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ABELIAN GROUPS DEFINABLE IN p-ADICALLY CLOSED FIELDS

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Abstract. Recall that a group G has finitely satisfiable generics (*fsg*) or definable f-generics (*dfg*) if there is a global type p on G and a small model M_0 such that every left translate of p is finitely satisfiable in M_0 or definable over M_0, respectively. We show that any abelian group definable in a p-adically closed field is an extension of a definably compact *fsg* definable group by a *dfg* definable group. We discuss an approach which might prove a similar statement for interpretable abelian groups. In the case where G is an abelian group definable in the standard model $\mathbb{Q}_p$, we show that $G^0 = G^{00}$, and that G is an open subgroup of an algebraic group, up to finite factors. This latter result can be seen as a rough classification of abelian definable groups in $\mathbb{Q}_p$.

§1. Introduction. In this paper we study abelian groups definable in p-adically closed fields. Our first main result is a decomposition of these groups into *fsg* and *dfg* components. Recall that a definable group G has *finitely satisfiable generics* (*fsg*) if there is a global type on G, finitely satisfiable in a small model, with boundedly many left translates. Similarly, G has *definable f-generics* (*dfg*) if there is a definable global type on G with boundedly many left translates. If G is definable in an o-minimal expansion of RCF or a p-adically closed field, then G has *fsg* iff it is definably compact [7, 12, 15], and G has *dfg* iff it is totally non-compact [4, 19].

The central theorem of this paper is the following decomposition of abelian definable groups in p-adically closed fields into *dfg* and *fsg* components.

Theorem 1.1. *Suppose that M is a p-adically closed field and G is an abelian group definable in M. Then there is a short exact sequence of definable groups*

$$1 \to H \to G \to C \to 1,$$

where H has dfg and C is definably compact and has fsg.

The proof, and several examples of the decomposition, appear in Section 3.

Remark 1.2. An analogous decomposition for abelian groups in o-minimal structures was proved by Conversano and Pillay [4, Propositions 4.6–7]. In fact, they give the decomposition in the more general setting of definably amenable groups. Recall from [7, Section 5] that a definable group G is *definably amenable* if there is a finitely additive probability measure on definable subsets of G, invariant under left translation. Solvable definable groups are definably amenable, because they are amenable in the classical sense from analysis. Definable groups in stable

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theories are also known to be definably amenable. In NIP theories, a definable group G is definably amenable iff there is a global type on G with boundedly many left translates [3]. In particular, *dfg* and *fsg* groups are definably amenable.

As mentioned above, Conversano and Pillay proved a decomposition analogous to Theorem 1.1 for definably amenable groups in o-minimal structures [4, Propositions 4.6–7] (see also the discussion around [18, Fact 1.18]). Pillay and the second author asked whether such a decomposition exists for any definably amenable group in a distal theory [18, Question 1.19]. Theorem 1.1 can be seen as evidence towards a positive answer.

When $M = \mathbb{Q}_p$, we obtain two useful consequences from Theorem 1.1:

THEOREM 1.3. *Suppose that G is an abelian group definable in $\mathbb{Q}_p$.*

(i) $G^{00} = G^0$ *(in a monster model extending $\mathbb{Q}_p$).*

(ii) *There is a finite index definable subgroup $E \subseteq G$ and a finite subgroup $F \subseteq E$ such that E/F is isomorphic to an open subgroup of an algebraic group.*

The proof appears in Section 4. We will generalize Theorem 1.3(i) to non-abelian groups (over $\mathbb{Q}_p$) in a future paper. Note that Theorem 1.3(i) needn't be true when the group is definable over a non-standard model. In fact, Onshuus and Pillay show that $E^0 \neq E^{00}$ for "nonstandard Tate [elliptic] curves" over monster models of pCF [15, Proposition 3.7].

Theorem 1.3(ii) yields a loose "classification" of abelian definable groups in $\mathbb{Q}_p$—up to finite factors, they are exactly the open subgroups of commutative algebraic groups.[1]

For the case of *one-dimensional* definable groups in *arbitrary* p-adically closed fields, a similar classification up to finite factors was obtained by Acosta López [1]. It seems possible that his methods could be used to give another proof of Theorem 1.3 or generalize it beyond the specific field $\mathbb{Q}_p$.

1.1. Outline. In Section 2, we review some tools needed in the proof. In Section 3 we prove the decomposition in Theorem 1.1 and give some examples. In Section 4 we obtain the consequences for $\mathbb{Q}_p$-definable groups listed in Theorem 1.3. In Section 5 we discuss our original strategy for Theorem 1.1, which suggests a generalization of Theorem 1.1 to interpretable groups (Conjecture 5.14).

There are also two appendices. Appendix A proves a technical statement about topological properties of ict patterns in interpretable groups, needed in Lemma 5.8. Appendix B is on *dfg* in short exact sequences, and generalizes some facts in Section 2.1 beyond the context of pCF.

1.2. Notation and conventions. "Definable" means "definable with parameters." We work in a monster model $\mathbb{M}$, which will usually be NIP. We say that a set A is "small" or "bounded" if it is smaller than the degree of saturation of $\mathbb{M}$. If X is definable over a small model M, then $X(M)$ denotes the set of M-points of X, i.e., the corresponding definable set in M.

A "type" is a complete type, and a "partial type" is a partial type. Every type over a small subset is realized in $\mathbb{M}$. A set in $\mathbb{M}$ is "type-definable" if it is a small intersection

[1] Open subgroups of commutative algebraic groups over $\mathbb{Q}_p$ are automatically definable by a recent result of Pillay and the second author [20].

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of definable sets, or equivalently, it is defined by a type over a small subset. Tuples are finite by default. We usually write tuples as a, b, x, y rather than $\bar{a}, \bar{b}, \bar{x}, \bar{y}$. We distinguish between "real" elements or tuples (in $\mathbb{M}$) and "imaginaries" (in $\mathbb{M}^{\mathrm{eq}}$), and we distinguish between "definable" (in $\mathbb{M}$) and "interpretable" (in $\mathbb{M}^{\mathrm{eq}}$). The exception is Appendix B, where we work in $\mathbb{M}^{\mathrm{eq}}$. If D is a definable set, then $\ulcorner D \urcorner$ denotes its code, a tuple in $\mathbb{M}^{\mathrm{eq}}$. If p is a definable type, then $\ulcorner p \urcorner$ denotes its code, an infinite tuple in $\mathbb{M}^{\mathrm{eq}}$.

Throughout, pCF means the complete theory of $\mathbb{Q}_p$, and a "p-adically closed field" is a model of this theory, or equivalently, a field elementarily equivalent to $\mathbb{Q}_p$. We do not consider "p-adically closed fields" in the broader sense (fields elementarily equivalent to finite extensions of $\mathbb{Q}_p$), though we strongly suspect that all the results generalize to these theories. We write the language of pCF as $\mathcal{L}$. The language $\mathcal{L}$ should be one-sorted; the choice of $\mathcal{L}$ is otherwise irrelevant. We write the valuation as $v(x)$ and the value group as Γ.

§2. Tools. In this section, we review a few tools that will be needed in the proof of the main theorems. In Section 2.1 we show that certain properties ($G^0 = G^{00}$, *dfg*) behave well in short exact sequences. In Section 2.2 we show that we can take quotients by certain *dfg* groups without leaving the definable category. Throughout this section, work in a monster model $\mathbb{M}$ of some complete theory.

2.1. Extensions. Let G be a group definable in $\mathbb{M}$. Following standard terminology, we say that "G^0 exists" if the family of definable subgroups of finite index is small, in which case G^0 is defined to be the intersection of this family. Similarly, we say that G^{00} *exists* if the family of type-definable subgroups of bounded index is small, in which case G^{00} is defined to be the intersection of this family. When they exist, G^{00} and G^0 are type-definable normal subgroups of bounded index, and $G^{00} \subseteq G^0 \subseteq G$. In NIP theories, G^0 and G^{00} always exist [7, Proposition 6.1].

Recall that if X is a type-definable set and E a type-definable equivalence relation on X with X/E bounded, then we can define a topology on X/E called the *logic topology*. Let $\pi : X \to X/E$ be the natural projection. Then $Z \subseteq X/E$ is closed in the logic topology iff $\pi^{-1}(Z) \subseteq X$ is type-definable. This makes X/E into a compact Hausdorff space. See [17, Section 2] for more about the logic topology.

When G^0 or G^{00} exists, the logic topology makes the quotient G/G^0 or G/G^{00} into a compact Hausdorff topological group, and G/G^0 is even profinite. When G^{00} exists, G/G^0 is the largest profinite quotient of G/G^{00}. Consequently, $G^0 = G^{00}$ if and only if G/G^{00} is profinite.

LEMMA 2.1 (Assuming NIP). *Let $\pi : G \to X$ be a surjective homomorphism of definable groups. Then $\pi(G^{00}) = X^{00}$.*

PROOF. There is a surjection $G/G^{00} \to X/\pi(G^{00})$, so $X/\pi(G^{00})$ is bounded and $\pi(G^{00}) \supseteq X^{00}$. There is a bijection $G/\pi^{-1}(X^{00}) \to X/X^{00}$, so $G/\pi^{-1}(X^{00})$ is bounded and $G^{00} \subseteq \pi^{-1}(X^{00})$. This implies $\pi(G^{00}) \subseteq X^{00}$. ⊣

LEMMA 2.2 (Assuming NIP). *Let $1 \to H \to G \xrightarrow{\pi} X \to 1$ be a short exact sequence of definable groups. If $H^0 = H^{00}$ and $X^0 = X^{00}$, then $G^0 = G^{00}$.*

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PROOF. The fact that $H^0 = H^{00}$ and $X^0 = X^{00}$ means that H/H^{00} and X/X^{00} are profinite. The short exact sequence

$$1 \to H/(H \cap G^{00}) \to G/G^{00} \to X/X^{00} \to 1 \qquad (*)$$

shows that $H/(H \cap G^{00})$ is bounded, and then $(*)$ is continuous in the logic topology. As $H/(H \cap G^{00})$ is bounded, it must be a quotient of H/H^{00} which is profinite. Therefore $H/(H \cap G^{00})$ is profinite. In the category of compact Hausdorff groups, an extension of a profinite group by a profinite group is profinite. Therefore G/G^{00} is profinite, which implies $G^0 = G^{00}$. ⊣

Recall that pCF has definable Skolem functions.

LEMMA 2.3. *Suppose that* $\mathbb{M}$ *is a saturated model of* pCF. *Let*

$$1 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 1$$

be a short exact sequence of definable groups. Then B has dfg iff A and C do.

PROOF. We prove the following:

1. If B has *dfg*, then C has *dfg*.
2. If B has *dfg*, then A has *dfg*.
3. If A and C have *dfg*, then B has *dfg*.

By definable Skolem functions, there is a definable function $f : C \to B$ which is a set-theoretic section of π, in the sense that $\pi(f(c)) = c$ for $c \in C$. Now we proceed with the proofs:

1. If $\mathrm{tp}(b/\mathbb{M})$ is a definable f-generic type in B, then $\mathrm{tp}(\pi(b)/\mathbb{M})$ is a definable f-generic type in C.
2. The proof is nearly identical to [19, Lemmas 2.24 and 2.25]. In an elementary extension $\mathbb{M}' \succeq \mathbb{M}$, take $b_0 \in B(\mathbb{M}')$ realizing a definable f-generic type in B. Write b_0 as $a_0 \cdot f(\pi(b_0))$ for some $a_0 \in A(\mathbb{M}')$. Then $a_0 \in \mathrm{dcl}(\mathbb{M}b_0)$, so $\mathrm{tp}(a_0/\mathbb{M})$ is definable. We claim that $\mathrm{tp}(a_0/\mathbb{M})$ has boundedly many left translates, and is therefore a definable f-generic type in A. Note that $A^{00} \subseteq B^{00}$ because $A/(A \cap B^{00}) \cong AB^{00}/B^{00}$ is bounded. If $\delta \in A^{00}(\mathbb{M})$, then $\mathrm{tp}(\delta \cdot b_0/\mathbb{M}) = \mathrm{tp}(b_0/\mathbb{M})$, and therefore

   $$\mathrm{tp}(\delta \cdot b_0 \cdot f(\pi(\delta \cdot b_0))^{-1}/\mathbb{M}) = \mathrm{tp}(b_0 \cdot f(\pi(b_0))^{-1}/\mathbb{M}) = \mathrm{tp}(a_0/\mathbb{M}).$$

   But $\pi(\delta \cdot b_0) = \pi(b_0)$, and so

   $$\mathrm{tp}(\delta \cdot b_0 \cdot f(\pi(\delta \cdot b_0))^{-1}/\mathbb{M}) = \mathrm{tp}(\delta \cdot b_0 \cdot f(\pi(b_0))^{-1}/\mathbb{M}) = \mathrm{tp}(\delta \cdot a_0/\mathbb{M}).$$

   Therefore $\mathrm{tp}(a_0/\mathbb{M})$ is invariant under left translation by any $\delta \in A^{00}$, and it has boundedly many left translates.
3. Let $p(x) \in S_A(\mathbb{M})$ and $q(y) \in S_C(\mathbb{M})$ be *dfg* types of A and C respectively. Let M_0 be a small model defining the section f, the short exact sequence, and all the left translates of p and q.

   In some elementary extension $\mathbb{M}' \succeq \mathbb{M}$, take $c_0 \models q$ and $a_0 \models p|\mathbb{M}c_0$. Then $\mathrm{tp}(a_0, c_0/\mathbb{M})$ is M_0-definable—it is the Morley product of p and q. Let $b_0 = f(c_0) \cdot a_0$. Then $\mathrm{tp}(b_0/\mathbb{M})$ is again M_0-definable. We claim that every left translate of $\mathrm{tp}(b_0/\mathbb{M})$ is M_0-definable.

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Fix some $\delta \in B(\mathbb{M})$. Let $b_1 = \delta \cdot b_0$. Let $c_1 = \pi(\delta) \cdot c_0$. Let $\delta' = f(c_1)^{-1} \cdot \delta \cdot f(c_0)$. Note

$$\pi(\delta') = \pi(f(c_1))^{-1} \cdot \pi(\delta) \cdot \pi(f(c_0)) = c_1^{-1} \cdot \pi(\delta) \cdot c_0 = 1,$$

so $\delta' \in A(\mathbb{M}')$. Let $a_1 = \delta' \cdot a_0$. Then

$$b_1 = \delta \cdot b_0 = \delta \cdot f(c_0) \cdot a_0 = f(c_1) \cdot \delta' \cdot a_0 = f(c_1) \cdot a_1.$$

Now $\operatorname{tp}(c_1/\mathbb{M}) = \operatorname{tp}(\pi(\delta) \cdot c_0/\mathbb{M})$ is a left-translate of the *dfg* type $\operatorname{tp}(c_0/\mathbb{M}) = q$, and so $\operatorname{tp}(c_1/\mathbb{M})$ is M_0-definable. If U is $\operatorname{dcl}(\mathbb{M}c_0) = \operatorname{dcl}(\mathbb{M}c_1)$, then $\operatorname{tp}(a_1/U) = \operatorname{tp}(\delta' \cdot a_0/U)$ is a left translate of the *dfg* type $\operatorname{tp}(a_0/U) = p|U$ (because $\delta' \in U$). Therefore $\operatorname{tp}(a_1/U)$ is again M_0-definable. As $b_1 = f(c_1) \cdot a_1$, we see that $\operatorname{tp}(\delta \cdot b_0/\mathbb{M}) = \operatorname{tp}(b_1/\mathbb{M})$ is M_0-definable for the same reason that $\operatorname{tp}(b_0/\mathbb{M})$ is M_0-definable, essentially because $\operatorname{tp}(c_1/\mathbb{M})$ and $\operatorname{tp}(a_1/\mathbb{M}c_1)$ are M_0-definable. ⊣

See Theorem B.6 in Appendix B for an alternate proof of (3) not using definable Skolem functions.

2.2. Codes and quotients. Let G be a definable group and let H be a normal subgroup. A priori, the quotient group G/H is interpretable, not definable.

EXAMPLE 2.4. If X is the multiplicative group $\mathbb{M}^\times$ and G is the subgroup $\{x \in X : v(x) = 0\}$, then X/G is the value group Γ, which is interpretable but *not* definable, since $\dim(\Gamma) = 0 \neq \text{dp-rk}(\Gamma) = 1$, for example. On definable sets, dp-rank agrees with dimension.

In this section, we show that for certain *dfg* groups H in pCF, the quotient G/H is automatically definable (Corollary 2.10). The key is to show that certain definable types are coded by *real* tuples (Theorem 2.8). Both of these results are proved in greater generality in a later paper with Andújar Guerrero [2, Theorems 3.4 and 4.1].

If D is a definable set in a model M, let $\ulcorner D \urcorner$ denote "the" code of D in M^{eq}, which is well-defined up to interdefinability. If $\sigma \in \operatorname{Aut}(M)$, then

$$\sigma(D) = D \iff \sigma(\ulcorner D \urcorner) = \ulcorner D \urcorner,$$

and this property characterizes $\ulcorner D \urcorner$ when M is sufficiently saturated and homogeneous.

LEMMA 2.5. *Let K be a field and let $V \subseteq K^n$ be Zariski closed. Then the definable set V is coded by a tuple in K (rather than K^{eq}). In particular, finite subsets of K^n are coded by tuples in K.*

PROOF. Passing to an elementary extension, we may assume K is $\aleph_1$-saturated and strongly $\aleph_1$-homogeneous. Let $M = K^{alg}$. Let $\overline{V}$ be the Zariski closure of V in M^n. Note $V = \overline{V} \cap K^n$. By elimination of imaginaries in ACF, there is a tuple $b \in M$ which codes $\overline{V}$ in the structure M^n. If $\sigma \in \operatorname{Aut}(M/K)$ then σ fixes V setwise, so it also fixes the Zariski closure $\overline{V}$. Therefore $\sigma(b) = b$, for any $\sigma \in \operatorname{Aut}(M/K)$. By Galois theory, b is in the perfect closure of K. Replacing b with b^{p^n} if necessary, we may assume b is a tuple in K.

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We claim that b codes V in the structure K. Suppose $\sigma_0 \in \mathrm{Aut}(K)$. Extend σ_0 to an automorphism $\sigma \in \mathrm{Aut}(M)$ arbitrarily. Then b codes V because

$$\sigma_0(V) = V \iff \sigma(V) = V \overset{*}{\iff} \sigma(\overline{V}) = \overline{V} \iff \sigma(b) = b \iff \sigma_0(b) = b.$$

The starred $\overset{*}{\iff}$ requires some explanation. The direction $\Rightarrow$ holds because the formation of Zariski closures is automorphism invariant. The direction $\Leftarrow$ holds because σ fixes K setwise and $V = \overline{V} \cap K^n$. ⊣

LEMMA 2.6. *Work in a monster model $\mathbb{M}$ of pCF.*

1. *If an imaginary tuple a is algebraic over a real tuple b, then a is definable over b.*
2. *If an imaginary tuple a is interalgebraic with a real tuple b, then a is interdefinable with some real tuple b'.*

More generally, both statements hold if we work over a set of real parameters $C \subseteq \mathbb{M}$.

PROOF. 1. Note that $\mathrm{dcl}(b) \preceq \mathbb{M}$ by definable Skolem functions, and so $\mathrm{dcl}^{\mathrm{eq}}(b) \preceq \mathbb{M}^{\mathrm{eq}}$. Submodels are algebraically closed, so $\mathrm{acl}^{\mathrm{eq}}(b) = \mathrm{dcl}^{\mathrm{eq}}(b)$ and $a \in \mathrm{dcl}^{\mathrm{eq}}(b)$.

2. By part (1), $a \in \mathrm{dcl}^{\mathrm{eq}}(b)$. Write a as $f(b)$ for some $\varnothing$-definable function f. Let $S \subseteq \mathbb{M}^n$ be the set of realizations of $\mathrm{tp}(b/a)$. Then S is finite as $b \in \mathrm{acl}^{\mathrm{eq}}(a)$. Moreover, S is a-definable, and so the code $\ulcorner S \urcorner$ is in $\mathrm{dcl}^{\mathrm{eq}}(a)$. By Lemma 2.5, we can take the code $\ulcorner S \urcorner$ to be a real tuple. For any $c \in S$, we have $f(c) = a$, which implies $a \in \mathrm{dcl}^{\mathrm{eq}}(\ulcorner S \urcorner)$. Then a is interdefinable with the real tuple $\ulcorner S \urcorner$.

The "more general" statements follow by the same proofs. Indeed, we can name the elements of C as constants without losing definable Skolem functions or codes for finite sets. ⊣

If p is a definable n-type over M, let $\ulcorner p \urcorner$ denote the infinite tuple $(\ulcorner D_\varphi \urcorner : \varphi \in L)$, where

$$D_\varphi = \{b \in M^m : \varphi(x, b) \in p(x)\}.$$

For $\sigma \in \mathrm{Aut}(M)$, we have

$$\sigma(p) = p \iff \sigma(\ulcorner p \urcorner) = \ulcorner p \urcorner,$$

and this property determines $\ulcorner p \urcorner$ up to interdefinability when M is sufficiently saturated and homogeneous.

LEMMA 2.7. *If $q \in S_1(\mathbb{M})$ is definable, then $\ulcorner q \urcorner$ is interdefinable with a (finite) real tuple.*

PROOF. By [13, Proposition 2.24], the type q must accumulate at some point c in the projective line $\mathbb{P}^1(\mathbb{M})$, because $\mathbb{P}^1(\mathbb{M})$ is definably compact. If necessary, we can push q forward along the map $x \mapsto 1/x$ to ensure $c \neq \infty$. Then $c \in \mathbb{M}$. Note $c \in \mathrm{dcl}^{\mathrm{eq}}(\ulcorner q \urcorner)$. There are only boundedly many types concentrating at c by [9, Corollary 7.5] or [13, Fact 2.20], so $\ulcorner q \urcorner$ has a small orbit under $\mathrm{Aut}(\mathbb{M}/c)$. Then $\ulcorner q \urcorner \in \mathrm{acl}^{\mathrm{eq}}(c)$. As in the proof of Lemma 2.6(1), $\ulcorner q \urcorner \in \mathrm{dcl}^{\mathrm{eq}}(c)$, so $\ulcorner q \urcorner$ is interdefinable with c. ⊣

THEOREM 2.8. *Suppose $q \in S_n(\mathbb{M})$ is a definable type, and $\dim(q) = 1$. Then $\ulcorner q \urcorner$ is interdefinable with a real tuple.*

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PROOF. Take an elementary extension $\mathbb{M}' \succeq \mathbb{M}$ containing a realization $\bar{a}$ of q. Then $\operatorname{tr.deg}(\bar{a}/\mathbb{M}) = \dim(q) = 1$, so there is some i such that a_i is a transcendence basis of $\bar{a}$ over $\mathbb{M}$, implying that $\bar{a}$ is field-theoretically algebraic over $\mathbb{M}$ and a_i. Then there is a Zariski-closed set $V_0 \subseteq \mathbb{M}^n$ such that there are only finitely many $\bar{b} \in V_0(\mathbb{M}')$ with $b_i = a_i$.

Let $V \subseteq \mathbb{M}^n$ be the smallest Zariski-closed set such that $\bar{a} \in V(\mathbb{M}')$, or equivalently, the smallest Zariski-closed set on which q concentrates. Any automorphism of $\mathbb{M}$ which fixes q fixes V, and so

$$\ulcorner V \urcorner \in \operatorname{dcl}^{\mathrm{eq}}(\ulcorner q \urcorner). \tag{1}$$

As $V \subseteq V_0$, there are only finitely many $\bar{b} \in V(\mathbb{M}')$ with $b_i = a_i$. Therefore $\bar{a} \in \operatorname{acl}^{\mathrm{eq}}(\ulcorner V \urcorner a_i)$. By Lemma 2.5, we may assume $\ulcorner V \urcorner$ is a real tuple in $\mathbb{M}$, and then $\bar{a} \in \operatorname{dcl}^{\mathrm{eq}}(\ulcorner V \urcorner a_i)$ by Lemma 2.6(1). Therefore $\bar{a}$ and a_i are interdefinable over $\ulcorner V \urcorner$.

Take a bijection f defined over $\ulcorner V \urcorner$ such that $\bar{a} = f(a_i)$. Then $q = \operatorname{tp}(\bar{a}/\mathbb{M})$ is the pushforward of the definable type $r := \operatorname{tp}(a_i/\mathbb{M})$ along the $\ulcorner V \urcorner$-definable bijection f. Therefore

$$\ulcorner q \urcorner \in \operatorname{dcl}^{\mathrm{eq}}(\ulcorner V \urcorner \ulcorner r \urcorner). \tag{2}$$

Likewise, r is the pushforward of q along the 0-definable coordinate projection $\pi(\bar{x}) = x_i$, so

$$\ulcorner r \urcorner \in \operatorname{dcl}^{\mathrm{eq}}(\ulcorner q \urcorner). \tag{3}$$

Combining equations (1)–(3), we see that $\ulcorner q \urcorner$ is interdefinable with $\ulcorner V \urcorner \ulcorner r \urcorner$. But $\ulcorner V \urcorner$ is a real tuple by Lemma 2.5 as noted above, and $\ulcorner r \urcorner$ is a real tuple by Lemma 2.7. ⊣

Using a different argument, one can show that Theorem 2.8 holds for *any* definable n-type, without the assumption $\dim(q) = 1$ [2, Theorem 3.4]. However, the real tuple may need to be infinite [2, Proposition 3.7].

PROPOSITION 2.9. *If a one-dimensional dfg group G acts on a definable set X, then the quotient space X/G is definable* (*not just interpretable*).

PROOF. Take a global definable type p on G with boundedly many right translates. Take a small model M_0 over which everything is defined, including the boundedly many right translates of p. It suffices to show that every element of the interpretable set X/G is interdefinable over M_0 with a real tuple. By Lemma 2.6(2), it suffices to show that every element of X/G is *interalgebraic* over M_0 with a real tuple. Fix some element $e = G \cdot a \in X/G$, where $a \in X$. Let $p \cdot a$ denote the pushforward of p along the map $x \mapsto x \cdot a$. Note that the global types p and $p \cdot a$ both have dimension 1 (or less). By Theorem 2.8, the code $\ulcorner p \cdot a \urcorner$ can be taken to be a real tuple. We claim that $\ulcorner p \cdot a \urcorner$ is interalgebraic with e over M_0.

In one direction, $p \cdot a$ is contained in the collection

$$\begin{aligned} \mathfrak{S} &= \{p \cdot a' : a' \in G \cdot a\} \\ &= \{p \cdot (g \cdot a) : g \in G\} = \{(p \cdot g) \cdot a : g \in G\}, \end{aligned}$$

which is $\operatorname{Aut}(\mathbb{M}/M_0 e)$-invariant by the first line, and small by the second line. It follows that $p \cdot a$ has a small number of conjugates over $M_0 e$, and so $\ulcorner p \cdot a \urcorner \in \operatorname{acl}^{\mathrm{eq}}(M_0 e)$.

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In the other direction, the type $p \cdot a$ concentrates on $G \cdot a$, so its pushforward along the M_0-definable map $X \to X/G$ is the constant type $x = e$. Therefore $e \in \mathrm{dcl}^{\mathrm{eq}}(M_0 \ulcorner p \cdot a \urcorner)$. This completes the proof that e is interalgebraic with $\ulcorner p \cdot a \urcorner$ over M_0. ⊣

Again, this holds without the assumption $\dim(G) = 1$. See [2, Theorem 4.1].

COROLLARY 2.10. *Let G be a definable group and let H be a one-dimensional definable normal subgroup. If H has dfg, then G/H is definable and* $\dim(G/H) = \dim(G) - 1$.

§3. Proof of Theorem 1.1.

Work in a model $M \models p\mathrm{CF}$.

THEOREM 1.1. *Suppose that M is a p-adically closed field and G is an abelian group definable in M. Then there is a short exact sequence of definable groups*

$$1 \to H \to G \to C \to 1,$$

where H has dfg and C is definably compact and has fsg.

PROOF. For definable groups, *fsg* is equivalent to definable compactness [12, Theorem 1.2]. Say a definable subgroup $H \subseteq G$ is "good" if G/H is definable and H has *dfg*. For example, $H = \{1\}$ is good. Take a good subgroup H maximizing $\dim(H)$. If G/H is definably compact then we are done. Otherwise, G/H is not definably compact. By [13, Corollary 6.11], there is a one-dimensional definable *dfg* subgroup of G/H. This subgroup has the form H'/H for some definable subgroup H' of G. The short exact sequence

$$1 \to H \to H' \to H'/H \to 1$$

shows that H' has *dfg* by Lemma 2.3, and that

$$\dim(H') = \dim(H) + \dim(H'/H) = \dim(H) + 1 > \dim(H).$$

The quotient $G/H' = (G/H)/(H'/H)$ is definable by Corollary 2.10, and so H' is a good subgroup, contradicting the choice of H. ⊣

Here are some examples of this decomposition.

EXAMPLE 3.1. Suppose $p = 2$ or $p \equiv 3 (\mathrm{mod}\ 4)$. Then -1 is not a square in $\mathbb{Q}_p$. Let G be the multiplicative group of the quadratic extension $\mathbb{Q}_p(\sqrt{-1})$, regarded as a definable group in $\mathbb{Q}_p$. Explicitly, G is the set $\mathbb{Q}_p^2 \setminus \{(0,0)\}$ with the group operation given by

$$(x, y) \cdot (x', y') := (xx' - yy', xy' + x'y).$$

The decomposition of G into *dfg* and *fsg* groups is

$$1 \to \mathbb{Q}_p^\times \to G \to C \to 1,$$

where C is the subgroup $\{(x, y) \in G : x^2 + y^2 = 1\} \subseteq G$, and the two maps are

$$\begin{aligned} \mathbb{Q}_p^\times &\to G, & G &\to C, \\ x &\mapsto (x, 0), & (x, y) &\mapsto \left(\frac{x^2 - y^2}{x^2 + y^2}, \frac{2xy}{x^2 + y^2} \right). \end{aligned}$$

We leave the verification of these details as an exercise to the reader.

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EXAMPLE 3.2. Let M be a p-adically closed field, and let a be an element of positive valuation. Let G be the set of pairs $(x, y) \in M^\times \times M^\times$ such that $0 \leq v(y) < v(a)$. The operation

$$(x, y) \star (x', y') = \begin{cases} (xx', yy'), & \text{if } v(yy') < v(a) \\ \left(\frac{xx'}{p}, \frac{yy'}{a}\right), & \text{if } v(yy') \geq v(a) \end{cases}$$

makes G into a group. The decomposition of G into *dfg* and *fsg* components looks like

$$1 \to M^\times \to G \to C \to 1,$$

where C is $\{y \in M^\times : 0 \leq v(y) < v(a)\}$ with the group operation

$$y \star y' = \begin{cases} yy', & \text{if } v(yy') < v(a), \\ \frac{yy'}{a}, & \text{if } v(yy') \geq v(a). \end{cases}$$

The inclusion $M^\times \to G$ is $x \mapsto (x, 1)$, while the projection $G \to C$ is $(x, y) \mapsto y$. Again, we leave the verification as an exercise.

EXAMPLE 3.3. If we understand correctly, abelian varieties can have extensions by powers of the additive group or the multiplicative group. More precisely, there are non-trivial short exact sequences of abelian algebraic groups of the forms

$$0 \to (\mathbb{G}_m)^n \to G \to A \to 0,$$
$$0 \to (\mathbb{G}_a)^n \to G \to A \to 0,$$

where $\mathbb{G}_m$ is the multiplicative group, $\mathbb{G}_a$ is the additive group, and A is an abelian variety. Extensions by the multiplicative group are semiabelian varieties. For extensions by the additive group, see [21].

We can find such sequences over p-adic fields, because of the well-known fact that any finitely generated field of characteristic zero embeds into at least one p-adic field.

These sequences of algebraic groups yield sequences of definable groups:[2]

$$0 \to (\mathbb{Q}_p^\times)^n \to G(\mathbb{Q}_p) \to A(\mathbb{Q}_p) \to 0,$$
$$0 \to \mathbb{Q}_p^n \to G(\mathbb{Q}_p) \to A(\mathbb{Q}_p) \to 0.$$

In each case, this is a decomposition as in Theorem 1.1. The group $A(\mathbb{Q}_p)$ has *fsg* because it is definably compact, while the additive and multiplicative groups have *dfg*.

REMARK 3.4. The sequence in Theorem 1.1 need not split. That is, there are non-split short exact sequences of definable groups

$$1 \to H \to G \to C \to 1, \qquad (*)$$

[2]The map $G(\mathbb{Q}_p) \to A(\mathbb{Q}_p)$ is surjective in each case because of the vanishing of the Galois cohomology groups $H^1(\mathrm{Gal}(\mathbb{Q}_p), \mathbb{Q}_p)$ and $H^1(\mathrm{Gal}(\mathbb{Q}_p), \mathbb{Q}_p^\times)$, i.e., Hilbert's Theorem 90. The vanishing cohomology ensures that if B is $\mathbb{Q}_p$ or $\mathbb{Q}_p^\times$, then any $\mathbb{Q}_p$-definable B-torsor must have a $\mathbb{Q}_p$-point. The fibers of $G \to A$ over points of $A(\mathbb{Q}_p)$ are $\mathbb{Q}_p$-definable B-torsors.

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where H has *dfg* and C has *fsg*. A simple example is

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.$$

(Finite groups have both *dfg* and *fsg*.)

Nevertheless, one could ask whether the sequence splits "up to finite factors" in some sense. *This does not happen either.* Let M be a moderately saturated elementary extension of $\mathbb{Q}_p$, and take an element $a \in M$ with positive valuation, such that a is an nth power for every n. Using this value of a in Example 3.2, we get groups G and C and a short exact sequence of definable groups

$$1 \to M^\times \to G \to C \to 1.$$

This sequence does not split, even non-definably (as a sequence of abstract groups), even "up to finite factors." More precisely, the following is true:

CLAIM 3.5. *There is no injective homomorphism from C to G. More generally, let $\sim$ be the smallest equivalence relation on abelian groups such that $A \sim B$ whenever there is a homomorphism $f : A \to B$ with finite kernel and cokernel. If $G' \sim G$ and $C' \sim C$, then there is no injective homomorphism from C' to G'.*

PROOF SKETCH Say that an abelian group A is "spacious" if it has a subgroup isomorphic to $\mathbb{Q}/\mathbb{Z}$. If $g : A \to B$ is an injective homomorphism and A is spacious, then B is spacious. It is an exercise in homological algebra to show that if $f : A \to B$ is a homomorphism with finite kernel and cokernel, then A is spacious if and only if B is spacious. Therefore, it suffices to show that C is spacious and G is not.

Using saturation of M and divisibility of the element a in $M^\times$, extend the homomorphism

$$\begin{aligned} \mathbb{Z} &\to M^\times \\ n &\mapsto a^n \end{aligned}$$

to a homomorphism

$$\begin{aligned} \mathbb{Q} &\to M^\times \\ q &\mapsto \text{“}a^q\text{”}. \end{aligned}$$

Then there is an injective group homomorphism

$$\begin{aligned} \mathbb{Q}/\mathbb{Z} &\to C \\ q &\mapsto a^q. \end{aligned}$$

This homomorphism shows that C is spacious.

Next, consider G. Suppose that $(x, y) \in G$ and $(x, y)^n = (1, 1)$ for some $n > 0$. Then

$$(1, 1) = (x, y)^n = \left(\frac{x^n}{p^m}, \frac{y^n}{a^m} \right)$$

for some integer $m \geq 0$. Then $y^n = a^m$ and $x^n = p^m$. The fact that $0 \leq v(y) < v(a)$ implies that

$$v(a^m) = v(y^n) < v(a^n),$$

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so that $m < n$. On the other hand, $n \cdot v(x) = v(x^n) = v(p^m) = m$, so m is a multiple of n. This forces $m = 0$, so that $x^n = y^n = 1$. In other words, we have shown

$$(x, y)^n = (1, 1) \implies x^n = y^n = 1.$$

Therefore, the torsion subgroup of G is isomorphic to the torsion subgroup of $M^\times \times M^\times$. But the torsion subgroup of $M^\times$ is finite, owing to the isomorphisms

$$\mathbb{Z}_2^\times \cong \mathbb{Z} \times \mathbb{Z}/2 \times \mathbb{Z}_2$$
$$\mathbb{Z}_p^\times \cong \mathbb{Z} \times \mathbb{Z}/(p-1) \times \mathbb{Z}_p \text{ for } p > 2.$$

Therefore the groups $M^\times$, $M^\times \times M^\times$, and G have finite torsion, and G is not spacious. ⊣

REMARK 3.6. As a different example, we suspect that the exact sequences of Example 3.3 do not split *definably*, even up to finite factors, provided that the underlying sequence of algebraic groups doesn't split. But at least in the additive case, the sequence

$$0 \to \mathbb{Q}_p^n \to G(\mathbb{Q}_p) \to A(\mathbb{Q}_p) \to 0$$

splits non-definably, because the group $\mathbb{Q}_p^n$ is divisible, hence injective as a $\mathbb{Z}$-module. We don't know whether the sequence splits non-definably in the semiabelian case.

§4. Abelian groups over $\mathbb{Q}_p$.

FACT 4.1. *Let G be a definably amenable group definable over $\mathbb{Q}_p$. There is an algebraic group H over $\mathbb{Q}_p$ and a definable finite-to-one group homomorphism from G^{00} to H.*

PROOF. This follows from [14, Theorem 2.19] via the proof of [14, Corollary 2.22]. ⊣

THEOREM 1.3. *Suppose that G is an abelian group definable in $\mathbb{Q}_p$.*

(i) *$G^{00} = G^0$ (in a monster model extending $\mathbb{Q}_p$).*
(ii) *There is a finite index definable subgroup $E \subseteq G$ and a finite subgroup $F \subseteq E$ such that E/F is isomorphic to an open subgroup of an algebraic group.*

PROOF. For part (i), Theorem 1.1 gives a short exact sequence

$$1 \to H \to G \to C \to 1,$$

where H has *dfg* and C is definably compact. Then $C^0 = C^{00}$ because C is definably compact and defined over $\mathbb{Q}_p$ [15, Corollary 2.4], and $H^0 = H^{00}$ because H is *dfg* [18, Proof of Lemma 1.15]. Then $G^0 = G^{00}$ by Lemma 2.2.

By (i), we have $G^0 = G^{00}$. Since G is an abelian group, it is amenable as an abstract group, so it is also definably amenable. By Fact 4.1, there is an algebraic group H and a finite-to-one definable homomorphism $f : G^0 \to H$. Now G^0 is the intersection of all $\mathbb{Q}_p$-definable finite-index subgroups of G. By compactness there is a $\mathbb{Q}_p$-definable finite-index subgroup $E \subseteq G$ such that f extends to a finite-to-one definable homomorphism $f' : E \to H$. Changing f', we may assume f' is $\mathbb{Q}_p$-definable, in case this wasn't provided by Fact 4.1. Replacing H with the Zariski closure of the image of f', we may assume the image is an open subgroup of H. ⊣

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§5. Interpretable groups. In this section, we discuss our original approach to Theorem 1.1, which yielded a weaker result, only giving an interpretable group. However, this approach is more general in one way—one can *start* with an interpretable group. Unfortunately, in the interpretable case we don't know how to prove the termination of the recursive process implicit in the proof of Theorem 1.1.

PROPOSITION 5.1. *Let G be an abelian definable group, let H be a definable subgroup, and let $X = G/H$ be the interpretable quotient group. Consider the canonical definable manifold topology on G, and the quotient topology on X.*

1. *The quotient map $\pi : G \to X$ is an open map.*
2. *The quotient topology on X is definable.*
3. *The quotient topology on X is a group topology.*
4. *The quotient topology on X is Hausdorff.*

PROOF. 1. If $U \subseteq G$ is open, then $\pi^{-1}(\pi(U)) = U \cdot H = \bigcup_{h \in H}(U \cdot h)$ which is open. By definition of the quotient topology, $\pi(U)$ is open.
2. If $\mathcal{B}$ is a definable basis of opens on G, then $\{\pi(U) : U \in \mathcal{B}\}$ is a definable basis of opens on X, because π is an open map.
3. We claim $(x, y) \mapsto x \cdot y^{-1}$ is continuous on X. Fix $a, b \in X$. Let $U \subseteq X$ be an open neighborhood of $a \cdot b^{-1}$. Take $\tilde{a}, \tilde{b} \in G$ lifting a and b. Then $\tilde{a} \cdot \tilde{b}^{-1} \in \pi^{-1}(U)$, which is open. By continuity of the group operations on G, there are open neighborhoods $V \ni \tilde{a}$ and $W \ni \tilde{b}$ such that $x \in V,\ y \in W \implies x \cdot y^{-1} \in \pi^{-1}(U)$. Because π is an open map, $\pi(V)$ and $\pi(W)$ are open neighborhoods of a and b, respectively. If $x \in \pi(V)$ and $y \in \pi(W)$, then $x \cdot y^{-1} \in U$, because we can write $x = \pi(\tilde{x}),\ y = \pi(\tilde{y})$ for $\tilde{x} \in V,\ \tilde{y} \in W$, and then $x \cdot y^{-1} = \pi(\tilde{x} \cdot \tilde{y}^{-1}) \in \pi(\pi^{-1}(U)) = U$. This proves continuity of $x \cdot y^{-1}$ at (a, b).
4. Because the quotient topology is a group topology, it suffices to show that $\{1_X\}$ is closed. By definition of the quotient topology, it suffices to show that H is closed in G. On definable manifolds, the frontier of a set is lower-dimensional than the set itself [5, Theorem 3.5]:

$$\dim(\overline{H} \setminus H) < \dim(H).$$

But $\overline{H} \setminus H$ is a union of cosets of H, and each coset has dimension $\dim(H)$. Therefore $\overline{H} \setminus H$ must be empty, and H is closed. ⊣

DEFINITION 5.2. A *manifold-dominated group* is an interpretable group X with a Hausdorff definable group topology such that there is a definable manifold $\tilde{X}$ and an interpretable surjective continuous open map $f : \tilde{X} \to X$.

In the setting of Proposition 5.1, X is manifold dominated via the map $G \to X$.

REMARK 5.3. If X is *any* interpretable group, then there is a definable group topology τ on X making (X, τ) be manifold-dominated [11, Theorem 5.10]. Moreover, τ is uniquely determined, though the manifold $\tilde{X}$ is not. This motivates working in the more general context of manifold-dominated abelian groups, rather than the special case of quotient groups G/H.

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THEOREM 5.4. *Let X be a manifold-dominated interpretable abelian group. Suppose X is not definably compact. Then there is an interpretable subgroup $X' \subseteq X$ with the following properties*:

1. *X' is not definably compact.*
2. dp-rk$(X') = 1$.
3. *X' has dfg.*

Theorem 5.4 is an analogue of [13, Theorem 6.8 and Corollary 6.11], and the proof is similar. Nevertheless, we sketch the proof for completeness.

For the rest of the section, work in a monster model $\mathbb{M}$. Fix a definable manifold $\tilde{X}$, an interpretable abelian group X with a Hausdorff definable group topology, and an interpretable continuous surjective open map $\pi : \tilde{X} \to X$. Also fix a small model K over which everything is defined.

DEFINITION 5.5. If S is an interpretable topological space (in pCF) and $x_0 \in S$, then a *good neighborhood basis* of x_0 is an interpretable family $\{O_t\}_{t\in\Gamma}$ with the following properties:

1. $\{O_t\}_{t\in\Gamma}$ is a neighborhood basis of x_0.
2. $t \leq t' \implies O_t \subseteq O_{t'}$.
3. Each set O_t is clopen and definably compact.
4. $\bigcup_t O_t = S$.

This is more general than the definition in [13, Definition 2.27], since we are considering topological spaces rather than topological groups. The definition here is slightly weaker, since we do not require $O_t^{-1} = O_t$ when S is a group.

Fix some element $\tilde{1} \in \tilde{X}$ lifting $1 \in X$. By the proof of [13, Proposition 2.28], there is a good neighborhood basis $\{O_t\}_{t\in\Gamma}$ of $\tilde{1}$ in $\tilde{X}$. Let $V_t = \pi(O_t)$. Then $\{V_t\}_{t\in\Gamma}$ is a good neighborhood basis of 1 in X. The analogue of [13, Proposition 2.29] holds, via the same proof:

1. For any $t \in \Gamma$, there is $t' \in \Gamma$ such that $V_{t'} \cdot V_{t'}^{-1} \subseteq V_t$.
2. For any $t \in \Gamma$, there is $t'' \in \Gamma$ such that $V_t \cdot V_t^{-1} \subseteq V_{t''}$.

Say that a set $S \subseteq X$, not necessarily interpretable, is *bounded* if $S \subseteq V_t$ for some $t \in \Gamma$. As in [13, Proposition 2.10], S is bounded if and only if S is contained in a definably compact subset of X. If $A, B \subseteq X$, let $A \diamond B$ denote the set

$$\{g \in A : gB \cap A = \varnothing\},$$

as in [13, Section 4.1]. Let $A \diamond B \setminus C$ mean $A \diamond (B \setminus C)$.

LEMMA 5.6. *Let $I \subseteq X$ be an unbounded interpretable set. Let $A \subseteq X$ be bounded, but not necessarily interpretable. Then there is $t \in \Gamma_{\mathbb{M}}$ such that $I \diamond V_t \setminus A$ is bounded.*

PROOF. The proofs of Lemmas 4.9–4.11 in [13] work here, after making a couple trivial changes. The interpretable group X has finite dp-rank because dp-rk$(X) \leq$ dp-rk$(\tilde{X}) = \dim(\tilde{X}) < \infty$. ⊣

Recall our assumption that $\pi : \tilde{X} \to X$ is K-interpretable for some small model K. Fix $|K|^+$-saturated L with $K \preceq L \preceq \mathbb{M}$. If Σ is a definable type or definable partial type over K, then Σ^L denotes its canonical extension over L. (See [16, Definition 2.12] for definability of partial types. When Σ is complete, Σ^L is the heir of Σ.)

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LEMMA 5.7. *There is a one-dimensional definable type $p \in S_{\tilde{X}}(K)$ whose pushforward $q = \pi_* p$ has the following properties*:

1. *q is "unbounded" over K, in the sense that q does not concentrate on any K-interpretable bounded set, or equivalently, q does not concentrate on V_t for any $t \in \Gamma_K$.*
2. *Similarly, the heir q^L is unbounded over L.*
3. *If $b \in X$ realizes q and $b \notin V_t$ for any $t \in \Gamma_L$, then b realizes q^L.*

PROOF. Take $u \in \mathbb{M}$ with $v(u) > \Gamma_K$. In other words, u is infinitesimally close to 0 over K. Then $\mathrm{tp}(u/K)$ is definable. Let $\gamma = v(u)$. As X is not definably compact, $V_\gamma \neq X$. The set $\pi^{-1}(X \setminus V_\gamma)$ is a non-empty Ku-definable subset of $\tilde{X}$. By definable Skolem functions, there is $\beta_0 \in \pi^{-1}(X \setminus V_\gamma)$ with $\beta_0 \in \mathrm{dcl}(Ku)$. Then $\beta_0 = f(u)$ for some K-definable function f. Let $p = \mathrm{tp}(\beta_0/K)$. Then $p = f_*(\mathrm{tp}(u/K))$, so p is definable. Let $b_0 = \pi(\beta_0)$ and let $q = \pi_* p = \mathrm{tp}(b_0/K)$. By choice of β_0, $b_0 = \pi(\beta_0) \notin V_\gamma$, which implies $b_0 \notin V_t \subseteq V_\gamma$ for any $t \in \Gamma_K$. Thus q is unbounded over K. As q^L is the heir, it is similarly unbounded over L.

Finally, suppose that b satisfies the assumptions of (3). Then $\mathrm{tp}(b/K) = q = \mathrm{tp}(b_0/K)$, so there is $\sigma \in \mathrm{Aut}(\mathbb{M}/K)$ with $\sigma(b_0) = b$. Let $\beta = \sigma(\beta_0)$. Then $(b, \beta) \equiv_K (b_0, \beta_0)$, and in particular β realizes p and $\pi(\beta) = b$. Recall the sets O_t used to define V_t. If $\beta \in O_t$ for some $t \in \Gamma_L$, then $b = \pi(\beta) \in \pi(O_t) = V_t$, contradicting the assumptions. Therefore, $\beta \notin O_t$ for any $t \in \Gamma_L$. By [13, Lemma 2.25], β realizes p^L. Then $b = \pi(\beta)$ realizes $\pi_*(p^L) = q^L$. ⊣

Fix p, q as in Lemma 5.7. Fix $\beta \in \tilde{X}$ realizing p^L and let $b = \pi(\beta) \in X$. Then b realizes q^L.

We will make use of the notation and facts from [13, Section 5], applied to the group X and the definable type q. In particular, μ is the infinitesimal partial type of X over K, μ^L is the infinitesimal partial type of X over L, and $\mathrm{st}_L^{\mathbb{M}}$ is the standard part map, a partial map from X to $X(L)$. The domain of $\mathrm{st}_L^{\mathbb{M}}$ is the subgroup $\mu^L(\mathbb{M}) \cdot X(L)$ of points in X infinitesimally close to points in $X(L)$. If $Y \subseteq X$, then $\mathrm{st}_L^{\mathbb{M}}(Y)$ denotes the image of $Y \cap (\mu^L(\mathbb{M}) \cdot X(L))$ under $\mathrm{st}_L^{\mathbb{M}}$.

The following lemma takes the place of [13, Fact 6.3].

LEMMA 5.8. *Suppose $Y \subseteq X$ is β-interpretable.*

1. *The set $\mathrm{st}_L^{\mathbb{M}}(Y) \subseteq X(L)$ is interpretable (in the structure L).*
2. $\mathrm{dp\text{-}rk}(\mathrm{st}_L^{\mathbb{M}}(Y)) \leq \mathrm{dp\text{-}rk}(Y)$.

See Remark A.1 for the definition of *ict pattern* and *dp-rank*.

PROOF. 1. Fix some interpretable basis of opens for X. Let $\mathcal{F}$ be the collection of L-interpretable basic open sets which intersect Y. Then $\mathcal{F}$ is interpretable in the structure L, because $\mathcal{F}$ is defined externally using β, but $\mathrm{tp}(\beta/L)$ is definable. Now if $a \in X(L)$, the following are equivalent:
   (a) $a \in \mathrm{st}_L^{\mathbb{M}}(Y)$.
   (b) There is $a' \in Y$ such that for every L-interpretable basic open neighborhood $U \ni a$, we have $a' \in U$.
   (c) For every L-interpretable basic open neighborhood $U \ni a$, there is $a' \in Y$ such that $a' \in U$.
   (d) Every L-interpretable basic open neighborhood of a is in $\mathcal{F}$.

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Indeed, (a) $\iff$ (b) by definition, (b) $\iff$ (c) by saturation of $\mathbb{M}$, and (c) $\iff$ (d) by definition of $\mathcal{F}$. Condition (d) is definable because $\mathcal{F}$ is.

2. Let r be the dp-rank of the interpretable set $D := \mathrm{st}_L^{\mathbb{M}}(Y)$. It is finite, bounded by dp-rk(X). There is an ict-pattern of depth r in D. That is, there are uniformly interpretable sets $S_{i,j} \subseteq D$ for $i < r$ and $j < \omega$, and points $b_\eta \in D$ for $\eta \in \omega^r$, such that $b_\eta \in S_{i,j} \iff j = \eta(i)$. By Theorem A.6 in Appendix A, we can also ensure that $S_{i,j}$ is open and $j \neq \eta(i) \implies b_\eta \notin \overline{S_{i,j}}$. As L is $\aleph_1$-saturated, we can arrange for all the data to be L-interpretable. Then each b_η is $\mathrm{st}_L^{\mathbb{M}}(b'_\eta)$ for some $b'_\eta \in Y$. Since $S_{i,j}$ is open and L-interpretable, we have $b'_\eta \in S_{i,j}$ for $j = \eta(i)$. Since $\overline{S_{i,j}}$ is closed and L-interpretable, we have $b'_\eta \notin \overline{S_{i,j}}$ for $j \neq \eta(i)$. Then the sets $S_{i,j}$ and elements b'_η are an ict-pattern of depth r in Y, showing dp-rk$(Y) \geq r =$ dp-rk(D). ⊣

LEMMA 5.9. *The following subsets of $X(L)$ are equal*:

1. stab$(\mu^L \cdot q^L)$.
2. $\bigcap_{\varphi \in \mathcal{L}} \mathrm{stab}_\varphi(\mu \cdot q)(L)$.
3. $\mathrm{st}_L^{\mathbb{M}}(q^L(\mathbb{M})b^{-1})$.
4. $\bigcap_{\psi \in q^L} \mathrm{st}_L^{\mathbb{M}}(\psi(\mathbb{M})b^{-1})$.
5. $\bigcap_{\psi \in q} \mathrm{st}_L^{\mathbb{M}}(\psi(\mathbb{M})b^{-1})$.

See [13, Definition 5.3] for the definition of stab$_\varphi(-)$.

PROOF. The equivalence of (1)–(4) is Remark 5.12 and Lemma 5.13 in [13]. The equivalence of (4) and (5) follows by a similar argument to the proof of [13, Lemma 6.2], using Lemma 5.7(3) instead of [13, Lemma 2.25]. ⊣

LEMMA 5.10. *If $I \subseteq X$ is L-interpretable and contains b, then $\mathrm{st}_L^{\mathbb{M}}(Ib^{-1})$ is unbounded in $X(L)$.*

PROOF. If not, take $t \in \Gamma_L$ such that $\mathrm{st}_L^{\mathbb{M}}(Ib^{-1}) \subseteq V_t$. By Lemma 5.7(2), b is not in any L-interpretable bounded sets. Therefore I is unbounded. By Lemma 5.6, we can find $t' \in \Gamma_L$ such that $I \diamond V_{t'} \setminus V_t$ is bounded. Then $b \notin I \diamond V_{t'} \setminus V_t$. This means that

$$b \cdot (V_{t'} \setminus V_t) \cap I \neq \varnothing.$$

Therefore there is $a \in V_{t'} \setminus V_t$ such that $ba \in I$. Then there is $\alpha \in O_{t'}$ with $\pi(\alpha) = a$. The conditions on α and a are definable over $\mathrm{dcl}(Lb) \subseteq \mathrm{dcl}(L\beta)$ (where β is the realization of p^L). By definable Skolem functions, we can assume $\alpha \in \mathrm{dcl}(L\beta)$. Then $\mathrm{tp}(\alpha/L)$ is a pushforward of $\mathrm{tp}(\beta/L)$, so $\mathrm{tp}(\alpha/L)$ is a one-dimensional definable type on $\tilde{X}$. This type $\mathrm{tp}(\alpha/L)$ concentrates on the definably compact set $O_{t'} \subseteq \tilde{X}$, and therefore $\mathrm{tp}(\alpha/L)$ specializes to some point $\gamma \in G(L)$ by [13, Lemma 2.23]. As the map $\pi : \tilde{X} \to X$ is continuous, $\mathrm{tp}(a/L)$ specializes to $c := \pi(\gamma) \in X(L)$. Thus $\mathrm{st}_L^{\mathbb{M}}(a)$ exists and equals c. Since $V_{t'} \setminus V_t$ is closed, $\mathrm{st}_L^{\mathbb{M}}(a) \in V_{t'} \setminus V_t$. But $a \in b^{-1}I = Ib^{-1}$, and

$$\mathrm{st}_L^{\mathbb{M}}(a) \in \mathrm{st}_L^{\mathbb{M}}(Ib^{-1}) \subseteq V_t,$$

a contradiction. ⊣

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We can now complete the proof of Theorem 5.4. By Lemma 5.9,

$$\bigcap_{\varphi\in\mathcal{L}} \mathrm{stab}_\varphi(\mu\cdot q)(L) = \bigcap_{\psi\in q} \mathrm{st}^{\mathbb{M}}_L(\psi(\mathbb{M})b^{-1}). \tag{$*$}$$

The groups $\mathrm{stab}_\varphi(\mu\cdot q)$ are K-interpretable because $\mu\cdot q$ is a K-definable partial type. The sets $\mathrm{st}^{\mathbb{M}}_L(\psi(\mathbb{M})b^{-1})$ are interpretable by Lemma 5.8(1). Both intersections involve at most $|K|$ terms, and both intersections are filtered.

If some $\mathrm{stab}_\varphi(\mu\cdot q)(L)$ is bounded, then by $|K|^+$-saturation of L we have $\mathrm{st}^{\mathbb{M}}_L(\psi(\mathbb{M})b^{-1}) \subseteq \mathrm{stab}_\varphi(\mu\cdot q)(L)$ for some $\psi(x)\in q(x)$, contradicting Lemma 5.10. Therefore, every group $\mathrm{stab}_\varphi(\mu\cdot q)(L)$ is unbounded. Consequently, *no* $\mathrm{stab}_\varphi(\mu\cdot q)$ is definably compact.

Since $\mathrm{tp}(\beta/K)$ has dimension 1, there is some K-definable set $D\ni\beta$ of dimension 1. Then $\text{dp-rk}(\pi(D))\le \text{dp-rk}(D) = \dim(D) = 1$. If $\psi(x)$ defines $\pi(D)$, then $\psi(x)\in q = \mathrm{tp}(b/K)$, and $\mathrm{st}^{\mathbb{M}}_L(\psi(\mathbb{M})b^{-1})$ has dp-rank at most 1 by Lemma 5.8(2). By $|K|^+$-saturation, $(*)$ gives some φ such that $\mathrm{stab}_\varphi(\mu\cdot q)(L)\subseteq \mathrm{st}^{\mathbb{M}}_L(\psi(\mathbb{M})b^{-1})$. Then $\mathrm{stab}_\varphi(\mu\cdot q)$ has dp-rank at most 1. On the other hand, $\mathrm{stab}_\varphi(\mu\cdot q)$ is infinite, since it is not definably compact. Therefore $X' := \mathrm{stab}_\varphi(\mu\cdot q)$ has dp-rank at least 1.

It remains to show that the interpretable subgroup $X'\subseteq X$ has *dfg*. The proof of [13, Lemma 6.10] works with minor changes. For completeness, we give the details. For abelian groups of dp-rank 1, "not *fsg*" implies *dfg* as in the proof of [19, Lemma 2.9]. It suffices to show that X' does *not* have *fsg*. Assume for the sake of contradiction that X' has *fsg*. By [7, Proposition 4.2], non-generic sets form an ideal, and there is a small model M_0 such that every generic set contains an M_0-point. Take t large enough that V_t contains every point in $X(M_0)$. Then $X'\setminus V_t$ is not generic in X', so $X'\cap V_t$ *is* generic, meaning that finitely many translates of $X'\cap V_t$ cover X'. But $X'\cap V_t$ and its translates are bounded (as subsets of X), so then X' is bounded, a contradiction. This completes the proof of Theorem 5.4.

COROLLARY 5.11. *Let X be an abelian interpretable group. Then there is $\alpha\le\omega$ and an increasing chain of dfg subgroups $(Y_i : i<\alpha)$ with $Y_0 = 0$ such that the quotients Y_i/Y_{i+1} have dp-rank 1. In the case when $\alpha<\omega$, the quotient $X/Y_{\alpha-1}$ is definably compact and has fsg.*

PROOF. Any interpretable group is manifold-dominated [11, Theorem 5.10], so we can apply Theorem 5.4 to any interpretable group. The first application gives Y_1; applying the theorem to X/Y_1 gives Y_2, and so on. The process terminates if any quotient X/Y_i is definably compact. Definably compact groups have *fsg* [11, Theorem 7.1]. To prove that the groups Y_i have *dfg*, we can no longer use Lemma 2.3, as $p\mathrm{CF}^{\mathrm{eq}}$ lacks definable Skolem functions. But Theorem B.6 in Appendix B works. ⊣

REMARK 5.12. If we start with a quotient group G/H, we can replace the use of [11, Theorem 5.10] with Proposition 5.1.

REMARK 5.13. If X is *definable*, then the quotients Y_i/Y_j are definable by induction on $i-j$, using Corollary 2.10. Then $\dim(Y_{i+1}/Y_i) = \text{dp-rk}(Y_{i+1}/Y_i) = 1$, which implies $\dim(Y_{i+1}) > \dim(Y_i)$. Therefore, the sequence *must* terminate, as we saw in the proof of Theorem 1.1. In the general interpretable case, it's unclear whether this works, so we make a conjecture:

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CONJECTURE 5.14. In Corollary 5.11, α is finite. Therefore, any abelian interpretable group X sits in a short exact sequence $1 \to Y_{\alpha-1} \to X \to X/Y_{\alpha-1} \to 1$ where $Y_{\alpha-1}$ has *dfg* and $X/Y_{\alpha-1}$ has *fsg* and is definably compact.

Pillay and Yao asked whether any definably amenable group G in a distal theory sits in a short exact sequence $1 \to H \to G \to C \to 1$ with C having *fsg* and H having *dfg* [18, Question 1.19]. If Conjecture 5.14 is true, it would provide further evidence for this.

§A. Nice ict patterns.

REMARK A.1. Following [22, Definition 4.21], an *ict-pattern* of depth κ in a partial type $\Sigma(x)$ is a sequence of formulas $\varphi_i(x; y_i)$ and an array $(b_{i,j} : i < \kappa,\ j < \omega)$ with $|b_{i,j}| = |y_i|$ such that for any function $\eta : \kappa \to \omega$, the following partial type is consistent:

$$\Sigma(x) \cup \{\varphi_{i,\eta(i)}(x, b_{i,\eta(i)}) : i < \kappa\} \cup \{\neg\varphi_{i,j}(x, b_{i,j}) : i < \kappa,\ j \neq \eta(i)\}.$$

Abusing notation, we say that $(\varphi_i(x; b_{i,j}) : i < \kappa,\ j < \omega)$ is an ict-pattern to mean that the pair $((\varphi_i : i < \kappa), (b_{i,j} : i < \kappa,\ j < \omega))$ is an ict-pattern. Sometimes we consider ict-patterns where the columns are indexed by an infinite linear order I other than ω. The definition is analogous, and ict-patterns of this sort can be converted to ict-patterns indexed by ω via a compactness argument.

Finally, the *dp-rank* of $\Sigma(x)$ is the supremum of cardinals κ such that there is an ict-pattern of depth κ in $\Sigma(x)$, possibly in an elementary extension.

Work in $\mathbb{M}^{\mathrm{eq}}$ for some monster model $\mathbb{M} \models p\mathrm{CF}$. There is a well-behaved notion of dimension on $\mathbb{M}^{\mathrm{eq}}$ [6], which gives rise to a notion of independence:

$$a \overset{\dim}{\underset{C}{\downarrow}} b \iff \dim(a/Cb) = \dim(a/C) \iff \dim(b/Ca) = \dim(b/C).$$

This notion satisfies many of the usual properties [11, Section 2.1].[3] Say that a sequence $\{a_i : i \in I\}$ is *dimensionally independent* over a set B if $a_i \overset{\dim}{\underset{B}{\downarrow}} a_{<i}$ for $i \in I$, where $a_{<i} = \{a_j : j < i\}$. As usual, this is independent of the order on I.

LEMMA A.2. *If* $\mathrm{tp}(a/Cb)$ *is finitely satisfiable in* C, *then* $a \overset{\dim}{\underset{C}{\downarrow}} b$.

PROOF. Suppose not. Let $n = \dim(b/Ca) < \dim(b/C)$. By [6, Proposition 3.7], there is a Ca-interpretable set X containing b with $\dim(X) = n$. Write X as $\varphi(a, \mathbb{M})$ for some $\mathcal{L}^{\mathrm{eq}}_C$-formula $\varphi(x, y)$. By [11, Proposition 2.12], the set $\{a' \in \mathbb{M} : \dim(\varphi(a', \mathbb{M})) = n\}$ is definable, defined by some $\mathcal{L}^{\mathrm{eq}}_C$-formula $\psi(x)$. Then $\mathbb{M} \models \varphi(a, b) \wedge \psi(a)$. As $\mathrm{tp}(a/Cb)$ is finitely satisfiable in C, there is some $a' \in C$ such that $\mathbb{M} \models \varphi(a', b) \wedge \psi(a')$. Then b is in the C-interpretable set $\varphi(a', \mathbb{M})$ which has dimension n as $\mathbb{M} \models \psi(a')$. Therefore $\dim(b/C) \leq n$, a contradiction. ⊣

COROLLARY A.3. *Suppose* $\ldots, b_{-1}, b_0, b_1, \ldots, \ldots, c_{-1}, c_0, c_1, \ldots$ *is* C_0*-indiscernible. Then the sequence* $\ldots, b_{-1}, b_0, b_1, \ldots$ *is dimensionally independent over* $C = C_0 \cup \{c_i : i \in \mathbb{Z}\}$.

[3] The one unusual property is that "$\dim(a/C) = 0$" is strictly weaker than "$a \in \mathrm{acl}(C)$."

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PROOF. For example, $p = \mathrm{tp}(b_n/Cb_1b_2\cdots b_{n-1})$ is finitely satisfiable in C; any formula in p is satisfied by c_i for $i \ll 0$. This argument shows that any finite subsequence of $\{b_i\}_{i\in\mathbb{Z}}$ is dimensionally independent over C. This implies the full sequence is dimensionally independent, by finite character of $\mathop{\smile\!\!\!\!\!\!|}^{\dim}$. ⊣

LEMMA A.4. *If* $\{b_i : i \in I\}$ *is dimensionally independent over* C, *and* $\dim(a/C) = n$, *then* $a \mathop{\smile\!\!\!\!\!\!|}^{\dim}_C b_i$ *for all but at most* n *values of* i.

The proof is standard, but we include it for completeness.

PROOF. Otherwise, passing to a subsequence, we could arrange for $b_1, \ldots, b_{n+1}$ to be dimensionally independent over C, but $a \not\mathop{\smile\!\!\!\!\!\!|}^{\dim}_C b_i$ for each i. The sequence $(\dim(a/Cb_1, \ldots, b_i) : 0 \le i \le n+1)$ cannot decrease $n+1$ times, so there is some $0 \le i \le n$ such that $\dim(a/Cb_1, \ldots, b_i) = \dim(a/Cb_1, \ldots, b_{i+1})$, i.e.,

$$a \mathop{\smile\!\!\!\!\!\!|}^{\dim}_{Cb_1,\ldots,b_i} b_{i+1}.$$

As $b_1, \ldots, b_i \mathop{\smile\!\!\!\!\!\!|}^{\dim}_C b_{i+1}$, left transitivity gives $a \mathop{\smile\!\!\!\!\!\!|}^{\dim}_C b_{i+1}$, a contradiction. ⊣

LEMMA A.5. *Let* X *be a* C*-interpretable set of parameters, with dp-rank* r. *Then there is* $C' \supseteq C$ *and an ict pattern of depth* r *in* X *of the form* $(\varphi_i(x; b_{i,j}) : i < r,\ j \in \mathbb{Z})$, *such that the array* $(b_{i,j} : i < r,\ j \in \mathbb{Z})$ *is mutually* C'*-indiscernible, and for each* i, *the sequence* $(b_{i,j} : j \in \mathbb{Z})$ *is dimensionally independent over* C'.

PROOF. Let $\mathbb{Z} + \mathbb{Z}'$ denote two copies of $\mathbb{Z}$ laid end to end, with the second copy denoted $\mathbb{Z}'$. Take an ict pattern $(\varphi_i(x; b^0_{i,j}) : i < r,\ j < \omega)$ in X. Let $(b_{i,j} : i < r,\ j \in \mathbb{Z} + \mathbb{Z}')$ be a mutually C-indiscernible array extracted from $(b^0_{i,j} : i < r,\ j < \omega)$. Then $(\varphi_i(x; b_{i,j}) : i < r,\ j \in \mathbb{Z} + \mathbb{Z}')$ is an ict pattern in X. Let $C' = C \cup \{b_{i,j}, i < r,\ j \in \mathbb{Z}'\}$. Then $(b_{i,j} : i < r,\ j \in \mathbb{Z})$ is mutually C'-indiscernible, and each row is dimensionally independent over C' by Corollary A.3. ⊣

THEOREM A.6. *Let* G *be a manifold-dominated interpretable group of dp-rank* r. *There is an ict-pattern* $(\varphi_i(x; b_{i,j}) : i < r,\ j < \omega)$ *in* G *such that if* $S_{i,j} = \varphi_i(\mathbb{M}; b_{i,j})$, *then the following properties hold*:

1. *Each set* $S_{i,j}$ *is open*.
2. *For each function* $\eta : r \to \omega$, *there is an element* $a_\eta \in G$ *such that*

$$j = \eta(i) \implies a_\eta \in S_{i,j},$$
$$j \ne \eta(i) \implies a_\eta \notin \overline{S_{i,j}}.$$

PROOF. By [11, Theorem 5.10], the topology on G is "admissible," and so

$$\dim(\overline{D} \setminus D) < \dim(D) \qquad \text{(Small boundaries property)}$$

for any interpretable subset $D \subseteq G$, by [11, Proposition 4.34]. By Lemma A.5, there is an ict-pattern $(\psi_i(x; b_{i,j}) : i < r,\ j \in \mathbb{Z})$ and a set of parameters C (over which G is interpretable) such that the $b_{i,j}$ are mutually indiscernible over C, and each row is dimensionally independent over C. Take some a such that $\mathbb{M} \models \psi_i(a; b_{i,j}) \Leftrightarrow j = 0$ for all $i < r$ and $j \in \mathbb{Z}$. By [6, Proposition 3.7] there is a formula $\theta_i(x; b_{i,0}, c_i)$ in $\mathrm{tp}(a/Cb_{i,0})$ such that $\dim(\theta_i(x; b_{i,0}, c_i)) = \dim(a/Cb_{i,0})$.

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Replacing $b_{i,j}$ with $b_{i,j}c_i$ and replacing $\psi_i(x;b_{i,j})$ with $\psi_i(x;b_{i,j}) \wedge \theta_i(x;b_{i,j},c_i)$, we may assume that $\dim(\psi_i(x;b_{i,0})) = \dim(a/Cb_{i,0}) =: k_i$. Let $V_{i,j} = \psi_i(\mathbb{M};b_{i,j})$. Then $\dim(V_{i,j}) = \dim(V_{i,0}) = k_i$ by indiscernibility.

For each i, we have $a \mathop{\smile\!\!\!\!\!\!|}_C^{\dim} b_{i,j}$ for all but finitely many j, by Lemma A.4. Throwing away the finitely many bad values of $b_{i,j}$ in each row, we may assume $a \mathop{\smile\!\!\!\!\!\!|}_C^{\dim} b_{i,j}$ for all $j \neq 0$. Thus $\dim(a/Cb_{i,j}) = \dim(a/C)$ for $j \neq 0$. By the Small Boundaries Property,

$$\begin{aligned}\dim(\overline{V_{i,j}} \setminus V_{i,j}) &< \dim(V_{i,j}) = \dim(V_{i,0}) = k_i = \dim(a/Cb_{i,0}) \\ &\leq \dim(a/C) = \dim(a/Cb_{i,j}),\end{aligned}$$

for $j \neq 0$. Then a cannot be in the $Cb_{i,j}$-interpretable set $\overline{V_{i,j}} \setminus V_{i,j}$. By choice of a, we also have $a \notin V_{i,j}$. So $a \notin \overline{V_{i,j}}$ for any $j \neq 0$. Thus

$$\begin{aligned}j = 0 &\implies a \in V_{i,j}, \\ j \neq 0 &\implies a \notin \overline{V_{i,j}}.\end{aligned}$$

By mutual indiscernibility, we can find a_η for any $\eta : r \to \mathbb{Z}$ such that

$$\begin{aligned}j = \eta(i) &\implies a_\eta \in V_{i,j}, \\ j \neq \eta(i) &\implies a_\eta \notin \overline{V_{i,j}}.\end{aligned}$$

Recall that the topology on G is a group topology, so every open neighborhood of a_η has the form $a_\eta \cdot N$ for some open neighborhood N of 1. For each i, j, η with $j \neq \eta(i)$, we can find an open neighborhood $N_{i,j,\eta} \ni 1$ such that $(a_\eta \cdot N_{i,j,\eta}) \cap V_{i,j} = \varnothing$. By saturation, there is an interpretable open neighborhood $N_0 \ni 1$ with $N_0 \subseteq N_{i,j,\eta}$ for all i, j, η. Because the topology is a group topology, there is a smaller interpretable open neighborhood $N \ni 1$ such that $N = N^{-1}$ and $N \cdot N \subseteq N_0$.

Let $U_{i,j} = V_{i,j} \cdot N = \{x \cdot y : x \in V_{i,j},\ y \in N\}$. Note that $U_{i,j}$ is open. If $j \neq \eta(i)$, then

$$(a_\eta \cdot N \cdot N) \cap V_{i,j} \subseteq a_\eta \cdot N_{i,j,\eta} \cap V_{i,j} = \varnothing.$$

The fact that $(a_\eta \cdot N \cdot N) \cap V_{i,j} = \varnothing$ implies that

$$(a_\eta \cdot N) \cap U_{i,j} = (a_\eta \cdot N) \cap (V_{i,j} \cdot N) = \varnothing.$$

The neighborhood $a_\eta \cdot N$ then shows that $a_\eta \notin \overline{U_{i,j}}$. On the other hand, $1 \in N$, so $V_{i,j} \subseteq U_{i,j}$. Therefore, if $j = \eta(i)$, then $a_\eta \in V_{i,j} \subseteq U_{i,j}$. Putting everything together, we get

$$\begin{aligned}j = \eta(i) &\implies a_\eta \in U_{i,j}, \\ j \neq \eta(i) &\implies a_\eta \notin \overline{U_{i,j}}.\end{aligned}$$

The sets $U_{i,j}$ are uniformly interpretable, so we can find some formula $\varphi(x;y)$ such that each $U_{i,j}$ has the form $\varphi(\mathbb{M};b_{i,j})$ for some $b_{i,j}$ (not the original ones). Then $(\varphi(\mathbb{M};b_{i,j}) : i < r,\ j < \omega)$ is the desired ict pattern. ⊣

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§B. Extensions and *dfg*. Work in a highly resplendent monster model $\mathbb{M}$. $\operatorname{acl}(-)$ always means $\operatorname{acl}^{\mathrm{eq}}$. All sets and parameters can come from $\mathbb{M}^{\mathrm{eq}}$ by default. "Definable" means "interpretable."

DEFINITION B.1. A definable set D is *almost A-definable* if it is $\operatorname{acl}(A)$-definable, or equivalently, $\{\sigma(D) : \sigma \in \operatorname{Aut}(\mathbb{M}/A)\}$ is finite. A global definable type p is *almost A-definable* if it is $\operatorname{acl}(A)$-definable, or equivalently, $\{\sigma(p) : \sigma \in \operatorname{Aut}(\mathbb{M}/A)\}$ is small.

The following is folklore; see [10, Lemma 3.13] for a proof.

FACT B.2. *Suppose b realizes $p|A$ for some almost A-definable global type p. Suppose c realizes $q|(Ab)$ for some almost Ab-definable global type q. Then c realizes $r|A$ for some almost A-definable global type r.*

DEFINITION B.3. Let G be an A-definable group. Say that G has *dfg over A* if there is a global definable type p on G such that p and all its left-translates are almost A-definable.

LEMMA B.4. *Let G be a definable dfg group and let S be a definable set with a regular right action of G. Suppose everything is A-definable, and G has dfg over A. Then there is a global type on S that is almost A-definable.*

PROOF. For $b \in S$, let $b \cdot p$ denote the pushforward of the A-definable type p along the map $x \mapsto b \cdot x$ from G to S. Note that $b \cdot p$ is a definable type on S.

The set $\mathfrak{S} = \{b \cdot p : b \in S\}$ is small, because it is $\{b_0 \cdot g \cdot p : g \in G\}$ for any fixed $b_0 \in S$. If $\sigma \in \operatorname{Aut}(\mathbb{M}/\operatorname{acl}(A))$, then σ fixes p and σ fixes $\mathfrak{S}$ setwise, since $\mathfrak{S}$ was defined in an invariant way. Therefore any $b \cdot p$ has small orbit under $\operatorname{Aut}(\mathbb{M}/\operatorname{acl}(A))$, implying that $b \cdot p$ is almost A-definable. ⊣

If G is a $\varnothing$-definable group, let $\mathbb{M} \ltimes G$ be the new structure obtained by adding a copy of G as a new sort S, and putting no structure on S other than the regular right action of G. For any $g \in G$, there is an automorphism of $\mathbb{M} \ltimes G$ fixing $\mathbb{M}$ and acting as left translation by g on the new sort S. In fact, $\operatorname{Aut}(\mathbb{M} \ltimes G) \cong \operatorname{Aut}(\mathbb{M}) \ltimes G$.

This construction is called "Construction C" in [8, Section 1], where it is attributed to Hrushovski's thesis. It also appears in [22] above Lemma 8.19. As mentioned in [22], $\mathbb{M} \ltimes G$ is a conservative extension of $\mathbb{M}$, in the sense that it introduces no new $\varnothing$-definable or definable sets on $\mathbb{M}$. After naming the element $1 \in S$, the two structures are bi-interpretable. Since we assumed $\mathbb{M}$ was very resplendent, $\mathbb{M} \ltimes G$ will be too.

LEMMA B.5. *Let $A \subseteq \mathbb{M}$ be a small set of parameters. Suppose that in $\mathbb{M} \ltimes G$, there is a global type p on S that is almost A-definable. Then G has dfg over A.*

PROOF. For $b, s \in S$, let $b^{-1} \cdot s$ denote the unique $x \in G$ such that $s = b \cdot x$. Let $b^{-1} \cdot p$ denote the pushforward of p along the map $x \mapsto b^{-1} \cdot x$ from S to G. Then $b^{-1} \cdot p$ is a definable type on G. If $\sigma \in \operatorname{Aut}(\mathbb{M}/A)$, we can extend σ to $\hat{\sigma} \in \operatorname{Aut}((\mathbb{M} \ltimes G)/A)$ fixing b. Then

$$\sigma(b^{-1} \cdot p) = \hat{\sigma}(b^{-1} \cdot p) = b^{-1} \cdot \hat{\sigma}(p).$$

There are only a small number of possibilities for $\hat{\sigma}(p)$, and so $b^{-1} \cdot p =: q$ is almost A-definable.

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If $g \in G$, then $g \cdot b^{-1} \cdot x = (b \cdot g^{-1})^{-1} \cdot x$ for $x \in S$, and so

$$g \cdot q = g \cdot b^{-1} \cdot p = (b \cdot g^{-1})^{-1} \cdot p = (b')^{-1} \cdot p$$

for $b' = b \cdot g^{-1}$. Replacing b with b' in the argument above, we see that $(b')^{-1} \cdot p = g \cdot q$ is almost A-definable. In other words, every translate $g \cdot q$ of q is almost A-definable, showing G has *dfg* over A. ⊣

Lemmas B.4 and B.5 are formally analogous to [22, Lemma 8.19], replacing "non-forking over A" with "almost A-definable."

THEOREM B.6. *If $1 \to N \to G \to H \to 1$ is a short exact sequence of definable groups, and N, H have dfg, then G has dfg.*

PROOF. Naming parameters, we may assume the whole sequence is $\varnothing$-definable, and that N and H have *dfg* over $\varnothing$. Construct $\mathbb{M} \ltimes G$. Let S be the new sort with a regular right action of G. Let S' be the quotient S/N. Then S' has a regular right action by H. By Lemma B.4, there is an almost $\varnothing$-definable global type p on S'. Take b realizing $p|\varnothing$. Let S'' be the fiber of $S \to S'$ over $b \in S'$. Then S'' is a b-definable set with a b-definable regular right action by N. By Lemma B.4, there is an almost b-definable global type q on S''. Let c realize $q|b$. Note $c \in S$. By Fact B.2, there is an almost $\varnothing$-definable global type r on S such that c realizes $r|\varnothing$. By Lemma B.5, G has *dfg*. ⊣

Theorem B.6 generalizes one direction of Lemma 2.3. We cannot expect the reverse direction to hold (if G has *dfg*, then N and H have *dfg*). For example, in $p\mathrm{CF}^{\mathrm{eq}}$, the short exact sequence

$$0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0$$

is a counterexample: $\mathbb{Q}_p$ has *dfg* but $\mathbb{Z}_p$ does not. So the use of definable Skolem functions in Lemma 2.3 is essential.

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