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The locally translation structure of flows on the wandering points set

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Abstract. A dynamical criterion of conjugacy of a flow with a shift and a counterpart of the Brouwer plane translation theorem for flows on a normal topological space are established.

1. *Introduction*

According to the *Brouwer plane translation theorem* (see [**B, S, Fr**]), a fixed point free orientation preserving homeomorphism f of the plane possesses the following dynamical structure. *There exists an open f-invariant covering $\mathbb{R}^2 = \bigcup_\alpha X_\alpha$ such that the restrictions $f \mid X_\alpha$ are topologically conjugate to a shift on $\mathbb{R}^2$.* In [**S**] a version of this theorem related to the sphere S^2 is proved.

In the present paper we study the local translation structure for flows.

On a topological space X we consider a *uniformly continuous flow* of homeomorphisms $f^t : X \to X$, $t \in \mathbb{R}$. (We call this a *flow* for short). By definition, $f^{t+s} = f^t \circ f^s$, $f^0 = \mathrm{id}$ and the corresponding mapping $F : X \times \mathbb{R} \to X$, $F(x,t) = f^t x$ is continuous.

A flow $f^t : X \to X$ is called *conjugate* to a flow $g^t : Y \to Y$ if there exists a homeomorphism $\Phi : X \to Y$ such that $\Phi f^t = g^t \Phi$ for all $t \in \mathbb{R}$.

The simplest example of a flow is a *shift*. This means that Y is a direct product, $Y = Z \times \mathbb{R}$, Z is a topological space and

$$g^t(z,s) = (z, s+t) \quad (z \in Z;\ s,t \in \mathbb{R}).$$

In the case when Z is a singleton we can identify Y with $\mathbb{R}$ and get the standard *one-dimensional shift* $g^t(s) = s + t$. *Any one-dimensional flow $f^t : \mathbb{R} \to \mathbb{R}$ is conjugate to the one-dimensional shift if and only if f^t is fixed point free: $f^t(s) \neq s$ for all s and all $t \neq 0$.*

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Example 1.1. Consider $X = \mathbb{C} \setminus \mathbb{R}_-$ where $\mathbb{R}_- = \{\xi \in \mathbb{R} : \xi \le 0\}$. It is clear that X is diffeomorphic to the plane $\mathbb{R}^2$. Define $f^t(x) = e^{-t}\xi + ie^t\eta,\ x = \xi + i\eta$. This flow is fixed point free. However, it is not conjugate to any shift $g^t(x) = x + tz$ on $\mathbb{C}$. Indeed, otherwise the functional equation

$$\Phi(e^{-t}\xi + ie^t\eta) = \Phi(\xi + i\eta) + tz$$

would have a continuous solution Φ. By substituting $\xi + i\eta = 1 + ie^{-t}$ we obtain

$$\Phi(e^{-t} + i) = \Phi(1 + ie^{-t}) + tz$$

but the latter is impossible for large $t > 0$. On the other hand, $X = X_1 \bigcup X_2$ where the sets $X_1 = \{\xi + i\eta : \eta \ne 0\}$ and $X_2 = \{\xi + i\eta : \xi > 0\}$ are f^t-invariant. The restriction $f^t \mid X_1$ is conjugate to the shift g^t on $Z \times \mathbb{R}$, where $Z = \{-1, 1\} \times \mathbb{R}$, the conjugating homeomorphism is $\Phi_1(\xi + i\eta) = (\operatorname{sign}\eta, \eta, \ln|\eta|)$. The restriction $f^t \mid X_2$ is conjugate to the shift on $\mathbb{R} \times \mathbb{R}$, the conjugating homeomorphism is $\Phi_2(\xi + i\eta) = (\xi\eta, \ln\xi)$.

THEOREM 1.2. *For any flow $f^t : X \to X$ on a normal topological space X there exists an open invariant covering $\bigcup X_\alpha$ of the wandering points set such that every restriction $f \mid X_\alpha$ is conjugate to a shift on a topological space $Z_\alpha \times \mathbb{R}$.*

Our proof of Theorem 1.2 is based on a criterion of conjugacy to a shift (Theorem 2.2).

We also consider the smooth flows (including real analytic ones) and extend all results to this situation.

2. *The topological case*

Let us recall that a flow $f^t : X \to X$ is said to be *semiconjugate* to a flow $g^t : Y \to Y$ if there is a continuous (not necessarily invertible) mapping $\Phi : X \to Y$ such that $\Phi \circ f^t = g^t \circ \Phi$ for all t.

LEMMA 2.1. *On an arbitrary topological space X a flow $f^t : X \to X$ is conjugate to a shift if and only if it is semiconjugate to the one-dimensional shift.*

Proof. Suppose a flow $f^t : X \to X$ is conjugate to a shift $g^t : Z \times \mathbb{R} \to Z \times \mathbb{R}$. Then there exists a homeomorphism $\Phi : X \to Z \times \mathbb{R}$ such that $\Phi(f^t x) = (\Phi_Z x, \Phi_{\mathbb{R}} x + t)$ where Φ_Z and $\Phi_{\mathbb{R}}$ are canonical projections. Hence, $\Phi_{\mathbb{R}}(f^t x) = \Phi_{\mathbb{R}}(x) + t$, i.e. f^t is semiconjugate to the one-dimensional shift.

Conversely, let a continuous function $h : X \to \mathbb{R}$ be such that

$$h(f^t x) = h(x) + t \quad (t \in \mathbb{R}). \tag{2.1}$$

Consider $Z = \{x \in X : h(x) = 0\}$ and the mapping $\Phi : X \to Z \times \mathbb{R},\ \Phi x = (f^{-s}x, s)$, where $s = h(x)$. Since $h(f^{-s}x) = h(x) - s = 0$, we get $f^{-s}x \in Z$. The mapping Φ is continuous and invertible, $\Phi^{-1}(y, s) = f^s(y)$. Thus, Φ is a homeomorphism.

Let g^t be the shift on $Z \times \mathbb{R}$. Then

$$\Phi(f^t x) = (f^{-(s+t)} f^t x, s + t) = (f^{-s}x, s + t) = g^t(\Phi x),$$

i.e. f^t is conjugate to g^t. □

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Let us recall that an open set $U \subset X$ is said to be *wandering* for a flow f^t if there exists a number $t_0 > 0$ such that

$$f^t(\overline{U}) \bigcap \overline{U} = \emptyset \quad (|t| \geq t_0)$$

($\overline{U}$ means the closure of U). We call a subset $S \subset X$ *wandering* if there is a wandering neighborhood of S.

If $X = \bigcup X_\alpha$ with all wandering X_α then this covering is also called *wandering*.

In the case $Z \times \mathbb{R}$ we have the countable expanding open covering

$$\bigcup \{Z \times (-n, n) : n \in \mathbb{N}\}$$

which is wandering with respect to the corresponding shift. Therefore *if a flow $f^t : X \to X$ is conjugate to a shift then there exists a countable extending open wandering (c.e.o.w.) covering $X = \bigcup \{X_n : n \in \mathbb{N}\}$.*

THEOREM 2.2. *Let X be a normal topological space. Then a flow $f^t : X \to X$ is conjugate to a shift if and only if there exists a c.e.o.w. covering of X.*

Proof. By the previous remark and Lemma 2.1 it is sufficient to prove that under the existence of a c.e.o.w. covering f^t is semiconjugate to the one-dimensional shift.

Let us introduce the sets

$$A = \bigcup_{k \geq 1} \bigcup_{t \geq t_k} f^t(\overline{X}_k), \quad N = \bigcup_{m \geq 1} \bigcup_{t \geq t_m} f^{-t}(\overline{X}_m),$$

where $0 < t_1 < t_2 < \cdots$ and

$$t \geq t_k \Rightarrow f^t(\overline{X}_k) \bigcap \overline{X}_k = \emptyset. \tag{2.2}$$

Then, first of all, $A \bigcap N = \emptyset$. Indeed, if, say, $k \geq m$ and $t \geq t_k$ then

$$f^t(\overline{X}_k) \bigcap f^{-s}(\overline{X}_m) = f^{-s}\left(f^{t+s}(\overline{X}_k) \bigcap \overline{X}_m\right) \subset f^{-s}\left(f^{t+s}(\overline{X}_k) \bigcap \overline{X}_k\right) = \emptyset$$

for all s.

In a natural sense A and N are an *absorber* and a *nozzle*, respectively (see [**BL**]). It is important that A and N are closed. In order to prove this we consider a closed set $M \subset X$. Note that for any closed segment $I \subset \mathbb{R}$ the corresponding arc of orbit $O_I(M) = \bigcup \{f^t(M) : t \in I\}$ is closed. Indeed, let $(f^{t_i} x_i)$ be a net with $x_i \in M$, $t_i \in I$ and let $\lim f^{t_i} x_i = y$. One can assume that $\lim_i t_i = t \in I$. Then $\lim_i x_i = f^{-t} y$. Since, M is closed, we have $f^{-t} y \in M$, hence $y \in f^t(M)$.

Now let us suppose that $x \notin A$. It is sufficient to find a neighborhood V of x such that $V \bigcap A = \emptyset$. Take a set X_n from the given covering such that $x \in X_n$. Then by (2.2)

$$f^t(\overline{X}_k) \bigcap \overline{X}_n = \emptyset \quad (k \geq n,\ t \geq t_k). \tag{2.3}$$

The subset

$$A_1 = \bigcup_{1 \leq k \leq n} \bigcup_{t_k \leq t \leq t_n} f^t(\overline{X}_k) \subset A$$

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is closed because of the above remark. Therefore, there exists a neighborhood U of x such that $U \bigcap A_1 = \emptyset$. On the other hand, $A \setminus A_1 \subset A_2 \bigcup A_3$ where

$$A_2 = \bigcup_{1 \le k \le n} \bigcup_{t \ge t_n} f^t(\overline{X}_k), \quad A_3 = \bigcup_{k>n} \bigcup_{t \ge t_k} f^t(\overline{X}_k).$$

By (2.3) we have $A_3 \bigcap \overline{X}_n = \emptyset$. Moreover, actually $A_2 = \bigcup\{f^t(\overline{X}_n) : t \ge t_n\}$ so that $A_2 \bigcap X_n = \emptyset$ as well. The neighborhood $V = U \bigcap X_n$ of x is just what we need. Thus, A is closed. Similarly, the set N is closed.

Since the space X is normal, there exists a continuous function γ_+ such that $\gamma_+ \mid A = 0$ and $\gamma_+ \mid N = 1$. Respectively, $\gamma_- \mid A = 1$ and $\gamma_- \mid N = 0$ for $\gamma_- = 1 - \gamma_+$.

The function $t \mapsto \gamma_+(f^t x)$ vanishes for $t \ge t_n$ if n is such that $x \in X_n$. This allows us to introduce the continuous function

$$h_+(x) = -\int_0^\infty \gamma_+(f^s x)\, ds.$$

Obviously,

$$h_+(f^t x) = h_+(x) + \int_0^t \gamma_+(f^s x)\, ds.$$

Similarly, for

$$h_-(x) = \int_{-\infty}^0 \gamma_-(f^s x)\, ds$$

we get

$$h_-(f^t x) = h_-(x) + \int_0^t \gamma_-(f^s x)\, ds.$$

Therefore, the function $h = h_+ + h_-$ satisfies equality (2.1), hence f^t is semiconjugate to the one-dimensional shift. □

Combining Lemma 2.1 and Theorem 2.2 we obtain the following.

COROLLARY 2.3. *Let X be a normal topological space with a flow $f^t : X \to X$. Then the following statements are equivalent:*

(a) *f^t is semiconjugate to the one-dimensional shift;*
(b) *f^t is conjugate to a shift;*
(c) *there exists a c.e.o.w. covering of X.*

In fact, (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c) is true for an arbitrary topological space X.

Note that in the situation described by Theorem 2.2 all *compact subsets are wandering*. Indeed, if a set $K \subset X$ is compact and $X = \bigcup X_n$ is a c.e.o.w. covering then $K \subset X_n$ for some n, so K is wandering along with X_n. Conversely, let a space X be locally compact and countable at infinity (l.c.c.i.). The latter means the existence of a countable covering $X = \bigcup K_j$, $K_j \subset \operatorname{int} K_{j+1}$ with compact K_j. For such a space X the condition (c) of Corollary 2.3 is equivalent to:

(c′) *all compact subsets are wandering.*

This property is violated in Example 1.1. Indeed, let

$$\Delta = \{x = \xi + i\eta : \xi + i\eta = 1,\ \xi \ge 0,\ \eta \ge 0\}.$$

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For any $t \neq 0$ the point

$$x_t = \frac{(e^t - 1) + i(1 - e^{-t})}{e^t - e^{-t}}$$

belongs to Δ and its f^t-image is

$$y_t = \frac{(1 - e^{-t}) + i(e^t - 1)}{e^t - e^{-t}} \in \Delta.$$

To conclude this section let us note that it may happen that two flows on the same space with property (c) are not conjugate. To construct an example, note that a shift $g_1^t : Y_1 \times \mathbb{R} \to Y_1 \times \mathbb{R}$ is conjugate to a shift $g_2^t : Y_2 \times \mathbb{R} \to Y_2 \times \mathbb{R}$ if and only if the space Y_1 is homeomorphic to Y_2. Therefore, it is sufficient to find a pair of non-homeomorphic spaces Y_1 and Y_2 such that $Y_1 \times \mathbb{R}$ is homeomorphic to $Y_2 \times \mathbb{R}$.

Example 2.4. Let $Y_1 = T(S^2)$ be the tangent bundle of the sphere and $Y_2 = S^2 \times \mathbb{R}^2$. Since S^2 is not parallelized (see [**MS**, §2]), Y_1 and Y_2 are not homeomorphic. However, the bundle $Y_1 \times \mathbb{R}$ is trivial, i.e. homeomorphic to $Y_2 \times \mathbb{R} = S^2 \times \mathbb{R}^3$.

3. *The smooth case*

In this section we study a C^k-flow $f^t : X \to X$ $(1 \leq k \leq \omega)$ on a manifold X of class C^ω, i.e. X is a real analytic manifold and the mapping $F(x, t) = f^t x$ is of the class C^k. In the case when $X = Z \times \mathbb{R}$ with real analytic Z, the corresponding shift is, obviously, of the class C^ω.

Two such *smooth* flows $f^t : X \to X$ and $g^t : Y \to Y$ are called *C^k-conjugate* if they are conjugate by a C^k-diffeomorphism $\Phi : X \to Y$.

Similarly, the *C^k-semiconjugacy* can be defined.

THEOREM 3.1. *Let f^t be a C^k-flow on a real analytic manifold X. Then the following statements are equivalent:*

(a) *f^t is C^0-semiconjugate to the one-dimensional shift;*
(b) *f^t is C^k-semiconjugate to the one-dimensional shift;*
(c) *f^t is C^k-conjugate to a shift;*
(d) *there exists a c.e.o.w. covering of X.*

Proof. Let us start with the non-analytic case, $k < \omega$.

(a) $\Rightarrow$ (b). It is sufficient to note that the function γ_+ in the proof of Theorem 2.2 can be chosen in the class C^k (see [**W**, Appendix III, §1]). This yields a C^k-solution of equation (2.1).

(b) $\Rightarrow$ (c). For the C^k-function $h : X \to \mathbb{R}$ which transforms f^t to the one-dimensional shift the zero set Z is a C^k-submanifold of X. Indeed, it follows from (2.1) that $dh(f^t x)/dt = 1$, hence $\operatorname{grad} h(s) \neq 0$ everywhere. The homeomorphism $\Phi : X \to Z \times \mathbb{R}$ constructed in the proof of Lemma 2.1 turns out to be a C^k-diffeomorphism.

(c) $\Rightarrow$ (d) is trivial and (d) $\Rightarrow$ (a) is contained in Corollary 2.3.

Now we pass to the case $k = \omega$. In this case the previous proof of (a) $\Rightarrow$ (b) fails because of absence of the Uryson-type functions. However, we can prove it using (a) $\Rightarrow$ (c) in the non-analytic case. Namely, let $\Phi : X \to Z \times \mathbb{R}$ be a C^∞-diffeomorphism which

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transforms f^t to the shift $g^t(z,s) = (z, s+t)$. By a well known Whitney theorem (see [**W**, Ch. IY, §2]) one can assume Z to be an analytic manifold.

The analytic diffeomorphisms $f = f^t|_{t=1},\ g = g^t|_{t=1}$ are smoothly conjugate.

Consider the orbit space $X \setminus f$. Since all compact subsets $K \subset X$ are wandering, for any non-equivalent points $x_1, x_2 \in X$ there are some neighborhoods $W_1 \ni x_1, W_2 \ni x_2$ such that $f^n(W_1) \bigcap f^m(W_2) = \emptyset$ $(n, m \in \mathbb{Z})$. Hence, $X \setminus f$ is a Hausdorff space.

Furthermore, let $p : X \to X \setminus f$ be the canonical projection. Since any point $x \in X$ is wandering, there exists a neighborhood W_x such that $f^n(W_x) \bigcap W_x = \emptyset$ $(n \in \mathbb{Z}, n \neq 0)$. Consider $\hat{W}_x = p(W_x)$. We have got an open covering $X \setminus f = \bigcup\{\hat{W}_x \mid x \in X\}$.

Let $(U_\alpha, \Theta_\alpha)$ be a local map on the manifold X such that $x \in U_\alpha,\ \Theta_\alpha : U_\alpha \to \mathbb{R}^d$ is a homeomorphism onto its image, $d = \dim X$. One can assume that $W_x \subset U_\alpha$ and then consider $\Phi_x = \Theta_\alpha \mid W_x$. Let $\hat{\Phi}_x(\{f^n z\}) = \Phi_x(z),\ z \in W_x$. The mapping $\hat{\Phi}_x : \hat{W}_x \to \Phi_x(W_x)$ is a well-defined homeomorphism, i.e. the pair $(\hat{W}_x, \hat{\Phi}_x)$ is a local map at the point $p(x) \in X \setminus f$. In such a way $X \setminus f$ turns out to be a C^ω-manifold which actually is diffeomorphic to the orbit space $Z \times \mathbb{R} \setminus g = Z \times \mathbb{T}$. Hence, the analytic manifolds $X \setminus f$ and $Z \times \mathbb{T}$ are analytically diffeomorphic (see [**F**, Ch. 3]), i.e. there exists a C^ω-diffeomorphism $X \setminus f \to Z \times \mathbb{T}$. Let $\Psi : X \to Z \times \mathbb{R}$ be its analytic lifting. Then $\Psi(fx) = g^n(\Psi(x))$ $(x \in X)$ where $n = n(x) \neq 0$ is constant on every connected component of X. In particular, the function $h(x) = \Psi_{\mathbb{R}} x$ is an analytic solution of the equation $h(fx) - h(x) = n(x)$. Setting

$$\tilde{h}(x) = \frac{1}{n(x)} \int_0^1 h(f^s x)\, ds,$$

we get an analytic solution of (3.1).

Thus, (a) $\Rightarrow$ (b) is proved in the analytic case. Other implications can be proved as in the non-analytic case. □

COROLLARY 3.2. *Under the assumptions of Theorem 3.1 there exists an open invariant covering $\bigcup X_\alpha$ of the wandering points set such that every restriction $f^t \mid X_\alpha$ is C^k-conjugate to a shift.*

It may happen that two analytic shifts are topologically but not smoothly conjugate.

Example 3.3. Let Z_1 be a Donaldson manifold [**D**] which is homeomorphic but not diffeomorphic to $Z_2 = \mathbb{R}^4$. Since the smooth structure on $\mathbb{R}^5$ is unique, the spaces $Y_1 = Z_1 \times \mathbb{R}$ and $Y_2 = Z_2 \times \mathbb{R}$ are diffeomorphic. Obviously, shifts g_1^t on Y_1 and g_2^t on Y_2 are topologically conjugate. But they are not smoothly conjugate, otherwise Y_1 and Y_2 would be diffeomorphic.

Now we consider some flows on a Lie group X. Let $\mathcal{A}$ be the corresponding Lie algebra. Then one can define the flow

$$f^t x = \exp(ta) \cdot x, \quad a \in \mathcal{A}. \tag{3.1}$$

Here $\exp(ta)$ means the one-parametric subgroup whose tangent vector at the unit point ε is a (see [**H**, Ch. II, §3]).

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THEOREM 3.4. *Flow (3.1) is semiconjugate to the one-dimensional shift if and only if the one-parametric subgroup* $f^t\varepsilon = \exp(ta)$ *has no limit points as* $t \to +\infty$.

Proof. Let us apply Corollary 2.3 with (c′) instead of (c). Assume that a compact subset $K \subset G$ is non-wandering. Then there exists a subsequence $t_n \to +\infty$ such that $f^{t_n}(K) \bigcap K \neq \emptyset$. Hence,

$$f^{t_n}\varepsilon \in KK^{-1}.$$

Since KK^{-1} is also a compact set, $\{f^{t_n}\varepsilon\}$ has a limit point.

Conversely, let all compact subsets be wandering. Suppose that $\exp(t_n a) \to g$ as $t_n \to +\infty$. Let U be a compact neighborhood of ε. Then $\exp(t_n a) \in Ug$, if t_n is large enough. Hence, $g^{-1} \in \exp(-t_n a)U = f^{-t_n}(U)$, so U is non-wandering. □

As an application we consider flow (3.1) on $X = GL(p)$, the group of all invertible $(p \times p)$ matrices over $\mathbb{C}$ or $\mathbb{R}$. Let us say that a matrix h is *quasi-unitary* if $h = v^{-1}uv$ where u is a unitary matrix, $v \in GL(p)$. (In other words, h is a diagonalizable matrix with unitary spectrum.)

COROLLARY 3.5. *Flow (3.1) on* $X = GL(p)$ *is semiconjugate to the one-dimensional shift if and only if the matrix* $h = \exp(a)$ *is not quasi-unitary.*

Proof. Now $\mathcal{A}$ is the Lie algebra of all $p \times p$ matrices and $\exp(ta)$ is just the usual matrix exponent e^{ta}. By Theorem 3.4 we have to prove that this exponent has no limit points as $t \to +\infty$ if and only if h is not quasi-unitary.

Let h be quasi-unitary. Then the subgroup $f^t\varepsilon = e^{ta}$ is bounded, hence there exists a limit point $g \in \mathcal{A}$, $g = \lim_n e^{t_n a}$, $t_n \to +\infty$. The matrix g is invertible since if $\det g = 0$ then $\det(e^{t_n a}) \to 0$; hence $\det(e^{-t_n a}) \to \infty$, contrary to the boundness of the exponent.

Conversely, let h not be quasi-unitary. Then $\max(\|e^{ta}\|, \|e^{-ta}\|) \to \infty$ as $t \to +\infty$. But if $\lim_n e^{t_n a} = g$, $t_n \to +\infty$, then $\lim_n e^{-t_n a} = g^{-1}$, so $\max(\|e^{t_n a}\|, \|e^{-t_n a}\|)$ is bounded. □

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