Student Problems

Students up to the age of 19 are invited to send solutions to either or both of the following problems to Stan Dolan, 126A Harpenden Road, St Albans, Herts., AL3 6BZ.

Two prizes will be awarded – a first prize of £25, and a second prize of £20 – to the senders of the most impressive solutions for either problem. It is, therefore, not necessary to submit solutions to both. Solutions should arrive by May 20th 2017. Please give your School year, the name and address of your School or College, and the name of a teacher through whom the award will be made. Please print your own name clearly! The names of all successful solvers will be published in the July 2017 edition.

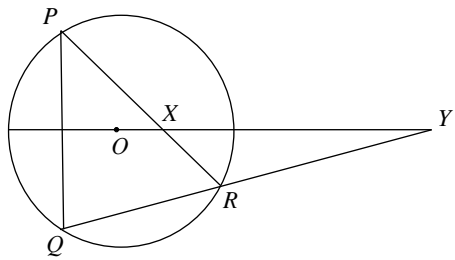
Problem 2017.1 (Anthony C. Robin)

A student is learning to drive and believes that her chance of failing her driving test will decrease with each possible future retest.

She denotes her chance of failing the driving test first time by q and assumes that, if she has to retake her driving test at least n times, then her chance of failing the n th test is $\frac{q}{n}$.

Find the mean and mode of the number of tests she will need to take to pass.

Problem 2017.2 (Geoff Strickland)



Let X be a fixed point on a diameter of a circle centre O and let PQ be a variable chord perpendicular to the diameter. The line PX meets the circle again at R and QR meets the extended diameter at Y.

Prove that the position of Y is independent of the position of PQ.

Solutions to 2016.5 and 2016.6

These two problems attracted some really high quality answers. Problem 2016.5 was solved by Arshad Fasiludeen (Bishop Wordsworth's School), Anton Grishechkin (Charterhouse), Thomas Jackson (Cedars Upper School, Leighton Buzzard), Yuhka Machino (Millfield School) and Richard

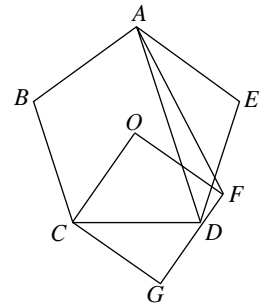
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Zhuowen (Charterhouse). Problem 2016.6 was solved by Robert Boyce (Bishop Wordsworth's School) and Nicholas Go (Dulwich College). Both problems were solved by Len Ma (Dulwich College), Andrew Ng (Dulwich College) and James Thorn (Bishop Wordsworth's School).

Problem 2016.5 (Abdilkadir Altintaş)

Let $ABCDE$ be a regular pentagon with centre O and let $OCGF$ be a square, as shown.

Find $\angle DAF$.



Solution

Anton successfully used an algebraic method based upon the cosine rule whereas all the other solutions were geometric in nature. Andrew, James and Richard gave particularly neat solutions using isosceles triangles. The succinct short solution which follows is that of Yuhka and is similar to the equally good solution from Thomas.

A, B, C, D, E and F are all equidistant from O and are therefore on a circle centre O.

$$\angle DOF = \angle COF - \angle COD = 90° - 72° = 18°$$

$$\angle DAF = \frac{1}{2}\angle DOF = 9° \qquad \left(\text{Angle at centre} = 2 \times \text{Angle at circumference}\right)$$

Problem 2016.6 (Stan Dolan)

Let x, y, z and t be four non-zero real numbers such that

$$x - \frac{1}{y} = y - \frac{1}{z} = z - \frac{1}{t} = t - \frac{1}{x}.$$

Show that either $x = y = z = t$ or $xyzt = 1$.

Solution

Most of the solvers started by clearing the fractions. For example, Andrew's solution proceeded as follows.

$$x - \frac{1}{y} = z - \frac{1}{t}$$

$$\Rightarrow yt(x - z) = t - y \qquad (1)$$

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similarly

$$xz(y - t) = x - z \qquad (2)$$
$$yz(x - y) = z - y \qquad (3)$$

Andrew then considered two cases separately.

Case 1: $x = z$

From (1), $t = y$. Then, from (3), either $x = y$ and therefore $x = y = z = t$ or $yz = 1$ and therefore $xyzt = (yz)^2 = 1$.

Case 2: $x \neq z$

Multiplying (1) and (2), $xyzt = -1$. Andrew dealt with this case by obtaining a quadratic equation for x which he then showed had no solutions.

An alternative is to note that in the cycle $x \to y \to z \to t \to x$ there must be a place where we move from a positive variable to a negative one. Then the common value of

$$x - \frac{1}{y},\ y - \frac{1}{z},\ z - \frac{1}{t} \text{ and } t - \frac{1}{x}$$

must be positive. However, there must be a place in the cycle where we move from a negative variable to a positive one and so the common value must be negative.

James adopted the very different approach of rewriting the equations in terms of a real parameter k

$$x - \frac{1}{y} = \ldots = -k \Rightarrow y = \frac{1}{k+x}, z = \frac{1}{k+y}, t = \frac{1}{k+z}, x = \frac{1}{k+t}.$$

Eliminating the variables y, z and t then gives the continued fraction equation

$$x = [0; k, k, k, k + x].$$

James multiplied this out and obtained the simplified equation

$$k\left(k^2 + 2\right)\left(x^2 + kx - 1\right) = 0.$$

Since k is real, there are just two possibilities:

Case 1: $k = 0$

Then $xy = 1$ and $zt = 1$. Therefore $xyzt = 1$.

Case 2: $x^2 + kx - 1 = 0$

Then $y = \dfrac{1}{k + x} = x$. Similarly, $z = \dfrac{1}{k + y} = \dfrac{1}{k + x} = x$ and $t = \dfrac{1}{k + z} = \dfrac{1}{k + x} = x$. Therefore $x = y = z = t$.

The first prize of £25 is awarded to James Thorn. The second prize of £20 is awarded to Andrew Ng.

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