$ is of finite length over A (corresponding to the fact that W is a component of $U \cap V$), and $\operatorname{supp}_B M' \cap \operatorname{supp}_B N' = \{\mathfrak{q}\}$ (corresponding to the transversality). Thus we are led to

(2·3·1) THEOREM. *Let A be a regular local ring with maximal ideal $\mathfrak{m}$. Let M, N be two A-modules of finite type. Set $B = \operatorname{gr}_{\mathfrak{m}} A$, $M' = \operatorname{gr}_{\mathfrak{m}} M$, $N' = \operatorname{gr}_{\mathfrak{m}} N$, $\mathfrak{q}$ = the augmentation ideal of B, and suppose that* $\operatorname{supp}_B M' \cap \operatorname{supp}_B N' = \{\mathfrak{q}\}$. *Then $M \otimes_A N$ is of finite length over A, and*

$$\chi^A(M,N) = \chi^B_{\mathfrak{q}}(M', N').$$

(That is, $I(\operatorname{Spec} A, \tilde{M}.\tilde{N}, \{\mathfrak{m}\}) = I(\operatorname{Spec} B, \tilde{M}'.\tilde{N}', \{\mathfrak{q}\})$, where $\sim$ denotes the process of taking the quasi-coherent sheaf associated to a module.)

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Proof. We first note that we can construct a spectral sequence of 'homological' type

$$E^1_{p,q} \Rightarrow \mathrm{Tor}^A_{p+q}(M, N)$$

such that the associated single complex of $E^1_{p,q}$ is

$$E^1_{(n)} = \oplus_{p+q=n} E^1_{p,q} = \mathrm{Tor}^B_n(M', N').$$

As in (12), p. 11–17, this arises from a filtered complex $K_.$ (of degree -1), constructed as follows. We first pick a resolution $X_. \to M$ of M by filtered free A-modules (direct sums of copies of A equipped with some translation of its $\mathfrak{m}$-adic filtration) such that the boundary morphisms of $X_.$ are strict morphisms for the filtrations. Then setting $K_. = X_. \otimes_A N$, the spectral sequence arises from $K_.$ in the usual way ((3), 0_{III} (11·2)). Since A is noetherian and the modules are of finite type, the spectral sequence converges.

We now see that the Euler–Poincaré characteristic of the single complex $E^1_{(\cdot)}$ exists. For, since $\mathrm{supp}_B M' \cap \mathrm{supp}_B N' = \{\mathfrak{q}\}$ and $\mathfrak{q}$ is maximal, each B-module $\mathrm{Tor}^B_n(M', N')$ has support contained in $\{\mathfrak{q}\}$ (since the formation of Tor commutes with localization), and so is of finite length; and moreover

$$l_B(\mathrm{Tor}^B_n(M', N')) = l_{B_\mathfrak{q}}(\mathrm{Tor}^{B_\mathfrak{q}}_n(M'_\mathfrak{q}, N'_\mathfrak{q}))$$

by (2), IV, § 2·5, cor. to prop. 8. Thus

$$\chi(E^1_{(\cdot)}) = \sum_{n \in \mathbb{N}} (-1)^n l_B(\mathrm{Tor}^B_n(M', N')) = \chi^B_\mathfrak{q}(M', N'),$$

the sum being in fact finite, since, by (1·5) and corollary 3 of (1·4), B is just a polynomial ring (in finitely many variables) over the field $A/\mathfrak{m}$.

By a general property of spectral sequences ((3), 0_{III} (11·10·3)) it follows that the Euler–Poincaré characteristic of the abutment

$$\sum_{n \in \mathbb{N}} (-1)^n l_A(\mathrm{Tor}^A_n M, N)) = \chi^A(M, N)$$

exists (so that in particular $M \otimes_A N$ is of finite length over A) and is equal to $\chi^B_\mathfrak{q}(M', N')$, as required.

(2·4) Corollary. *The result* (2·2·1) *holds.*

(2·5) *Remarks.* (*a*) As is usual, the hypothesis of regularity on A could probably be replaced by one of finiteness of homological dimension on M; compare (10).

(*b*) Since the right-hand side of (2·2·1) can be regarded as an intersection multiplicity of cones in affine space, the result can be interpreted as reducing, under the relatively mild transversality condition on the normal cones, a general intersection multiplicity to one of this kind; and thus it generalizes the 'criterion for multiplicity one' (see, for example, (11)).

(*c*) Whatever the regular local ring A is, the right-hand side of (2·3·1) is an intersection multiplicity defined over an equicharacteristic local ring (even an algebra over a field), and so 'Serre's conjecture' (6) holds for it. Thus it can be used to provide examples where Serre's conjecture holds although A is not equicharacteristic nor unramified nor of dimension $\leqslant 4$.

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(2·6) We end with an example of an application of (2·4).

Suppose that the following equations in $y, x_1, x_2, \ldots, x_n$ have coefficients in an algebraically closed field k

$$\begin{aligned} y^{m_1}x_1^{n_1} &= f_1(x_1, \ldots, x_n), \\ y^{m_2}x_2^{n_2} &= f_2(x_1, \ldots, x_n), \\ &\cdots \\ y^{m_n}x_n^{n_n} &= f_n(x_1, \ldots, x_n), \end{aligned}$$

and that they are all homogeneous, and each $m_i > 0$. Then we claim that they have a solution over k other than the solution $(y, x_1, \ldots, x_n) = (1, 0, \ldots, 0)$.

For, consider the subscheme of $\mathbb{P}_k^n$ that they define. If it has dimension > 0, such solutions exist; otherwise, it is a set of points counted with multiplicities, and by Bezout's theorem (for a family of divisors: see, for example, (11), II, §6·7) the total multiplicity is the product of the degrees of the equations, that is

$$\prod_{i=1}^{n} (m_i + n_i).$$

However, the multiplicity of the solution $(1, 0, \ldots, 0)$ is $\prod_{i=1}^{n} n_i$, by (2·4), since the tangent cone at this point to the divisor defined by the ith equation is given by $x_i^{n_i} = 0$. Since this is strictly less than the total multiplicity, there must be other solutions.

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