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The virtually generating graph of a profinite group

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Abstract. We consider the graph $\Gamma_{\mathrm{virt}}(G)$ whose vertices are the elements of a finitely generated profinite group G and where two vertices x and y are adjacent if and only if they topologically generate an open subgroup of G. We investigate the connectivity of the graph $\Delta_{\mathrm{virt}}(G)$ obtained from $\Gamma_{\mathrm{virt}}(G)$ by removing its isolated vertices. In particular, we prove that for every positive integer t, there exists a finitely generated prosoluble group G with the property that $\Delta_{\mathrm{virt}}(G)$ has precisely t connected components. Moreover, we study the graph $\widetilde{\Gamma}_{\mathrm{virt}}(G)$, whose vertices are again the elements of G and where two vertices are adjacent if and only if there exists a minimal generating set of G containing them. In this case, we prove that the subgraph $\widetilde{\Delta}_{\mathrm{virt}}(G)$ obtained removing the isolated vertices is connected and has diameter at most 3.

1 Introduction

The generating graph of a finite group is an interesting notion and has been much studied in the literature. In [7], the investigation on this topic was extended to the context of profinite groups. Let G be a 2-generated profinite group. The generating graph $\Gamma(G)$ is defined with vertex set G where two elements x and y are connected if and only if x and y (topologically) generate G. An open conjecture is that if G is a finite group, then the subgraph $\Delta(G)$ obtained from $\Gamma(G)$ by removing its isolated vertices is connected. In [7], it was proved that the conjecture is true if G is prosoluble, but an example is given showing that it fails for arbitrary 2-generated profinite groups. Recently, during the workshop 'Groups with Geometrical and Topological Flavours', Y. Barnea proposed to investigate a related graph. Following his suggestion, we define the virtually generating graph $\Gamma_{\mathrm{virt}}(G)$ associated with a profinite group G as the graph whose vertices are the elements of G and in which two elements x and y are connected if and only if x and y generate (topologically) an open subgroup of G. In particular, we address the question whether the graph $\Delta_{\mathrm{virt}}(G)$ obtained from $\Gamma_{\mathrm{virt}}(G)$ by removing its isolated vertices is connected. The answer is negative. Let $G_p = (\mathrm{SL}(2, 2^p))^{\delta_p}$, where p is a prime and δ_p is the largest positive integer with the property that the direct power $(\mathrm{SL}(2, 2^p))^{\delta_p}$ can be generated by 2-elements and consider $G = \prod_p G_p$. In [7, Section 3], it was proved that the graph $\Delta(G)$ has $2^{\aleph_0}$ connected components. This is a consequence of the fact that the diameter of $\Delta(G_p)$ tends to infinity with p. In Section 2, we will see that $\Delta_{\mathrm{virt}}(G)$ also has

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$2^{\aleph_0}$ connected components. In Section 3, we will prove a more surprising result, considering the graph $\Delta_{\mathrm{virt}}(G)$, when G is prosoluble. In [6], it was proved that if G is a finite soluble group, then $\Delta(G)$ is connected, with diameter at most 3, and this statement was generalized in [7, Theorem 3] to the case of prosoluble groups. However, this result cannot be extended to $\Delta_{\mathrm{virt}}(G)$. Indeed, we prove the following theorem.

Theorem 1.1 *For every positive integer t, there exists a finitely generated prosoluble group G with the property that $\Delta_{\mathrm{virt}}(G)$ has precisely t connected components.*

In the second part of the paper, we extend to the profinite groups the notion of independence graph, given in [8] for finite groups, and we introduce the new definition of virtually independence graph of a profinite group. Let G be a finitely generated profinite group. A generating set X of G is said to be minimal if no proper subset of X (topologically) generates G. If G is a finite group, we denote by $d(G)$ and $m(G)$, respectively, the smallest and the largest cardinality of a minimal generating set of a finite group G. A nice result in universal algebra, the Tarski irredundant basis theorem (see, for example, [1, Theorem 4.4]), implies that if G is a finite group, then, for every positive integer k with $d(G) \leq k \leq m(G)$, G contains a minimal generating set of cardinality k. In Section 4, we will prove that this result can be extended to the profinite case. In this case, we define $m(G)$ as the supremum of the sizes of the minimal generating set of G.

Theorem 1.2 *Let G be a finitely generated profinite group. For every positive integer k with $d(G) \leq k \leq m(G)$, G contains a minimal generating set of cardinality k. Moreover, $m(G)$ is finite if and only if the Frattini subgroup of G has finite index in G.*

We denote by $\widetilde{\Gamma}(G)$ the graph whose vertices are the elements of G and in which two vertices x and y are joined by an edge if and only if $x \neq y$ and there exists a finite minimal generating set of G containing x and y. Roughly speaking, x and y are adjacent vertices of $\widetilde{\Gamma}(G)$ if they are "independent", so we call $\Gamma(G)$ the independence graph of G. In Section 5, extending results proved in [8] in the case of finite groups, we show that if G is a finitely generated profinite group and g is an isolated vertex of $\widetilde{\Gamma}(G)$, then either g belongs to the Frattini subgroup of G or it is a topological generator of G. Moreover, the graph obtained from $\widetilde{\Gamma}(G)$ by removing the isolated vertices is always connected. Also, in this case, instead of generating subsets, we can consider virtually generating subsets, *i.e.,* subsets generating an open subgroup of G. In particular, we say that two elements x and y are virtually independent if there exists a minimal generating set of an open subgroup of G containing them. In Section 6, we define the virtually independent graph $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ of G as the graph whose vertices are the elements of G and in which two vertices are joined by an edge if and only if they are virtually independent. Moreover, we denote by $\widetilde{\Delta}_{\mathrm{virt}}(G)$ the subgraph induced by the non-isolated vertices of the virtually independent graph of G. In particular, we prove the following results.

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Theorem 1.3 *Let G be a finitely generated profinite group. If $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ contains a non-trivial isolated vertex g, then either g is a topological generator of G or one of the following occurs:*

(i) *$G \cong C_{p^n}$ is a cyclic group of p-power order and all the vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ are isolated.*

(ii) *$G \cong \mathbb{Z}_p$, the group of the p-adic integers, and all the vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ are isolated.*

(iii) *$G \cong Q_{2^n} = \langle x, y \mid x^{2^{n-1}}, y^2 = x^{2^{n-2}}, x^{-1}yx = y^{-1} \rangle$ is a generalized quaternion group and $g = y^2$ is the unique non-trivial isolated vertex of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$.*

Theorem 1.4 *If G is a finitely generated profinite group, then $\widetilde{\Delta}_{\mathrm{virt}}(G)$ is connected and* $\mathrm{diam}(\widetilde{\Delta}_{\mathrm{virt}}(G)) \leq 3$.

We will exhibit an example, showing that the bound $\mathrm{diam}(\widetilde{\Delta}_{\mathrm{virt}}(G)) \leq 3$ given by the previous theorem is best possible.

2 An Example of a Profinite Group G such that $\Delta_{\mathbf{inv}}(G)$ has $2^{\aleph_0}$ Connected Components

Let $G_p := (\mathrm{SL}(2, 2^p))^{\delta_p}$, where p is a prime and δ_p is the largest positive integer with the property that the direct power $(\mathrm{SL}(2, 2^p))^{\delta_p}$ can be generated by 2-elements. The graph $\Delta(G_p)$ is connected for every prime p, and, by [2, Theorem 1.3], there exists an increasing sequence $(p_n)_{n\in\mathbb{N}}$ of odd primes, such that $\mathrm{diam}(\Delta(G_{p_n})) \geq 2^n$ for every $n \in \mathbb{N}$. Consider the cartesian product

$$G := \prod_{n\in\mathbb{N}} G_{p_n},$$

with the product topology, and for any $n \in \mathbb{N}$, let $\pi_n : G \to G_{p_n}$ be the projection homomorphism.

Lemma 2.1 *Let $x = (x_n)_{n\in\mathbb{N}}$, $y = (y_n)_{n\in\mathbb{N}} \in G$. The subgroup $\langle x, y \rangle$ is open in G if and only if there exists $m \in \mathbb{N}$ such that $\langle x_n, y_n \rangle = G_{p_n}$ for any $n \geq m$.*

Proof Let $H := \langle x, y \rangle$ and, for $n \in \mathbb{N}$, set $H_n := \pi_n(G)$. Clearly, $H \leq \prod_{n\in\mathbb{N}} H_n$ and $|G : H| \geq |G : \prod_{n\in\mathbb{N}} H_n| = \prod_{n\in\mathbb{N}} |G_{p_n} : H_n|$. If H is open in G, then $|G : H|$ is finite and therefore $H_n = G_{p_n}$ for all but finitely many n. Conversely, assume that $H_n = G_{p_n}$ for any $n \leq m$ and set $K := \prod_{n<m} G_{p_n}$. It follows from the Goursat's Lemma (see, for example, [4, Theorem 5.5.1]) that $HK = G$, and so H is open in G. ∎

Let $V_{\mathrm{virt}}(G)$ be the set of the non-isolated vertices of $\Gamma_{\mathrm{virt}}(G)$. Moreover, given $x = (x_n)_{n\in\mathbb{N}} \in G$, define $\Lambda(x) := \{n \in \mathbb{N} \mid x_n \in V(G_{p_n})\}$, where $V(G_{p_n})$ is the set of the non-isolated vertices of $\Gamma(V(G_{p_n}))$.

Corollary 2.2 *$x = (x_n)_{n\in\mathbb{N}}$ is a non-isolated vertex of $\Gamma_{\mathrm{virt}}(G)$ if and only if $n \in \Lambda(x)$ for all but finitely many $n \in \mathbb{N}$.*

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Lemma 2.3 *Let $x = (x_n)_{n\in\mathbb{N}} \in V_{\mathrm{virt}}(G)$ and let Ω_x be the connected component of $\Delta_{\mathrm{virt}}(G)$ containing x. Then $y = (y_n)_{n\in\mathbb{N}}$ belongs to Ω_x if and only if*

$$\sup_{n\in\Lambda(x)\cap\Lambda(y)} \mathrm{dist}_{\Delta(G_{p_n})}(x_n, y_n) < \infty.$$

Proof Assume that $y = (y_n)_{n\in\mathbb{N}} \in \Omega_x$ and let $m = \mathrm{dist}_{\Delta(G)}(x, y)$. It follows from Lemma 2.1 that $\mathrm{dist}_{\Delta(G_{p_n})}(x_n, y_n) \le m$ for all but finitely many $n \in \Lambda(x) \cap \Lambda(y)$. Conversely, assume $y = (y_n)_{n\in\mathbb{N}} \in V_{\mathrm{virt}}(G)$ and $\mathrm{dist}_{\Delta(G_{p_n})}(x_n, y_n) \le m$ for every n in a cofinite subset Λ of $\Lambda(x) \cap \Lambda(y)$. If $n \in \Lambda$, then there is a path

$$x_n = x_{n,0}, x_{n,1}, \ldots, x_{n,\mu_n} = y_n,$$

with $\mu_n \le m$, joining x_n and y_n in the graph $\Delta(G_{p_n})$. For $0 \le i \le m$, set

$$\widetilde{x}_{n,i} := \begin{cases} x_n & \text{if } n \notin \Lambda, \\ x_{n,i} & \text{if } n \in \Lambda,\ i < \mu_n, \\ x_{n,\mu_n} & \text{if } n \in \Lambda,\ i \ge \mu_n \text{ and } i - \mu_n \text{ is even}, \\ x_{n,\mu_n-1}x_{n,\mu_n} & \text{if } n \in \Lambda,\ i \ge \mu_n \text{ and } i - \mu_n \text{ is odd}. \end{cases}$$

Then

$$x = \widetilde{x}_0 = (\widetilde{x}_{n,0})_{n\in\mathbb{N}},\ \widetilde{x}_1 = (\widetilde{x}_{n,1})_{n\in\mathbb{N}}, \ldots, y = \widetilde{x}_m = (\widetilde{x}_{n,m})_{n\in\mathbb{N}}$$

is a path joining x and y in the graph $\Delta_{\mathrm{virt}}(G)$, so $y \in \Omega_x$. ∎

Proposition 2.4 *$\Delta_{\mathrm{virt}}(G)$ has $2_0^{\aleph}$ different connected components.*

Proof Fix $x = (x_n)_{n\in\mathbb{N}} \in \Delta_{\mathrm{virt}}(G)$. Let τ be a real number with $\tau > 1$. Since

$$\mathrm{diam}(\Delta(G_{p_n})) \ge 2^n \ge 1 + \lfloor n/\tau \rfloor,$$

for every $n \in \mathbb{N}$, there exists $y_{\tau,n} \in G_{p_n}$ such that $\mathrm{dist}_{\Delta(G_{p_n})}(x_n, y_{\tau,n}) = 1 + \lfloor n/\tau \rfloor$. If $\tau_2 > \tau_1$, then

$$\mathrm{dist}(y_{\tau_2,n}, y_{\tau_1,n}) \ge \mathrm{dist}(x_n, y_{\tau_1,n}) - \mathrm{dist}(x_n, y_{\tau_2,n}) = \lfloor n/\tau_1 \rfloor - \lfloor n/\tau_2 \rfloor$$

tends to infinity with n, so, by Lemma 2.3, $\Omega_{y_{\tau_1}} \ne \Omega_{y_{\tau_2}}$. ∎

3 Proof of Theorem 1.1

We consider the set Ω of the elements $(x_1, x_2, \ldots, x_{2t-1}, x_{2t}) \in \mathbb{F}_2^{2t}$ such that $(x_{2i-1}, x_{2i}) \ne (0, 0)$ for $1 \le i \le t$. With any $\omega \in \Omega$, we associate a different odd prime number p_ω. Let $H := \langle y_1, y_2, \ldots, y_{2t-1}, y_{2t}\rangle$ be an elementary abelian 2-group of rank $2t$. For any $\omega = (x_1, x_2, \ldots, x_{2t-1}, x_{2t}) \in \Omega$, we define an action of H on the direct product $N_\omega := (\mathbb{Z}_{p_\omega})^3$ of three copies of the group $\mathbb{Z}_{p_\omega}$ of the p_ω-adic integers as follows:

$$(z_1, z_2, z_3)^{y_i} = \big(z_1, z_2, (-1)^{x_i} \cdot z_3\big).$$

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We consider the semidirect product

$$G := \left(\prod_{\omega \in \Omega} N_\omega \right) \rtimes H.$$

Notice that G is a prosoluble group that can be generated by $2t$ elements, as it follows easily from the following lemma.

Lemma 3.1 *Let* $\omega = (x_1, x_2, \dots, x_{2t-1}, x_{2t}) \in \Omega$ *and* $v_\omega := (1,0,1)$, $w_\omega := (0,1,-1) \in N_\omega$. *Then* $N_\omega \leq \cap_{1 \leq i \leq t} \langle v_\omega y_{2i-1}, w_\omega y_{2i} \rangle$.

Proof We prove $N_\omega \leq \langle v_\omega y_1, w_\omega y_2 \rangle$ (the other inclusions $N_\omega \leq \langle v_\omega y_{2i-1}, w_\omega y_{2i} \rangle$ can be proved with a similar argument). Let $\rho_1 := v_\omega y_1$ and $\rho_2 := w_\omega y_2$. Notice that

$$\rho_1^2 = (2, 0, 0^{x_1} \cdot 2), \qquad \rho_2^2 = (0, 2, -0^{x_2} \cdot 2),$$

$$[\rho_1, \rho_2] = \begin{cases} (0,0,4) & \text{if } (x_1, x_2) = (1,1), \\ (0,0,-2) & \text{otherwise.} \end{cases}$$

Since $p_\omega \neq 2$, it follows that $N_\omega = \langle (1,0,0), (0,1,0), (0,0,1) \rangle \leq \langle \rho_1, \rho_2 \rangle$. ∎

Now, for $1 \leq i \leq t$, set

$$\sigma_{2i-1} := \left((v_\omega)_{\omega \in \Omega} \right) y_{2i-1}, \quad \sigma_{2i} := \left((w_\omega)_{\omega \in \Omega} \right) y_{2i}.$$

It follows immediately from Lemma 3.1 that $\langle \sigma_1, \sigma_2 \rangle, \dots, \langle \sigma_{2t-1}, \sigma_{2t-2} \rangle$ are open subgroups of G (with index 2^{t-2}), and, therefore, $(\sigma_1, \sigma_2), \dots, (\sigma_{2t-1}, \sigma_{2t-2})$ are edges of $\Gamma_{\mathrm{virt}}(G)$. Now let

$$N := \prod_{\omega \in \Omega} N_\omega \quad \text{and} \quad \Sigma_i := N \rtimes \langle y_{2i-1}, y_{2i} \rangle \text{ for } 1 \leq i \leq t.$$

Proposition 3.2 *Suppose that* g_1 *and* g_2 *are two adjacent vertices of* $\Gamma_{\mathrm{virt}}(G)$. *Then* $\{g_1, g_2\} \subseteq \Sigma_i \backslash N$ *for some* $1 \leq i \leq t$. *More precisely, for* $g_1 = n_1 h_1$, $g_2 = n_2 h_2$ *with* $n_i \in N$, $h_1 = \prod_{1 \leq j \leq 2t} y_j^{a_j}$ *and* $h_2 = \prod_{1 \leq j \leq 2t} y_j^{b_j}$, *there exists* $1 \leq i \leq t$ *such that the* 2×2*-matrix*

$$C_i := \begin{pmatrix} a_{2i-1} & a_{2i} \\ b_{2i-1} & b_{2i} \end{pmatrix}$$

is invertible, and, moreover, for all $j \notin \{2i-1, 2i\}$, *one has* $a_j = b_j = 0$.

Proof We claim that the system

$$\begin{cases} a_1 x_1 + \cdots + a_{2t} x_{2t} = 0, \\ b_1 x_1 + \cdots + b_{2t} x_{2t} = 0, \end{cases} \tag{3.1}$$

where the coefficients are viewed as elements of $\mathbb{F}_2$, has no solution in Ω. Indeed, if $\omega = (x_1, \dots, x_{2t})$ is a solution of the previous system, then h_1 and h_2 centralize N_ω, and consequently, $\langle g_1, g_2 \rangle \cap N_\omega = \langle n_1, n_2 \rangle$ is a 2-generated subgroup of $N_\omega = (\mathbb{Z}_{p_\omega})^3$.

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But this would imply that

$$|G : \langle g_1, g_2\rangle| \geq |N_\omega\langle g_1, g_2\rangle : \langle g_1, g_2\rangle| = |N_\omega : N_\omega \cap \langle g_1, g_2\rangle| = \infty,$$

and therefore $\langle g_1, g_2\rangle$ is not an open subgroup of G.

For $1 \leq i \leq t$, consider the following matrices, with coefficients in $\mathbb{F}_2$:

$$C_i := \begin{pmatrix} a_{2i-1} & a_{2i} \\ b_{2i-1} & b_{2i} \end{pmatrix}, \quad D_i := \begin{pmatrix} a_1 & \dots & a_{2i-2} & a_{2i+1} & \dots & a_{2t} \\ b_1 & \dots & b_{2i-2} & b_{2i+1} & \dots & b_{2t} \end{pmatrix}.$$

Our statement, for given g_1, g_2, is equivalent to saying that there exists i such that C_i is invertible and $D_i = 0$. First, we prove that if C_i is invertible, then $D_i = 0$. Let Ω^* be the set of the elements $(r_1, r_2, \dots, r_{t-3}, r_{t-2}) \in \mathbb{F}_2^{2(t-1)}$ such that $(r_{2i-1}, r_{2i}) \neq (0, 0)$ for $1 \leq i \leq t-1$. Assume, by contradiction, that $D_i \neq 0$. Since $\langle \Omega^* \rangle = \mathbb{F}_2^{2t-2}$, there exists an element $(x_1, \dots, x_{2i-2}, x_{2i+1} \dots, x_{2t}) \in \Omega^*$ such that

$$D_i \begin{pmatrix} x_1 \\ \vdots \\ x_{2i-2} \\ x_{2i+1} \\ \vdots \\ x_{2t} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If C_i is invertible, then there exists $(x_{2i-1}, x_{2i}) \neq (0, 0)$ such that

$$C_i \begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

and $(x_1, \dots, x_{2t}) \in \Omega$ is a solution of (3.1). We remain with the case where none of the matrices $C_1, \dots, C_t$ is invertible. However, in this case, for any $1 \leq i \leq t$, there exists $(x_{2i-1}, x_{2i}) \neq (0, 0)$ such that

$$C_i \begin{pmatrix} x_{2i-1} \\ x_{2i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and again, $(x_1, \dots, x_{2t}) \in \Omega$ is a solution of (3.1). ∎

For $1 \leq i \leq t$, let Γ_i be the set of the non-isolated vertices of $\Gamma_{\text{virt}}(G)$ contained in Σ_i. By Proposition 3.2, Γ_i is a disjoint union of connected components of $\Gamma_{\text{virt}}(G)$. We are going to prove that indeed Γ_i is a connected component.

Lemma 3.3 *Let $g = \big((n_w)_{\omega\in\Omega}\big)h \in \Sigma_i$, with $h \in H$ and $n_\omega = (z_{1,\omega}, z_{2,\omega}, z_{3,\omega}) \in N_\omega$. If g is a non-isolated vertex of $\Gamma_{inv}(G)$, then the following conditions hold:*

(i) $h \neq 1$;
(ii) $(z_{1,\omega}, z_{2,\omega}) \neq (0, 0)$ *for any* $\omega \in \Omega$;
(iii) *if h centralizes N_ω, then* $z_{3,\omega} \neq 0$.

Proof We can assume $i = 1$. Suppose that there exists $g^* = \big((n^*_\omega)_{\omega\in\Omega}\big)h^*$ such that $X := \langle g, g^*\rangle$ is an open subgroup of G. By Proposition 3.2, $K := \langle h, h^*\rangle = \langle y_1, y_2\rangle$, and

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hence $h \neq 1$. Let

$$A_\omega := \{(z_1, z_2, 0) \mid z_1, z_2 \in \mathbb{Z}_{p_\omega}\} \leq N_\omega, \quad B_\omega := \{(0, 0, z_3) \mid z_3 \in \mathbb{Z}_{p_\omega}\} \leq N_\omega.$$

Since X is open in G, $(X \cap N_\omega)B_\omega/B_\omega = \langle (z_{1,\omega}, z_{2,\omega}, 0)B_\omega, (z^*_{1,\omega}, z^*_{2,\omega}, 0)B_\omega \rangle$ has finite index in $N_\omega/B_\omega \cong (\mathbb{Z}_{p_\omega})^2$, and this implies $(z_{1,\omega}, z_{2,\omega}) \neq (0, 0)$.

Finally, assume $[h, N_\omega] = 0$ and set $M_\omega := (\prod_{\widetilde{\omega} \neq \omega} N_{\widetilde{\omega}}) \times A_\omega$. Consider $G_\omega := G/M_\omega$. We have that XM_ω/M_ω is an open subgroup of G_ω, and this implies that $\langle (0, 0, z_{3,\omega})h, (0, 0, z^*_{3,\omega})h^* \rangle$ has finite index in $B_\omega K \cong \mathbb{Z}_{p_\omega} \rtimes K$. By definition, $[K, B_\omega] \neq 0$. Since $[h, B_\omega] = 0$ and $K = \langle h, h^* \rangle$, we deduce $[h^*, B_\omega] \neq 0$. This implies in particular that $((0, 0, z^*_{3,\omega})h^*)^2 = 1$. If $z_{3,\omega} = 0$, then $(0, 0, z_{3,\omega})h, (0, 0, z^*_{3,\omega})h^*$ are two commuting involutions and do not generate a subgroup of finite index of $B_\omega K$. ∎

Lemma 3.4 *If $g \in \Sigma_i$ satisfies conditions (i)–(iii) of Lemma 3.3, then g is a non-isolated vertex of $\Gamma_{inv}(G)$. In particular, if g_1 and g_2 are two elements of Σ_i satisfying (i)–(iii), then there exists $\widetilde{g}$ in Σ_i, adjacent to both g_1 and g_2 in $\Gamma_{inv}(G)$.*

Proof We can assume $i = 1$. Let $g_1 = ((n_{1,\omega})_{\omega \in \Omega})h_1$ and $g_2 = ((n_{2,\omega})_{\omega \in \Omega})h_2$, with $h_1, h_2 \in K = \langle y_1, y_2 \rangle$. Choose $\widetilde{h} \in K \backslash \{1, h_1, h_2\}$. We want to construct $\widetilde{n}_\omega$ in N_ω such that $((\widetilde{n}_\omega)_{\omega \in \Omega})\widetilde{h}$ is adjacent to both g_1 and g_2 in $\Gamma_{\mathrm{inv}}(G)$. It suffices to choose, for any $\omega \in \Omega$, $\widetilde{n}_\omega$ in N_ω with the property that $\langle n_{1,\omega}h_1, \widetilde{n}_\omega\widetilde{h} \rangle \cap N_\omega$ and $\langle n_{2,\omega}\widetilde{h}_2, \widetilde{n}_\omega\widetilde{h} \rangle \cap N_\omega$ have finite index in N_ω. Suppose $n_{1,\omega} = (z_{11}, z_{12}, z_{13})$ and $n_{2,\omega} = (z_{21}, z_{22}, z_{23})$. We can choose $(\widetilde{z}_1, \widetilde{z}_2) \in (\mathbb{Z}_{p_\omega})^2$ with the property that $\langle (\widetilde{z}_1, \widetilde{z}_2), (z_{11}, z_{12}) \rangle$ and $\langle (\widetilde{z}_1, \widetilde{z}_2), (z_{21}, z_{22}) \rangle$ have finite index in $(\mathbb{Z}_{p_\omega})^2$. For $j \in \{1, 2\}$, set $\eta_j = 0$ if $[h_j, N_\omega] = 0$ and $\eta_j = 1$ otherwise. Similarly, set $\widetilde{\eta} = 0$ if $[\widetilde{h}, N_\omega] = 0$, and $\widetilde{\eta}_j = 0$ otherwise. Choose $\widetilde{z}_3 \in \mathbb{Z}_{p_\omega}$ such that $\widetilde{z}_3 \notin \{\widetilde{\eta}(-1)^{\widetilde{\eta}+1}z_{13}, \widetilde{\eta}(-1)^{\widetilde{\eta}+1}z_{23}\}$. We claim that $\widetilde{n}_\omega := (\widetilde{z}_1, \widetilde{z}_2, \widetilde{z}_3)$ satisfies the requested property. More precisely, for $j \in \{1, 2\}$, the subgroup $\langle (n_{j,\omega}h_j)^2, (\widetilde{n}_\omega\widetilde{h})^2, [n_{1,\omega}h_j, \widetilde{n}_\omega\widetilde{h}] \rangle$ has finite index in N_ω. Indeed, we have the following two possibilities:

(a) $\eta_j = 0$. In this case, by hypothesis, $z_{j3} \neq 0$. Moreover, since $K = \langle h_j, \widetilde{h} \rangle$, we must have $[N_\omega, \widetilde{h}] \neq 0$. So

$$\begin{aligned} &\langle (n_{j,\omega}h_j)^2, (\widetilde{n}_\omega\widetilde{h})^2, [n_{j,\omega}h_j, \widetilde{n}_\omega\widetilde{h}] \rangle \\ &\quad = \langle (2z_{j1}, 2z_{j2}, 2z_{j3}), (2\widetilde{z}_1, 2\widetilde{z}_2, 0), (0, 0, -2z_{j3}) \rangle \\ &\quad = \langle (z_{j1}, z_{j2}, 0), (\widetilde{z}_1, \widetilde{z}_2, 0), (0, 0, z_{j3}) \rangle \end{aligned}$$

has finite index in N_ω.

(b) $\eta_j \neq 0$. In this case,

$$\begin{aligned} &\langle (n_{j,\omega}h_j)^2, (\widetilde{n}_\omega\widetilde{h})^2, [n_{j,\omega}h_1, \widetilde{n}_\omega\widetilde{h}] \rangle \\ &\quad = \langle (2z_{j1}, 2z_{j2}, 0), (2\widetilde{z}_1, 2\widetilde{z}_2, 2(1 - \widetilde{\eta})\widetilde{z}_3), (0, 0, 2\widetilde{\eta}z_{j3} + (-1)^{\widetilde{\eta}}2\widetilde{z}_3) \rangle \\ &\quad = \langle (z_{j1}, z_{j2}, 0), (\widetilde{z}_1, \widetilde{z}_2, (1 - \widetilde{\eta})\widetilde{z}_3), (0, 0, \widetilde{\eta}z_{j3} + (-1)^{\widetilde{\eta}}\widetilde{z}_3) \rangle \end{aligned}$$

has finite index in N_ω, since $\widetilde{\eta}z_{j3} + (-1)^{\widetilde{\eta}}\widetilde{z}_3 \neq 0$. ∎

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Combining Proposition 3.2 and Lemmata 3.3 and 3.4, we reach the following conclusion.

Proposition 3.5 *$\Gamma_1, \dots, \Gamma_t$ are the connected components of the graph $\Delta_{\mathrm{virt}}(G)$.*

4 Minimal Generating Sets

Let $G = \langle x_1 \dots, x_d \rangle$ be a finitely generated profinite group. Let $m(G) := \sup\{|Y| \mid Y \text{ is a finite minimal generating set of } G\}$. By [5, Theorem 5], $m(G)$ is finite if and only if G contains only finitely many maximal subgroups, *i.e.,* if and only if the Frattini subgroup of G has finite index in G.

Proposition 4.1 *Let G be a finitely generated profinite group. If $m(G)$ is infinite, then G contains an infinite minimal generating set.*

Proof Let $\{N_t\}_{t\in\mathbb{N}}$ be a descending chain of open normal subgroups of G with $\bigcap_{t\in\mathbb{N}} N_t = 1$ For any $t \in \mathbb{N}$, set $m_t := m(G/N_t)$ and let Ω_t be the set of the elements $(x_n)_{n\in\mathbb{N}} \in G^{\aleph_0}$ with the properties that $x_1N, \dots, x_{m_t}N$ is a minimal generating set of G/N_t and $x_n \in N_t$ for every $n > m_t$. It can be easily seen that Ω_t is a non-empty closed subset of $G^{\aleph_0}$. Let $t_1 > t_2 > \cdots > t_r$. Iterated applications of [5, Lemma 8] imply $\Omega_{t_1} \cap \Omega_{t_2} \cap \cdots \cap \Omega_{t_r} \neq \varnothing$. Since $G^{\aleph_0}$ is compact, $\bigcap_{t\in\mathbb{N}} \Omega_t \neq \varnothing$. Let $X := \{x_n \mid n \in \mathbb{N}\}$ with $(x_n)_{n\in\mathbb{N}} \in \bigcap_{t\in\mathbb{N}} \Omega_t$. We have $\langle X\rangle N_t = G$ for every $t \in \mathbb{N}$, so X is a generating set of G. We claim that X is a minimal generating set. Indeed, let $n \in \mathbb{N}$ and consider $Y = X\backslash\{x_n\}$. There exists t such that $n \leq m_t$, but then, since $x_1N, \dots, x_{m_t}N$ is a minimal generating set of G/N_t and $x_n \in N_t$ for every $n > m_t$, $\langle Y\rangle N_t = \langle x_1, \dots, x_{n-1}, x_{n+1}, \dots x_{m_t}\rangle N_t \neq G$. ∎

Theorem 4.2 *Let G be a finitely generated profinite group. For every positive integer k with $d(G) \leq k \leq m(G)$, there exists a minimal generating set of G of size k.*

Proof Let $k \in \mathbb{N}$ with $d(G) \leq k \leq m(G)$. By the way in which $m(G)$ is defined, there exists a minimal generating set $X = \{x_1, \dots, x_t\}$ of G with $k \leq t$. For $1 \leq i \leq t$, let $X_i := X\backslash\{x_i\}$. Since $\langle X_i\rangle \neq G$, there exists an open normal subgroup N_i of G such that $\langle X_i\rangle N_i \neq G$. Set $N := N_1 \cap \cdots \cap N_t$. We have $\langle X_i\rangle N \leq \langle X_i\rangle N_i < G$ for any $1 \leq i \leq t$, so $\{x_1N, \dots, x_tN\}$ is a minimal generating set for G/N. By the Tarski irredundant basis theorem [1, Theorem 4.4], since $d(G/N) \leq d(G) \leq k \leq t \leq m(G/N)$, there exists a minimal generating set $\{y_1N, \dots, y_kN\}$ of G/N. By the profinite version of the Gaschütz Lemma (see, for example, [3, 15.30]), there exists $n_1, \dots, n_k \in N$ such that $\{y_1n_1, \dots, y_un_k\}$ is a (minimal) generating set of G. ∎

Proposition 4.3 *If G is a finitely generated profinite group and $Y = \{y_1, \dots y_n\}$ is contained in a minimal generating set of G, then there is also a finite minimal generating set of G containing Y.*

Proof Let X be a minimal generating set of G containing Y and, for $1 \leq i \leq n$, let M_i be an open maximal subgroup of G containing $X\backslash\{y_i\}$. The intersection

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$H := M_1 \cap \cdots \cap M_n$ is an open subgroup of G, so there exists a finite set T generating H. We have $\langle Y, T\rangle = \langle Y, H\rangle = G$, so $Y \cup T$ contains a finite minimal generating set Z. For $1 \leq i \leq n$, $\langle Z \backslash \{y_i\}\rangle \leq M_i$, so $Y \subseteq Z$. ∎

5 The Independence Graph

We will denote by $\widetilde{V}(G)$ the set of the non-isolated vertices of the independence graph $\widetilde{\Gamma}(G)$ and by $\widetilde{\Delta}(G)$ the subgraph of $\widetilde{\Gamma}(G)$ induced by $\widetilde{V}(G)$.

Lemma 5.1 *Let G be a finitely generated profinite group and let $g \in G$. Then g is isolated in $\widetilde{\Gamma}(G)$ if and only if either $G = \langle g\rangle$ or $g \in \mathrm{Frat}(G)$.*

Proof Suppose $g \notin \mathrm{Frat}(G)$. There exists an open maximal subgroup M of G with $g \notin M$. Since M is an open subgroup of a finitely generated profinite group, there exists a finite subset Y of M with $M = \langle Y\rangle$. The set $X := \{g\} \cup Y$ contains a minimal generating X of G and $g \in X$ (otherwise $G = \langle X\rangle \leq M$). If $X \neq \{g\}$, then g is not isolated in $\widetilde{\Gamma}(G)$; otherwise, $\langle g\rangle = G$. ∎

Lemma 5.2 *Let N be an open normal subgroup of a finitely generated profinite group G. If $Y = \{y_1N, \ldots, y_tN\}$ is a minimal generating set of G/N, then there exist $n_1, \ldots, n_u \in N$ such that $\{y_1, \ldots, y_t, n_1, \ldots, n_u\}$ is a minimal generating set of G.*

Proof Let Z be a finite generating set of N. Since $G = \langle Y, Z\rangle$, $Y \cup Z$ contains a minimal generating set X of G, and the minimality property of Y implies $Y \subseteq X$. ∎

We will write $x_1 \sim_G x_2$ if x_1 and x_2 belong to the same connected component of $\widetilde{\Delta}(G)$. The following lemma is an immediate consequence of Lemma 5.2.

Lemma 5.3 *Let N be an open normal subgroup of a finite group G and let $x, y \in G$. If $xN, yN \in \widetilde{V}(G/N)$ and $xN \sim_{G/N} yN$, then $x \sim_G y$.*

Theorem 5.4 *If G is a finitely generated profinite group, then the graph $\widetilde{\Delta}(G)$ is connected.*

Proof Let $x_1, x_2 \in \widetilde{V}(G)$. If follows from Lemma 5.1 that there exists two open normal subgroups, N_1 and N_2 of G with $x_1N_1 \in V(G/N_1)$ and $x_2N_2 \in \widetilde{V}(G/N_2)$. Moreover, there exists an open normal subgroup N_3 with $x_1x_2^{-1} \notin N_3$. Let $N := N_1 \cap N_2 \cap N_3$. Since x_1N, x_2N are distinct elements of $\widetilde{V}(G/N)$, it follows from [8, Theorem 1] that $x_1N_1 \sim_{G/N} x_2N_2$. By Lemma 5.3, $x_1 \sim_G x_2$. ∎

6 The Virtually Independence Graph

Let G be a (finitely generated) profinite group. We say that two elements x and y are virtually independent if there exists a (finite) minimal generating set of an open subgroup of G containing them. We define the virtually independent graph $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ of G as the graph whose vertices are the elements of G and in which two vertices

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are joined by an edge if and only if they are virtually independent. Moreover, we denote by $\widetilde{\Delta}_{\mathrm{virt}}(G)$ the subgraph induced by the non-isolated vertices of the virtually independent graph of G.

Theorem 6.1 *Let G be a finite group. If $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ contains a non-trivial isolated vertex x, then either $G = \langle x \rangle$ or one of the following occurs:*

(i) *$G \cong C_{p^n}$ is a cyclic group of p-power order, and all the vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ are isolated.*

(ii) *$G \cong Q_{2^n} = \langle a, b \mid a^{2^{n-1}}, b^2 = a^{2^{n-2}}, a^{-1}ba = b^{-1} \rangle$ is a generalized quaternion group, and $x = b^2$ is the unique non-trivial isolated vertex of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$.*

Proof Let x be a non-trivial isolated vertex of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$. If $y \in G$, then $\{x, y\}$ is not a virtually independent subset, so either $x \in \langle y \rangle$ or $y \in \langle x \rangle$. In any case, $[x, y] = 1$. So $x \in Z(G)$. Assume $G \neq \langle x \rangle$ and let p be a prime divisor of $|x|$. Then $\langle x \rangle / \langle x^p \rangle$ is the unique minimal subgroup of $G/\langle x^p \rangle$. Indeed, if $\langle y \rangle / \langle x^p \rangle$ were another minimal subgroup of G, then x and y would be virtually independent. In particular, $G/\langle x^p \rangle$ is a non-trivial p-group and p is the unique prime divisor of $|x|$. All the elements of order p in G belong to $\langle x \rangle$, so G contains a unique minimal subgroup, and therefore, G is either cyclic of p-power order or generalized quaternion (see [9, 5.3.6]). ∎

Lemma 6.2 *Let G be a finitely generated profinite group and let N be an open normal subgroup of G. If xN and yN are adjacent vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G/N)$, then x and y are adjacent vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$.*

Proof Let $\{n_1, \ldots, n_t\}$ be a finite generating set of N and set $A := \{x, y, n_1, \ldots, n_t\}$. Notice that $H := \langle A \rangle$ is an open subgroup of G and that A contains a minimal generating set B of H. Since xN and yN are adjacent vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G/N)$, $\langle x, y \rangle N \neq \langle x \rangle N \neq \langle y \rangle N$, and therefore $x, y \in B$ and x and y are adjacent in $\Gamma_{\mathrm{virt}}(G)$. ∎

Theorem 6.3 *Let G be a finitely generated profinite group. If G is infinite and $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ contains a non-trivial isolated vertex, then G is procyclic.*

Proof Assume that x is a non-trivial isolated vertex in $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ and that G is not topologically generated by x. Let $\mathcal{N}$ be the set of the open normal subgroups N of G with the property that $x \notin N$ and $G \neq \langle x \rangle N$. Let $N \in \mathcal{N}$. By Lemma 6.2, xN is an isolated vertex of $\widetilde{\Gamma}_{\mathrm{virt}}(G/N)$, so, by Theorem 6.1, G/N is either a cyclic p-group or a generalized quaternion group. Assume that G/M is a generalized quaternion group for some $M \in \mathcal{N}$. Since $\cap_{N \in \mathcal{N}} N = 1$ and no proper epimorphic image of a generalized quaternion group is generalized quaternion, we would have $M = 1$ and G would be finite, against our assumption. So G/N is cyclic for every $N \in \mathcal{N}$, and therefore, G is procyclic. ∎

Lemma 6.4 *If G is a finite group, then the graph $\widetilde{\Delta}_{\mathrm{virt}}(G)$ is connected and* $\mathrm{diam}(\widetilde{\Delta}_{\mathrm{virt}}(G)) \leq 3$.

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Proof We prove the statement by induction. For this purpose, notice that it follows from Lemma 6.2 that if N is a normal subgroup of G and x_1N and x_2N are adjacent vertices of $\widetilde{\Delta}_{\mathrm{virt}}(G/N)$, then x_1 and x_2 are adjacent in $\widetilde{\Delta}_{\mathrm{virt}}(G)$ and $\mathrm{dist}_{\widetilde{\Delta}_{\mathrm{virt}}(G)}(x_1, x_2) \leq \mathrm{dist}_{\widetilde{\Delta}_{\mathrm{virt}}(G/N)}(x_1N, x_2N)$. Assume that x_1 and x_2 are distinct vertices of $\widetilde{\Delta}_{\mathrm{virt}}(G)$. If neither $\langle x\rangle \leq \langle y\rangle$ nor $\langle y\rangle \leq \langle x\rangle$, then $\{x, y\}$ is a minimal generating set of $\langle x, y\rangle$ and $\mathrm{dist}_{\Delta_{\mathrm{virt}}(G)}(x, y) = 1$. So we can assume $\langle x\rangle \leq \langle y\rangle$. Let $|x| = p_1^{a_1}\cdots p_t^{a_t}$, $|y| = p_1^{b_1}\cdots p_t^{b_t}$, $|G| = p_1^{c_1}\cdots p_t^{c_t}$, with $a_i \leq b_i \leq c_i$ for $1 \leq i \leq t$. First, assume $t \neq 1$. If $b_j = 0$ for some $1 \leq j \leq t$, let g be an element of G of order p_j: then g is adjacent to x and y and $\mathrm{dist}_{\widetilde{\Delta}_{\mathrm{virt}}(G)}(x, y) \leq 2$. So we can assume $b_j \neq 0$ for every $1 \leq j \leq t$. Since y is not isolated in $\widetilde{\Delta}_{\mathrm{virt}}(G)$, we have $G \neq \langle y\rangle$. In particular, there exists a prime p and a p-element $g \in G$ with $g \notin \langle y\rangle$. It is not restrictive to assume $p = p_t$. If $a_i \neq 0$ for some $1 \leq i \leq t-1$, then g is adjacent to x and y and $\mathrm{dist}_{\widetilde{\Delta}_{\mathrm{virt}}(G)}(x, y) \leq 2$. Otherwise, $x, y^{p_t^{b_t}}, g, y$ is a path in $\widetilde{\Delta}_{\mathrm{virt}}(G)$ and $\mathrm{dist}_{\widetilde{\Delta}_{\mathrm{virt}}(G)}(x, y) \leq 3$. We remain with the case when G is a p-group. If there exists a minimal subgroup $\langle g\rangle$ of G not contained in $\langle x\rangle$, then x and y are adjacent to g. So we can assume that G has a unique minimal subgroup, and consequently, G is either a cyclic p-group or a generalized quaternion group. We can exclude the first possibility, since in that case, all the vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ are isolated. In the second case, G has a unique minimal subgroup, say N, and xN, yN are distinct non-isolated vertices of G/N (which is a dihedral group), so $\mathrm{dist}_{\Delta_{\mathrm{virt}}(G)}(x, y) \leq \mathrm{dist}_{\widetilde{\Delta}_{\mathrm{virt}}(G/N)}(x, y) \leq 3$ by induction. ∎

The bound $\mathrm{diam}(\widetilde{\Delta}_{\mathrm{virt}}(G)) \leq 3$ is best possible. Consider

$$G = \langle a, b \mid a^4 = 1, b^3 = 1, b^a = b^{-1}\rangle$$

and let $x = a^2$, $y = a^2b$. The elements of G that are adjacent to y in $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ are precisely those of the form a^ib^j with i and odd integer. If g is one of these elements, then $g^2 = a^2 = x$, so g is not adjacent to x, and therefore, $\mathrm{diam}(\widetilde{\Delta}_{\mathrm{virt}}(G)) \geq 3$

Theorem 6.5 *If G is a finitely generated profinite group, then $\widetilde{\Delta}_{\mathrm{virt}}(G)$ is connected and* $\mathrm{diam}(\widetilde{\Delta}_{\mathrm{virt}}(G)) \leq 3$.

Proof By Proposition 6.4, we can assume that G is infinite and that G is not a procyclic pro-p group (otherwise all the vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G)$ are isolated). Assume that x and y are non-isolated distinct vertices of $\widetilde{\Delta}_{\mathrm{virt}}(G)$. There exist three open normal subgroups N_1, N_2, and N_3 with $x \notin N_1$, $y \notin N_2$, and $xy^{-1} \notin N_3$. Set $M := N_1 \cap N_2 \cap N_3$. Then xM and yM are two different non-trivial vertices of G/M. We can choose M in such a way that G/M is neither a cyclic p-group nor a generalized quaternion group (this is because a proper epimorphic image of a generalized quaternion group is not generalized quaternion). But then xM and yM are non-isolated vertices of $\widetilde{\Gamma}_{\mathrm{virt}}(G/M)$, so by Lemmata 6.2 and 6.4, $\mathrm{dist}_{\widetilde{\Delta}_{\mathrm{virt}}(G)}(x, y) \leq \mathrm{dist}_{\widetilde{\Delta}_{\mathrm{virt}}(G/N)}(xN, yN) \leq 3$. ∎

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