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Adjunction of roots, algebraic K-theory and chromatic redshift

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Abstract

Given an E_1-ring A and a class $a \in \pi_{mk}(A)$ satisfying a suitable hypothesis, we define a map of E_1-rings $A \to A(\sqrt[m]{a})$ realizing the adjunction of an mth root of a. We define a form of logarithmic topological Hochschild homology (THH) for E_1-rings, and show that root adjunction is log-THH-étale for tamely ramified root adjunctions, which provides a formula for $\mathrm{THH}(A(\sqrt[m]{a}))$ in terms of THH and log THH of A. If A is connective, we prove that the induced map $K(A) \to K(A(\sqrt[m]{a}))$ in algebraic K-theory is the inclusion of a wedge summand. Using this, we compute $V(1)_*K(ko_p)$ for $p > 3$ and also deduce that if $K(A)$ exhibits chromatic redshift, so does $K(A(\sqrt[m]{a}))$. We exhibit several maps of ring spectra as root adjunctions, and use this to obtain a new proof of the fact that Lubin–Tate spectra satisfy the redshift conjecture.

1. Introduction

Let A be an E_1-algebra spectrum, and let $a \in \pi_{mk}(A)$ with $m > 0$ and even $k \geq 0$. In this paper we define, under Hypothesis 4.1, an E_1-algebra extension $A \to A(\sqrt[m]{a})$ realizing the adjunction of an mth root of a in homotopy rings,

$$\pi_* A \to \pi_*(A(\sqrt[m]{a})) \cong \pi_*(A)[z]/(z^m - a),$$

and then study how the algebraic K-theory of $A(\sqrt[m]{a})$, or its topological Hochschild and cyclic homology, relates to that of A. Hypothesis 4.1 holds, for example, if A is an E_2-ring for which $\pi_* A$ is concentrated in even degrees.

In general, the existence of a suitable root adjunction to a ring spectrum and its effect on such invariants is an intriguing question. As shown in [SVW99, Proposition 2], there is no Gaussian sphere $\mathbb{S}[i]$; see [Law20] for a discussion of root adjunctions in degree 0.

For classes in positive degrees, examples exist and have been shown to be relevant, in particular in studying redshift for algebraic K-theory. We have the Adams splitting of connective complex K-theory ku completed at an odd prime p,

$$ku_p \simeq \bigvee_{1 \leq i < p-1} \Sigma^{2i} \ell_p,$$

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and the extension $\ell_p \to ku_p$ can be interpreted, on homotopy rings, as the root adjunction

$$\pi_*\ell_p \cong \mathbb{Z}_p[v_1] \to \mathbb{Z}_p[u] \cong \pi_* ku_p,$$

were $v_1 \mapsto u^{p-1}$, giving

$$\pi_* ku_p \cong (\pi_*\ell_p)[u]/(u^{p-1} - v_1).$$

Sagave [Sag14, 4.15] showed that ku_p can be constructed as an extension of ℓ_p, establishing how $\ell_p \to ku_p$ qualifies as a tamely ramified extension in E_∞-rings. In [AR02, Aus10], Rognes and the first author computed the algebraic K-theory of ℓ_p and ku_p with coefficients in a Smith–Toda complex $V(1) = \mathbb{S}/(p, v_1)$, for $p \geq 5$. Taking $T(2) = V(1)[v_2^{-1}]$, we have the formula

$$T(2)_* K(ku_p) \cong (T(2)_* K(\ell_p))[b]/(b^{p-1} + v_2)$$

relating the two computations, hinting at a chromatic shift (or redshift) of this tamely ramified root adjunction.

After our construction of root adjunctions in E_1-rings, we offer an investigation of how the resulting extension is reflected in algebraic K-theory. In particular, we have the following spectrum-level splitting of algebraic K-theory in the tamely ramified case, which applies to a wide array of examples.

Theorem 1.1 (Theorem 5.5). *Assume Hypothesis 4.1 with $p \nmid m$ and $|a| > 0$. Furthermore, assume that A is p-local and connective. In this situation, the map in algebraic K-theory*

$$K(A) \to K(A(\sqrt[m]{a}))$$

induced by the extension $A \to A(\sqrt[m]{a})$ is the inclusion of a wedge summand.

This is deduced from the corresponding result for topological cyclic homology, see Theorem 5.4. An analogous splitting result for algebraic K-theory in the non-connective case is provided in Corollary 5.6.

For an integer $n > 0$, we say that a spectrum E is *height* n if $L_{T(n)}E \not\simeq 0$ and $L_{T(m)}E \simeq 0$ for $m > n$, where $T(n)$ denotes a height-n telescope. We say an E_1-ring A of height n *exhibits redshift* if $K(A)$ is of height $n+1$. Owing to [LMM+24, Purity Theorem], $K(A)$ is of height at most $n+1$, so A will exhibit redshift if $L_{T(n+1)}K(A) \not\simeq 0$. The following result is thus an immediate consequence of our splitting results, Theorem 5.5 and Corollary 5.6.

Corollary 1.2. *Assume Hypothesis 4.1 with $p \nmid m$ and $|a| > 0$. If A exhibits redshift, then so does $A(\sqrt[m]{a})$.*

A key feature of the defined root adjunction is that $A(\sqrt[m]{a})$ is endowed with the structure of an E_1-algebra in the symmetric monoidal category $\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})$ of m-graded spectra, which is reflected in an Adams-type splitting of spectra

$$A(\sqrt[m]{a}) \simeq \bigvee_{0 \leq i < m} \Sigma^{ik} A.$$

Such a grading on a spectrum, compatible with further additional algebraic structures, has already proven to be very useful: let us mention the computations of Hesselholt and Madsen [HM97] of the K-theory of truncated polynomial algebras, and of the second and third authors [BM22] of the K-theory of the free E_1-algebras in degree 2 over finite fields. In the present paper, the grading corresponding to the root adjunction, and the induced splitting of the topological Hochschild homology (THH) as developed in [AMM+22, Appendix A], are essential ingredients in the proofs of our various splitting results.

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We use the theory of *logarithmic THH* for an in-depth study of the THH of $A(\sqrt[m]{a})$. Hesselholt and Madsen [HM03] introduced log THH for studying the algebraic K-theory of complete discrete valuation rings in mixed characteristic, and proved a descent property of log THH in the case of tamely ramified extensions. Rognes [Rog09] then initiated a study of logarithmic structures, logarithmic André–Quillen homology, and log THH in the context of E_∞-ring spectra. With Sagave and Schlichtkrull [RSS15, RSS18], they then established the existence of localization sequences for log THH and proved that it satisfies tamely ramified descent in the example of the extension $\ell_p \to ku_p$.

Here, we offer an alternative definition of log THH that applies to a more general class of ring spectra. More precisely, we define log THH for a given E_1-ring spectrum A and a class $a \in \pi_{mk}(A)$ satisfying Hypothesis 4.1, associated to the pre-log structure given by the monoid generated, under multiplication, by $a \in \pi_*(A)$ (see Definition 6.5); it is denoted $\mathrm{THH}(A \mid a)$. For instance, this applies to the Morava K-theory spectrum $k(n)$ for $v_n \in \pi_* k(n)$. We prove the following form of tamely ramified descent (see also Theorem 6.14).

Theorem 1.3 (Theorem 6.21). *If A is p-local and $p \nmid m$, then there is an equivalence of spectra*

$$\mathrm{THH}(A(\sqrt[m]{a})) \simeq \mathrm{THH}(A) \vee \left(\bigvee_{0<i<m} \Sigma^{ik} \mathrm{THH}(A \mid a) \right).$$

We also prove the existence of a localization cofiber sequence

$$\mathrm{THH}(A) \to \mathrm{THH}(A \mid a) \to \Sigma\, \mathrm{THH}(A/a),$$

see Theorem 6.19. This is an analogue in the present setting of the localization sequences constructed in [RSS15], but note that our definition only applies to the case of a pre-log structure given by a monoid on a single generator. We would like to point out also the recent preprint of Binda et al. [BLP+23], where a version of logarithmic Hochschild homology for simplicial commutative rings is constructed.

We now mention examples of application of the above results.

1.1 Topological K-theory

We prove in Theorems 4.6 and 7.5 that, at an odd prime p, there are equivalences of E_1-ring spectra

$$ku_p \simeq \ell_p(\sqrt[p-1]{v_1}) \quad \text{and} \quad ko_p \simeq \ell_p(\sqrt[(p-1)/2]{v_1}).$$

These equivalences upgrade the splitting result of Adams into a $\mathbb{Z}/(p-1)$-graded, respectively $\mathbb{Z}/(p-1/2)$-graded, E_1-ring structure. We prove that when $p=1$ in $\mathbb{Z}/m$, the splitting in Theorem 1.1 can be improved to a more refined splitting of $K(A(\sqrt[m]{a}))$. In the case of $ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$, this more refined splitting reads as

$$K(ku_p) \simeq \bigvee_{0\le i<p-1} K(ku_p)_i. \tag{1.1}$$

Here, $K(ku_p)_0 \simeq K(\ell_p)$ and, for $p>3$, we can compute the $V(1)$-homotopy groups of each of the ith-graded pieces $V(1)_* K(ku_p)_i$ using the first author's computation of $V(1)_* K(ku_p)$.

Using this refined splitting, the second author makes a simplified computation of $T(2)_* K(ku)$ in [Bay24]. We also obtain complete descriptions of $V(1)_* K(ko_p)$ and $T(2)_* K(ko_p)$; see Theorem 7.7.

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We note that the splitting (1.1) can be considered as a version of Adams-type splitting for the cohomology theory represented by $K(ku)$, with classes corresponding to 2-categorical complex vector bundles, as developed in [BDR+11].

1.2 Johnson–Wilson spectra and Morava *E*-theory

Let $n \geq 1$ be an integer, and let $E(n)$ and E_n be the Johnson–Wilson and Morava E-theory spectra. Let $\widehat{E(n)}$ be the $K(n)$-localization of $E(n)$. These spectra have coefficient rings given as

$$\pi_* E(n) \cong \mathbb{Z}_p[v_1, \dots, v_{n-1}, v_n^{\pm 1}], \quad \pi_* E_n \cong W(\mathbb{F}_{p^n})[|u_1, \dots, u_{n-1}|][u^{\pm 1}]$$

and $\pi_* \widehat{E(n)} \cong \pi_* E(n)^\wedge_{I_n}$, where $|u_i| = 0$, $|u| = -2$ and $I_n = (p, v_1, \dots, v_n)$. The Galois group $Gal = \mathrm{Gal}(\mathbb{F}_{p^n} | \mathbb{F}_p)$ acts on E_n, and let E_n^{hGal} be the homotopy fixed-point spectrum with coefficients $\pi_* E_n^{\mathrm{hGal}} \cong \mathbb{Z}_p[|u_1, \dots, u_{n-1}|][u^{\pm 1}]$.

We prove in Theorem 9.3 that there are equivalences of E_1-rings

$$\begin{aligned} E_n &\simeq \mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} \widehat{E(n)}(\sqrt[p^n-1]{v_n}) \\ E_n^{\mathrm{hGal}} &\simeq \widehat{E(n)}(\sqrt[p^n-1]{v_n}). \end{aligned} \tag{1.2}$$

This promotes the E_1-ring structure on Morava E-theories to a non-trivial $\mathbb{Z}/(p^n - 1)$-graded E_1-ring structure.

Using this description of E_n and the log-THH-étaleness of root adjunction, we show in Theorem 9.9 that the canonical map

$$\mathrm{THH}(\widehat{E(n)}) \wedge_{\widehat{E(n)}} E_n \xrightarrow{\simeq_p} \mathrm{THH}(E_n)$$

is an equivalence after p-completion. The relationship between such equivalences and the Galois descent question for THH is studied in [Mat17].

Applying Corollary 5.6 to these root adjunctions of non-connective spectra, we deduce the following result.

Theorem 1.4 (Theorem 9.4). *The canonical maps*

$$\begin{aligned} K(E(n)) &\to K(E_n^{\mathrm{hGal}}), \\ K(\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge E(n)) &\to K(E_n) \end{aligned}$$

are inclusions of wedge summands after $T(n+1)$-localization.

1.3 Lubin–Tate spectra

We can also apply our results to Lubin–Tate spectra that can be constructed, in several steps, from the truncated Brown–Peterson spectra $BP\langle n \rangle$, with coefficients $\pi_* BP\langle n \rangle \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$. In more precise terms, we consider an E_3 $MU[\sigma_{2(p^n-1)}]$-algebra form of $BP\langle n \rangle$ as constructed by Hahn and Wilson [HW22, Remark 2.1.2]. In this situation, we can construct $BP\langle n \rangle(\sqrt[p^n-1]{v_n})$ as an E_3 $MU[\sigma_2]$-algebra (Remark 4.7).

Let k be a perfect field of characteristic p, and let $\mathbb{S}_{W(k)}$ denote the spherical Witt vectors spectrum. We prove in Proposition 8.3 that the $MU[\sigma_2]$-orientation above provides a formal group law Γ of height n over k, and that there is an equivalence of E_3-rings

$$L_{K(n)}(\mathbb{S}_{W(k)} \wedge BP\langle n \rangle)(\sqrt[p^n-1]{v_n}) \simeq E_{(k,\Gamma)},$$

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where $E_{(k,\Gamma)}$ denotes the Lubin–Tate spectrum corresponding to Γ. Owing to [LMM+24, Purity Theorem], we have an equivalence

$$L_{T(n+1)}K(\mathbb{S}_{W(k)} \wedge BP\langle n\rangle(\sqrt[p-1]{v_n})) \simeq L_{T(n+1)}K(E_{(k,\Gamma)}).$$

By [HW22], $BP\langle n\rangle$ satisfies the redshift conjecture; following an argument suggested to us by Hahn, we show that $\mathbb{S}_{W(k)} \wedge BP\langle n\rangle$ also satisfies the redshift conjecture, cf. Proposition 8.1. By Corollary 1.2, we deduce that $E_{(k,\Gamma)}$ satisfies the redshift conjecture. Indeed, we deduce from this (see Theorem 8.6) a new proof of Yuan's result [Yua21] that all Lubin–Tate spectra satisfy the redshift conjecture. We also obtain the following from Corollary 5.6.

Theorem 1.5 (Theorem 8.5). *The induced map*

$$L_{T(n+1)}K(\mathbb{S}_{W(k)} \wedge BP\langle n\rangle) \to L_{T(n+1)}K(E_{(k,\Gamma)})$$

is the inclusion of a non-trivial wedge summand.

We expect the above result to be relevant also for explicit computations. For example, in [AAC+25], the authors compute $V(2)_*\,\mathrm{TC}(BP\langle 2\rangle)$ for $p \geq 7$, which provides an explicit description of $T(3)_*K(BP\langle 2\rangle)$. Through the inclusion above, we deduce that $T(3)_*K(BP\langle 2\rangle)$ maps isomorphically to a summand of $T(3)_*K(E_{(\mathbb{F}_p,\Gamma)})$ for a height-2 formal group law Γ over $\mathbb{F}_p$. To our knowledge, this is the first explicit, quantitative result on the algebraic K-theory groups of Lubin–Tate spectra for height larger than 1.

Note that it is not known if the E_3 MU-algebra forms of $BP\langle n\rangle$ constructed in [HW22] map into the Morava E-theories mentioned in the preceding subsection.

Remark 1.6. In [BSY22, Theorem G], the authors construct an E_∞-map $MU[\sigma_2] \to E_{(\overline{\mathbb{F}}_p,\Gamma')}$ for the unique height-n formal group law Γ' on $\overline{\mathbb{F}}_p$, solving an open question on the existence of orientations on Lubin–Tate spectra. Our constructions provide a similar orientation for a 'smaller' form of Lubin–Tate spectra. Namely, we obtain that $E_{(\mathbb{F}_p,\Gamma)}$ is an E_3 $MU[\sigma_2]$-algebra where σ_2 acts through $u^{-1} \in \pi_* E_{(\mathbb{F}_p,\Gamma)}$; see Example 8.4. Moreover, there is a grading on $E_{(\mathbb{F}_p,\Gamma)}$ that respects this structure. To be precise, $E_{(\mathbb{F}_p,\Gamma)}$ is an E_3 $MU[\sigma_2]$-algebra in the ∞-category of $\mathbb{Z}/(p^n-1)$-graded spectra, where $u^{-1} \in \pi_* E_{(\mathbb{F}_p,\Gamma)}$ is of weight 1.

Remark 1.7. The above construction of Lubin–Tate spectra by root adjunction is used in an essential way in the construction of a counter-example to the telescope conjecture in forthcoming work by Burklund, Hahn, Levy and Schlank.

1.4 Morava K-theory

In § 9.3, we construct two-periodic Morava K-theories from Morava K-theories through root adjunction. By Corollary 1.2, we deduce that two-periodic Morava K-theories satisfy the redshift conjecture if the redshift conjecture holds for Morava K-theories (Corollary 9.6).

For $p > 3$, the $V(1)$-homotopy of $K(k(1))$ is computed by the first author and Rognes in [AR12], where it is also shown that $k(1)$ satisfies the redshift conjecture. From this, we deduce that the two-periodic first Morava K-theory ku/p also satisfies the redshift conjecture (Corollary 9.7). Moreover, through the interpretation of ku/p as $k(1)(\sqrt[p-1]{v_1})$, the second author makes the first computation of $T(2)_*K(ku/p)$ in [Bay24].

1.5 Outline

We begin with a quick introduction of graded objects in § 2. In § 3, we construct a family of graded E_2 'polynomial' algebras and establish their even cell decompositions. In § 4, we provide

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our central construction for root adjunctions (Construction 4.3) and prove our first splitting result on the THH of ring spectra obtained via a root adjunction. In § 5, we prove Theorem 1.4. Section 6 is devoted to studying the variant of log THH we set forth, as well as the log-THH-étaleness of root adjunctions. Section 7 contains our results on the algebraic K-theory of real and complex topological K-theories. We apply our results to Lubin–Tate spectra in § 8. In § 9, we study the THH and the algebraic K-theory of Morava E-theories.

Notation 1.8.

(1) We work freely in the setting of ∞-categories and higher algebra from [Lur09, Lur17].
(2) For an E_2-algebra R in a symmetric monoidal ∞-category, when we say T is an R-algebra (or an E_1 R-algebra), we mean that it is an E_1-algebra in *right* R-modules. If we mean an E_1-algebra in left R-modules, we call this a *left* E_1 R-algebra. If R is E_∞, we do not need to denote the distinction.
(3) When we say E_n-ring, we mean an E_n-algebra in the ∞-category of spectra Sp.

2. Recollections on graded objects

Let $m \geq 0$ be an integer and let $\mathbb{Z}/m$ denote the discrete ∞-groupoid whose objects are the elements of the set of integers modulo m. For a presentably symmetric monoidal ∞-category $\mathcal{V}$, we define the *∞-category of m-graded objects in $\mathcal{V}$* to be the functor category $\mathrm{Fun}(\mathbb{Z}/m, \mathcal{V})$. For a functor F in $\mathrm{Fun}(\mathbb{Z}/m, \mathcal{V})$, we denote $F(i)$ by F_i for every $i \in \mathbb{Z}/m$. Since $\mathbb{Z}/m$ is discrete, we have

$$\mathrm{Fun}(\mathbb{Z}/m, \mathcal{V}) \simeq \prod_{i \in \mathbb{Z}/m} \mathcal{V}.$$

For $m = 0$, this is given by $\mathrm{Fun}(\mathbb{Z}, \mathcal{V})$, where $\mathbb{Z}$ is the corresponding discrete ∞-groupoid. In this case, we omit m and call $\mathrm{Fun}(\mathbb{Z}, \mathcal{V})$ the *∞-category of graded objects in $\mathcal{V}$*. We are mainly interested in the case $\mathcal{V} = \mathrm{Sp}$. We call an object of $\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})$ an *m-graded spectrum*; for $m = 0$, we drop m and call it a *graded spectrum*.

Using the symmetric monoidal structure on $\mathbb{Z}/m$ given by addition, we equip $\mathrm{Fun}(\mathbb{Z}/m, \mathcal{V})$ with the Day convolution closed symmetric monoidal structure [Gla16]. Since $\mathbb{Z}/m$ is discrete, this boils down to the following:

$$(F \otimes_{\mathrm{Day}} G)_k = \coprod_{i+j=k \text{ in } \mathbb{Z}/m} F_i \otimes G_j.$$

2.1 Algebras in graded spectra

We are interested in E_n-algebras in the ∞-category of m-graded spectra and the algebras over these E_n-algebras.

Definition 2.1. An *m-graded E_n-ring* A is an E_n-algebra in $\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})$. For $k < n$, an m-graded E_k A-algebra is an E_k A-algebra in $\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})$. Similarly, an m-graded (left) right A-module is a (left) right A-module in $\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})$.

Remark 2.2. Note that the notion of an m-graded E_k-algebra in Sp is, in general, different from the notion of an m-graded object in the ∞-category of E_k-algebras in Sp.

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2.2 Manipulations on graded objects

For a symmetric monoidal functor $\mathbb{Z}/m \to \mathbb{Z}/m'$ (i.e. a group homomorphism), there is an induced adjunction between the ∞-categories $\mathrm{Fun}(\mathbb{Z}/m, \mathrm{Sp})$ and $\mathrm{Fun}(\mathbb{Z}/m', \mathrm{Sp})$ where the left adjoint is symmetric monoidal and given by left Kan extension [Nik16, Corollary 3.8]. The right adjoint is given by restriction. This provides the following adjunctions, which allow us to move between various gradings. Let $n > 0$ and $s \geq 0$ be integers.

- We let $D^n_{sn} \dashv Q$ denote the adjunction induced by the quotient map $\mathbb{Z}/sn \to \mathbb{Z}/n$ sending 1 to 1.
- We often use D^n_{sn} for $s = 0$, which allows us to obtain an n-graded object out of a graded object X in Sp. We let D^n denote $D^n_0 \colon \mathrm{Fun}(\mathbb{Z}, \mathrm{Sp}) \to \mathrm{Fun}(\mathbb{Z}/n, \mathrm{Sp})$ and we have
$$D^n(X)_i \simeq \bigvee_{j \in \mathbb{Z} | j \equiv i \mod n} X_j.$$
- For $n = 1$, we denote D^1_s by D. In this case,
$$D \colon \mathrm{Fun}(\mathbb{Z}/s, \mathrm{Sp}) \to \mathrm{Sp}$$
is given by $D(X) \simeq \vee_{j \in \mathbb{Z}/s} X_j$, i.e. left Kan extension along $\mathbb{Z}/s \to 0$. For an s-graded (spectrum) E_n-ring X, we call $D(X)$ the *underlying (spectrum) E_n-ring* of X. We often omit D in our notation.
- For $s \in \mathbb{Z}$, let $L_s \dashv R_s$ denote the adjunction on $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$ induced by the map $\mathbb{Z} \xrightarrow{\cdot s} \mathbb{Z}$ given by multiplication by s. For a graded spectrum X, we have
$$L_s(X)_{si} \simeq X_i$$
for every i and $L_s(X)_j \simeq 0$ whenever $s \nmid j$; furthermore, $R_s(X)_j \simeq X_{sj}$ for every j.
- Let $F \dashv G$ denote the adjunction induced by the trivial map $0 \to \mathbb{Z}/m$. We have $G(X) = X_0$. For an m-graded E_n-ring A, $F(G(A))$ is given by A_0 in weight 0 and it is trivial on the other degrees. Therefore, we sometimes abuse notation and denote the m-graded E_n-ring $F(G(A))$ by A_0. The counit of this adjunction provides a map
$$A_0 \to A$$
of m-graded E_n-rings. If $A_i \simeq 0$ for $i \neq 0$, then this map is an equivalence and we say that A is *concentrated in weight 0*. The following lemma states that, in this situation, there is an equivalence of E_n-rings between the underlying E_n-ring of A and the weight-0 piece $G(A) = A_0$ of A. Therefore, we often do not distinguish between A, $G(A) = A_0$ and $D(A)$ in our notation when A is concentrated in weight 0.

Lemma 2.3. *Let A be an m-graded E_n-ring concentrated in weight 0. There is an equivalence of E_n-rings*
$$A_0 \simeq D(A),$$
where A_0 denotes $G(A)$. In particular, we have $F(D(A)) \simeq A$ as m-graded E_n-rings.

Proof. Since A is concentrated in weight 0, we have $D(A) \simeq DFG(A)$. As $D \circ F$ Kan extends through the composite $0 \to \mathbb{Z}/m \to 0$, it is equivalent to the identity functor. We obtain that $DFG(A) \simeq G(A) \simeq A_0$.

For the second statement, note that $F(D(A)) \simeq F(G(A))$ due to the first statement. Since A is concentrated in weight 0, we have $F(G(A)) \simeq A$. □

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3. A family of E_2 polynomial rings in graded spectra

In this section, we introduce the construction of a family of E_2-algebras in graded spectra. These have appeared in the work of Hahn and Wilson [HW22] and are also studied in greater depth by Devalpurkar and Moulinos (*Logarithmic structures for E_2-algebras*, in progress, 2022). These will be central to our constructions. For every $r, w \in \mathbb{Z}$, one constructs a graded E_2-ring $\mathbb{S}[\sigma_{2r}]$, which may be thought of as a 'polynomial' algebra with a generator in homotopical degree $2r$ and grading weight w. However, these are not polynomial algebras in the precise sense, as they are only demonstrated to admit E_2 structures. By mapping these E_2-rings into each other, we will be able to construct E_2-ring extensions.

3.1 Shearing preliminaries

The main mechanism underlying this construction is that of shearing, which we now briefly review. It has appeared in [Rak20], and is also studied by Devalpurkar and Moulinos, *Logarithmic structures for E_2-algebras*, in progress, 2022). In what follows, $\mathrm{Gr}(\mathrm{Sp})$ denotes $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$, i.e. the ∞-category of graded spectra.

Proposition 3.1. *There exists an endofunctor on graded spectra*

$$\mathrm{sh} : \mathrm{Gr}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp})$$

given by

$$\mathrm{sh}(M)_i := M_i[-2i]$$

with the following properties:

- sh *is an equivalence, with inverse given by* $\mathrm{sh}^{-1}(M_i) = M_i[2i]$*;*
- sh *admits an E_2-monoidal structure, with respect to the Day convolution product on* $\mathrm{Gr}(\mathrm{Sp})$.

Proof. This appears in the $\mathbb{Z}$-linear setting in [Rak20] and also appears in [DHL+23]. For the sake of completeness, we sketch the basic ideas underlying the construction. In [Lur15], Lurie constructs an E_2-monoidal map of spaces

$$\phi : \mathbb{Z} \to \mathrm{Pic}(\mathrm{Sp})$$

sending $n \mapsto \mathbb{S}^{-2n}$. We now define sh as the functor, obtained by adjunction, from the assignment

$$\mathbb{Z} \times \mathrm{Gr}(\mathrm{Sp}) \to \mathrm{Sp}$$

given by the composition

$$\mathbb{Z} \times \mathrm{Gr}(\mathrm{Sp}) \xrightarrow{(\phi, \mathrm{ev})} \mathrm{Pic}(\mathrm{Sp}) \times \mathrm{Sp} \xrightarrow{\otimes} \mathrm{Sp}\,.$$

Here, the first map sends $(n, M) \mapsto (\phi(n), M_n)$. The fact that this latter composition is E_2 follows from the fact that ϕ is itself E_2. This further implies that sh is itself E_2 monoidal. To see that this is an equivalence, one displays, as in [Rak20], an inverse in the same way by precomposing ϕ with the map $\mathbb{Z} \xrightarrow{-1\cdot} \mathbb{Z}$. □

Variant 3.2. One can precompose the map $\phi : \mathbb{Z} \to \mathrm{Pic}(\mathrm{Sp})$ with the map $\cdot(k) : \mathbb{Z} \to \mathbb{Z}$. We denote the composition by

$$\phi^k : \mathbb{Z} \to \mathrm{Pic}(\mathrm{Sp}).$$

As in the above, we use this to define an endofunctor

$$\mathrm{sh}^k : \mathrm{Gr}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp}).$$

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This acquires the same formal properties as above, e.g. it will be an E_2-monoidal autoequivalence on $\mathrm{Gr}(\mathrm{Sp})$. Furthermore, one has the description $(\mathrm{sh}^k M)_i \simeq M_i[-2ki]$.

3.2 Sheared polynomial algebras

Recall that there exists an E_∞-algebra $\mathbb{S}[t] \in \mathrm{Gr}(\mathrm{Sp})$, which gives a graded enhancement of the 'flat' polynomial algebra. One can obtain this, for example, by observing that the restriction map from filtered spectra to graded spectra

$$\mathrm{Res} : \mathrm{Fil}(\mathrm{Sp}) \to \mathrm{Gr}(\mathrm{Sp})$$

is lax symmetric monoidal. In more detail, this will be the restriction

$$\mathrm{Fil}(\mathrm{Sp}) = \mathrm{Fun}((\mathbb{Z}, \leq), \mathrm{Sp}) \to \mathrm{Fun}(\mathbb{Z}^{\mathrm{ds}}, \mathrm{Sp}) = \mathrm{Gr}(\mathrm{Sp})$$

along $\mathbb{Z} \hookrightarrow (\mathbb{Z}, \leq)$ so that, in particular, we forget the structure maps of the filtration; cf. [Lur15]. We remind the reader that this is different from the associated graded functor. One then sets $\mathbb{S}[t] := \mathrm{Res}(\mathbf{1})$, where 1 denotes the unit of the symmetric monoidal structure on $\mathrm{Fil}(\mathrm{Sp})$. Thus, $\mathbb{S}[t]$ (which is given by $\mathbb{S}$ in nonpositive weights and 0 in positive weights) acquires the structure of an E_∞-algebra in graded spectra.

Construction 3.3. As described in Proposition 3.1, there exists an E_2-monoidal autoequivalence sh. We set

$$\mathbb{S}[\sigma_2] := \mathrm{sh}(\mathbb{S}[t])$$

and more generally, for $k > 0$,

$$\mathbb{S}[\sigma_{2k}] := \mathrm{sh}^k(\mathbb{S}[t]);$$

that is, one applies sh^k to $\mathbb{S}[t]$ to obtain a family of E_2-algebras in graded spectra. For $k = 0$, we set $\mathbb{S}[\sigma_0] := \mathbb{S}[t]$. It follows by inspection that the underlying graded E_1-ring of $\mathbb{S}[\sigma_{2k}]$ is the free graded E_1-ring on $\mathbb{S}^{2k}(-1)$, where $\mathbb{S}^{2k}(-1)$ is $\mathbb{S}^{2k}$ concentrated in weight -1.

Remark 3.4. For $w \in \mathbb{Z}$ and even $k \geq 0$, we apply the functor L_{-w} which left Kan extends along the multiplication map

$$\cdot(-w) : \mathbb{Z} \to \mathbb{Z}$$

to obtain weight-sifted variants of $\mathbb{S}[\sigma_k]$. We often omit L_{-w} when the weight of σ_k is clear from the context but when we wish to be explicit, we write $\mathbb{S}[\sigma_{k,w}]$ for the graded E_2-ring $L_{-w}\mathbb{S}[\sigma_k]$, where $\sigma_{k,w}$ is in weight w. As before, the underlying graded E_1-ring of $\mathbb{S}[\sigma_{k,w}]$ is the free graded E_1-ring $\mathrm{Free}_{E_1}(\mathbb{S}^k(w))$.

To adjoin roots, we often start with $\mathbb{S}[\sigma_{mk}]$ and $\mathbb{S}[\sigma_k]$ with σ_{mk} and σ_k in weights m and 1, respectively, where $m > 0$ and $k > 0$ is even.

Proposition 3.5. *In the situation above, there exists a map of graded E_2-rings*

$$\mathbb{S}[\sigma_{mk}] \to \mathbb{S}[\sigma_k]$$

that carries σ_{mk} to σ_k^m in homotopy. This provides a map of m-graded E_2-rings

$$D^m(\mathbb{S}[\sigma_{mk}]) \to D^m(\mathbb{S}[\sigma_k]),$$

where $\sigma_k \in \pi_* D^m(\mathbb{S}[\sigma_k])$ is of weight 1 and $D^m(\mathbb{S}[\sigma_{mk}])$ is concentrated in weight 0.

Furthermore, we have $D^m(\mathbb{S}[\sigma_{mk}]) \simeq F(D(\mathbb{S}[\sigma_{mk}]))$ as m-graded E_2-rings; here, F Kan extends through $0 \to \mathbb{Z}/m$.

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Proof. We first remind the reader that there is an identification

$$\mathbb{S}[\sigma_{mk,-1}] := \mathrm{sh}^{mk}(\mathbb{S}[t]) \simeq R_m(\mathbb{S}[\sigma_{k,-1}])$$

of graded E_2-rings. This follows from the very definition of the kth shearing functor sh^k, in particular that it sends $\mathbb{S}[t]$ to the negative weight part of the graded spectrum given by the map

$$\phi^k : \mathbb{Z} \xrightarrow{\times k} \mathbb{Z} \to \mathrm{Pic}(\mathrm{Sp}).$$

Thus we may identify $\mathbb{S}[\sigma_{mk,-1}]$ with $R_m\mathbb{S}[\sigma_{k,-1}] \simeq R_m\,\mathrm{sh}^k(\mathbb{S}[t])$, negative-weight part of the graded spectrum given by the map

$$\phi^{mk} : \mathbb{Z} \to \mathrm{Pic}(\mathrm{Sp}).$$

Therefore, the counit of the adjunction $L_m \dashv R_m$ provides a map of graded E_2-rings

$$\mathbb{S}[\sigma_{mk,-m}] \to \mathbb{S}[\sigma_{k,-1}].$$

Here, $\mathbb{S}[\sigma_{mk,-m}]$ is the free graded E_1-algebra on a class $\sigma_{mk,-m}$ of degree mk and weight $-m$ and by inspection on the adjunction $L_m \dashv R_m$, this map carries $\sigma_{mk,-m}$ to $\sigma^m_{k,-1}$. Applying the functor L_{-1} to this map, we obtain the map of graded E_2-rings $\mathbb{S}[\sigma_{mk}] \to \mathbb{S}[\sigma_k]$ claimed in the proposition.

The functor D^m gives the desired map

$$D^m(\mathbb{S}[\sigma_{mk}]) \to D^m(\mathbb{S}[\sigma_k])$$

of m-graded E_2-algebras. Note that $\sigma_{mk} \in \pi_* D^m(\mathbb{S}[\sigma_{mk}])$ is of weight 0, which ensures that $D^m(\mathbb{S}[\sigma_{mk}])$ is concentrated in weight 0.

To see the last statement, we have

$$FDD^m(\mathbb{S}[\sigma_{mk}]) \simeq D^m(\mathbb{S}[\sigma_{mk}]) \tag{3.1}$$

due to Lemma 2.3, since $D^m(\mathbb{S}[\sigma_{mk}])$ is concentrated in weight 0. Furthermore, DD^m is the Kan extension through the composite $\mathbb{Z} \to \mathbb{Z}/m \to 0$, which is the same as the Kan extending through $\mathbb{Z} \to 0$. Therefore, $DD^m(\mathbb{S}[\sigma_{mk}]) \simeq D(\mathbb{S}[\sigma_{mk}])$. This, together with (3.1), provides the desired equivalence $D^m(\mathbb{S}[\sigma_{mk}]) \simeq F(D(\mathbb{S}[\sigma_{mk}]))$. □

We often omit the functor D^m in our notation and denote the map of m-graded E_2-rings $D^m(\mathbb{S}[\sigma_{mk}]) \to D^m(\mathbb{S}[\sigma_k])$ as $\mathbb{S}[\sigma_{mk}] \to \mathbb{S}[\sigma_k]$.

Remark 3.6. To adjoin a root to a degree-0 class, we need the $k = 0$ case of the proposition above. In other words, we need an analogous E_2-map $\mathbb{S}[\sigma_{mk}] \to \mathbb{S}[\sigma_k]$ for $k = 0$. For this, we start with the graded E_2-map $\mathbb{S}[\sigma_{m2}] \to \mathbb{S}[\sigma_2]$ and apply the functor sh. This procedure provides a graded E_2-map $\mathbb{S}[\sigma_{0,m}] \to \mathbb{S}[\sigma_{0,1}]$ that carries $\sigma_{0,m}$ to $\sigma^m_{0,1}$ as desired.

3.3 Cell structures on sheared polynomial algebras

A key technical result for us will be the even cell decomposition of $\mathbb{S}[\sigma_k]$ *as an* E_2*-algebra*. As we show in the remainder of this section, this is what will allow us to define E_2-algebra maps to a given E_2-algebra A, along which we will then adjoin roots.

Remark 3.7. In the second arXiv version of [HW22], Hahn and Wilson also construct E_2 even cell decompositions on the free E_1-algebra $\mathbb{S}[\sigma_k]$ using a Koszul duality argument for even $k \geq 0$. However, they removed this result in the later versions of their paper since they found a simpler argument for their redshift results that avoids the use of these even cell decompositions. Since this does not appear in the published version of [HW22], we give a proof of the E_2 even cell

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decompositions on $\mathbb{S}[\sigma_k]$ that we use. We would like to note that our methods are different from the ones used in [HW22, arXiv version 2].

Before doing this, we make precise what exactly we mean by even cell decomposition. The following notions are heavily inspired by § 6.3 of [GKR25].

Definition 3.8. Let $f : S \to R \in \mathrm{Alg}_{E_2}(\mathrm{Sp}^{\mathbb{Z}})$ be a map of E_2-algebras in graded spectra. We say f has a *filtered cellular decomposition* if there exists a tower in $\mathrm{Alg}_{E_2}(\mathrm{Fil}(\mathrm{Gr}(\mathrm{Sp})))$ for which

$$S = \mathrm{sk}_{-1}(f) \to \mathrm{sk}_0(f) \to \mathrm{sk}_1(f) \to \cdots \to \mathrm{colim}_i\, \mathrm{sk}_i(f) =: \mathrm{sk}(f) \simeq R$$

such that each $\mathrm{sk}_i(f)$ is obtained from $\mathrm{sk}_{i-1(f)}$ via the following pushout diagram.

$$\begin{array}{ccc} \mathrm{Free}_{E_2}(\bigsqcup_{\alpha\in I_{n_i}} \partial D^{g_\alpha,n_i}[i-1]) & \longrightarrow & \mathrm{sk}_{i-1}(f) \\ \downarrow & & \downarrow \\ \mathrm{Free}_{E_2}(\bigsqcup_{\alpha\in I_{n_i}} D^{g_\alpha,n_i}[i]) & \longrightarrow & \mathrm{sk}_i(f) \end{array}$$

The notation $X[n]$ in the above means that the object X is placed in filtering degree n.

In particular, in each degree i of the tower, we are adding cells in increasing dimension n_i. Thus, if $i \le j$, then $n_i \le n_j$ and I_{n_i} refers to the set of n_i cells of R. It may be the case that $n_i = i$, but we do not require this for the sake of flexibility of the definition, which is a point of departure from the notion in [GKR25]. If $f : \mathbf{1} \to R$ is the map from the unit, we call this a filtered cellular decomposition of R.

Definition 3.9. A map $f\colon S \to R$ of graded E_2-rings admits a cell decomposition if it is the colimit of a tower

$$S = \mathrm{sk}_{-1}(f) \to \mathrm{sk}_0(f) \to \mathrm{sk}_1(f) \to \cdots \to \mathrm{colim}_i\, \mathrm{sk}_i(f) \simeq R$$

in graded E_2-rings, where each stage is obtained from the previous via a cell attachement in graded E_2-rings. In particular, if f admits a filtered cellular decomposition, then taking levelwise colimits provides a cellular decomposition of f.

Our first step is to establish the decomposition for $\mathbb{S}[t]$.

Proposition 3.10. *As an E_2-algebra in graded spectra, $\mathbb{S}[t]$ admits a (filtered) cellular decomposition with cells in even degrees.*

Proof. The two key inputs for our argument are Theorems 11.21 and 13.7 of [GKR25]. The former result applied to the map $f : 0 \to I$ of non-unital E_2-algebras in graded spectra (where I is the augmentation ideal of the map $\mathbb{S}[t] \to \mathbb{S}$) says that there exists a relative CW decomposition

$$0 \to \mathrm{colim}\, \mathrm{sk}_n(f) \simeq I.$$

Moreover, the proof of this fact in the same work constructs a minimal cell structure, one that has the smallest possible number of cells in a given bidegree (the extra degree here arises since we are working in graded spectra). In particular, the colimit they construct, $\mathrm{colim}\, \mathrm{sk}_n(f)$, will have cells precisely in bidegree $b^{E_2}_{g,d}(\mathbb{S}[t]) := \dim_k H^{E_2}_{g,d}(\mathbb{S}[t], \mathbb{S}, k) \in \mathbb{N} \cup \{\infty\}$, where $H^{\mathcal{O}}_{g,d}(R; k)$ is the $\mathcal{O}$-homology of an $\mathcal{O}$-algebra R, with coefficients in the ring k; we will, in fact, set $k = \mathbb{Z}$.

We sketch the argument given there applied to our particular case for the sake of completeness. One proceeds by inductively constructing a factorization

$$0 = \mathrm{sk}_{-1} \xrightarrow{h_0} \cdots \xrightarrow{h_\epsilon} \mathrm{sk}_\epsilon \xrightarrow{f_\epsilon} I$$

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in the ∞-category of (increasingly) filtered objects of $\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp})$. Here, $h_e : \mathrm{sk}_{e-1} \to \mathrm{sk}_e$ comes with the structure of a (filtered) CW attachment of dimension e, where sk_i denotes the ith skeleton equipped with the skeletal filtration leading up to that degree. Taking the colimit along ϵ gives an induced map $f_\infty : \mathrm{colim}\, sk_\epsilon(f) \to I$, which is an equivalence.

For the inductive step in their argument, they show that the Hurewicz map

$$\pi_{*,\epsilon}(\mathbb{S}[t], \mathrm{sk}_{\epsilon-1}) \to H^{E_2}_{*,\epsilon}(\mathbb{S}[t], \mathrm{sk}_{\epsilon-1}; k)$$

from relative homotopy to relative E_2-homology with coefficients in k is surjective. Using this, one is able to choose a set of maps

$$\{E_\alpha : (D^\epsilon, \partial D^\epsilon) \to (\mathbb{S}[t](g), \mathrm{sk}_{\epsilon-1}(g))\},$$

whose images generate $H^{E_2}_{*,\epsilon}(\mathbb{S}[t], \mathrm{sk}_{\epsilon-1}; k)$ as a k-module. The boundary maps are then used to attach filtered cells (g, ϵ) to $\mathrm{sk}_{\epsilon-1}$ to form sk_ϵ, and the corresponding E_α is used to extend $f_{\epsilon-1}$ to f_ϵ. Putting all this together, we see that the attachment of the cells is parameterized by the dimensions of the E_2-algebra homology groups with coefficients in k. To see, in our particular setup, that this cell decomposition is concentrated in even degrees, it is therefore enough to verify that $H^{E_2}_{g,e}(\mathbb{S}[t], k)$ vanishes whenever $e \cong 1 \bmod 2$. For this, we apply [GKR25, Theorem 13.7], which states that the k-fold iterated bar construction of an E_k-algebra is equivalent to the k-suspension of the E_k-cotangent complex. By [Lur15, Proposition 5.4.9], there is an equivalence $\mathrm{Bar}^{(2)}(\mathbb{S}[t]) \simeq \mathrm{gr}(\mathbb{S}[\mathbb{C}P^n]_{n\geq 0})$ in graded spectra, where the right-hand side is the associated grading of the filtration on spherical chains on $\mathbb{C}P^\infty$, with filtration induced by the skeletal filtration on infinite projective space. Tensoring this with $k = \mathbb{Z}$ in our particular situation, we obtain (a 2-fold shift) of chains on $\mathbb{C}P^\infty$ with coefficients in $\mathbb{Z}$ which has a cell in each bidegree $(-n, 2n-2)$. By taking into account units, we conclude that $\mathbb{S}[t]$ may be constructed from $\mathbb{S}$ by attaching the same cells. □

We begin with the particular filtered cellular decomposition on $\mathbb{S}[t]$ given in the proof above and apply levelwise colimits to obtain a cellular decomposition for $\mathbb{S} \to \mathbb{S}[t]$ in the sense of Definition 3.9. In the following corollary, we use this particular cellular decomposition.

Corollary 3.11. *The degree-0 piece of the above cellular decomposition is the free algebra* $\mathrm{Free}_{E_2}(\mathbb{S}(-1))$*, i.e. the free* E_2*-algebra with generator in degree* 0 *and weight* -1*. Moreover, the map* $f_0 : \mathrm{Free}_{E_2}((\mathbb{S}(-1)) \to \mathbb{S}[t]$ *itself admits a cellular decomposition with even cells of positive dimension.*

Proof. In degree 0, we have the following pushout square in $\mathrm{Alg}_{E_2}(\mathrm{Fun}(\mathbb{Z}, \mathrm{Sp}))$.

$$\begin{array}{ccc} \mathbb{S} \simeq \mathrm{Free}_{E_2}(\emptyset) & \longrightarrow & \mathrm{sk}_{-1}(\mathbb{S}[t]) \simeq \mathbb{S} \\ \downarrow & & \downarrow \\ \mathrm{Free}_{E_2}(\mathbb{S}(-1)) \simeq \mathrm{Free}_{E_2}(D^0) & \longrightarrow & \mathrm{sk}_0(\mathbb{S}[t]) \end{array}$$

Since this is a pushout square, we obtain an equivalence

$$\mathrm{sk}_0(\mathbb{S}[t]) \simeq \mathrm{Free}_{E_2}(\mathbb{S}(-1)).$$

Moreover, by starting in degree 0 with the zero cells already attached, we may conclude that the map

$$f_0 : \mathrm{Free}_{E_2}(\mathbb{S}(-1)) \to \mathbb{S}[t]$$

itself admits a cellular decomposition with even cells of positive dimension. □

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By the above corollary, we have a cellular decomposition on the map of graded E_2-rings $\mathrm{Free}_{E_2}(\mathbb{S}(-1)) \to \mathbb{S}[t]$. We can apply shearing to this map to obtain a map

$$\mathrm{Free}_{E_2}(\mathbb{S}^k(-1)) \to \mathbb{S}[\sigma_k].$$

Proposition 3.12. *Let $k > 0$ be even. The map $f_0 : \mathrm{Free}_{E_2} \mathbb{S}^k(-1) \to \mathbb{S}[\sigma_k]$ admits a cell decomposition with cells concentrated in even degrees. Left Kan extending along the multiplication map $\mathbb{Z} \xrightarrow{\times -w} \mathbb{Z}$, we conclude that $f_0 : \mathrm{Free}_{E_2} \mathbb{S}^k(w) \to \mathbb{S}[\sigma_{k,w}]$ admits a cell decomposition with cells concentrated in even degrees.*

Proof. By Corollary 3.11, $\mathbb{S}[t]$ may be written as a filtered colimit of a diagram of E_2-algebras,

$$\mathrm{Free}_{E_2}(\mathbb{S}(-1)) \to \mathrm{sk}_1(f) \to \cdots \to \mathrm{sk}_{i-1}(f) \to \mathrm{sk}_i(f) \to \cdots,$$

where each $\mathrm{sk}_i(f)$ is formed as a pushout from sk_{i-1} along a map $\mathrm{Free}_{E_2}(\mathbb{S}^{2n+1}) \to \mathbb{S}$. We may apply $\mathrm{sh}^{k/2}$ to this diagram, and take note of the fact that this will commute with colimits along the filtered diagram, together with the free E_2-algebra functor. Thus, we conclude with an even cell presentation for the induced map:

$$\mathrm{sh}^{k/2}(\mathrm{Free}_{E_2}(\mathbb{S}(-1))) \simeq \mathrm{Free}_{E_2}(\mathbb{S}^k(-1)) \to \mathbb{S}[\sigma_k].$$

By left Kan extending along the multiplication by the $-w$ map on $\mathbb{Z}$ (i.e. applying L_{-w}), we conclude analogously for the map

$$\mathrm{Free}_{E_2}(\mathbb{S}^k(w)) \to \mathbb{S}[\sigma_{k,w}]. \qquad \square$$

Proposition 3.13. *Let A be a (graded) E_2-ring whose homotopy groups are concentrated in even degrees and let $a \in \pi_k A$ be a weight-w class for some even $k \geq 0$. Then there is a (graded) E_2-ring map*

$$\mathbb{S}[\sigma_{k,w}] \to A,$$

which carries σ_k to a.

Proof. By Proposition 3.12, the map

$$f : \mathrm{Free}_{E_2}(\mathbb{S}^k(w)) \to \mathbb{S}[\sigma_{k,w}]$$

admits an even cell decomposition. Let $a \in \pi_k(A)$ be as in the hypothesis of the proposition. This induces an E_2-algebra map $\mathrm{Free}_{E_2}(\mathbb{S}^k(w)) \to A$, which we would like to extend inductively along the above tower. In order to do this, it is enough to note that in degree i, we would need to trivialize the induced map $\mathrm{Free}_{E_2}(\mathbb{S}^{k+2i-1}) \to A$. Using the free/forgetful adjunction between $\mathrm{Alg}_{E_2}(\mathrm{Gr}(\mathrm{Sp}))$ and $\mathrm{Gr}(\mathrm{Sp})$ algebras and graded spectra, this will now follow from the fact that $\pi_{2n-1}(A) = 0$ for all n. $\square$

Remark 3.14. In Remark 3.6, we mentioned that we are going to use $\mathrm{sh}(\mathbb{S}[\sigma_{m2,m}])$ to adjoin roots to degree-0 classes. We remark that $\mathrm{sh}(\mathbb{S}[\sigma_{m2,m}])$ also satisfies the lifting property in the proposition above. This follows from the fact that the even cell decomposition for $\mathbb{S}[\sigma_{m2,m}]$ provides an even cell decomposition for $\mathrm{sh}(\mathbb{S}[\sigma_{m2,m}])$, since sh is an E_2-monoidal left adjoint functor.

Remark 3.15. We remark that another way to construct an E_2-algebra map $\mathbb{S}[t] \to A$ comes from the filtration on $\mathbb{S}[t]$ given its filtered cell decomposition. The mapping space

$$\mathrm{Map}_{\mathrm{Alg}_{E_2}}(\mathbb{S}[t], A)$$

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obtains a filtration from the filtration on the source; this will have associated grading

$$\operatorname{gr}\operatorname{Map}_{\operatorname{Alg}_{E_2}}(\mathbb{S}[t], A) \simeq \operatorname{Map}\left(\operatorname{Free}_{E_2}\left(\bigoplus_{n\geq 1}\mathbb{S}^{2n}, A\right)\right) \simeq \operatorname{Map}\left(\bigoplus_{n\geq 1}\mathbb{S}^{2n}, A\right).$$

Now, if X has even homotopy groups, then so does the associated grading, so the resulting spectral sequence computing the homotopy groups of the limit collapses. Thus, $a \in \pi_0(A)$ gives a class in $x \in \pi_0 \operatorname{Map}(\mathbb{S}^{2k}, X) \subset \pi_0 \operatorname{Map}(\bigoplus_{n\geq 1}\mathbb{S}^{2n+2nk-2}, X)$, which will be an infinite cycle, and thus corresponds to an E_2-algebra map $\mathbb{S}[t] \to A$. We remark that this approach should allow for one to define maps $\mathbb{S}[\sigma_k] \to A$ in the general case as well. We thank Oscar Randal-Williams for suggesting this approach.

4. Adjoining roots and THH

Here, we introduce our construction for adjoining roots to ring spectra and prove our first results on the THH of ring spectra obtained through this construction.

4.1 Background on algebras over E_n-algebras

Here is a quick background on some of the standard facts that we often use from [Lur17].

For an E_∞-algebra R in a presentably symmetric monoidal ∞-category $\mathcal{C}$, the ∞-category of E_n R-algebras is a symmetric monoidal ∞-category with the pointwise tensor product [Lur17, Example 3.2.4.4]. Therefore, for two E_n R-algebras A and B, $A \otimes_R B$ is an E_n R-algebra.

In this work, we often consider algebras over an E_n-algebra R and in this case, the ∞-category of E_m R-algebras (for $m \leq n-1$) are not known to carry an appropriate E_{n-1}-monoidal structure. To work around this problem, we use the following facts which can be extracted from [Lur17, Corollary 4.8.5.20].

The ∞-category of (left) right R-modules is an E_{n-1}-monoidal ∞-category. We call an E_m-algebra in the ∞-category of right R-modules an E_m R-algebra, where $m \leq n-1$.

Furthermore, for a map $f\colon R \to S$ of E_n-algebras in $\mathcal{C}$, one obtains an E_{n-1}-monoidal functor $- \otimes_R S$ between the respective ∞-categories of modules. For every $m \leq n-1$, this induces a functor

$$- \otimes_R S\colon \operatorname{Alg}_{E_m}(\operatorname{RMod}_R) \to \operatorname{Alg}_{E_m}(\operatorname{RMod}_S). \tag{4.1}$$

In particular, for an E_m R-algebra A, $A \otimes_R S$ is an E_m S-algebra. Furthermore, the forgetful functor induced by f, i.e. the right adjoint of $- \otimes_R S$, is E_{n-1}-lax monoidal, and therefore it induces a functor

$$\operatorname{Alg}_{E_m}(\operatorname{RMod}_S) \to \operatorname{Alg}_{E_m}(\operatorname{RMod}_R). \tag{4.2}$$

The unit of this adjunction provides a map of E_m R-algebras

$$A \to A \otimes_R S. \tag{4.3}$$

Since S is the monoidal unit in RMod_S, S is an E_{n-1} S-algebra, and forgetting through (4.2), it is an E_{n-1} R-algebra. In summary, an E_n-algebra map $R \to S$ equips S with the structure of an E_{n-1} R-algebra.

4.2 A construction for adjoining roots to ring spectra

We now introduce our construction for adjoining roots to ring spectra. For this, we use the following hypothesis. Recall that we often omit the functor D and let $\mathbb{S}[\sigma_k]$ denote the underlying E_2-ring of the graded E_2-ring $\mathbb{S}[\sigma_k]$.

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Hypothesis 4.1 (Root adjunction hypothesis). Given an E_1-ring A with $a \in \pi_{mk}A$, there is a chosen $\mathbb{S}[\sigma_{mk}]$-algebra structure on A for which the structure map $\mathbb{S}[\sigma_{mk}] \to A$ carries σ_{mk} to $a \in \pi_{mk}A$. Here, $m > 0$ and $k \geq 0$ is even. See Proposition 4.2 for the cases of interest where this is satisfied.

The hypothesis above may not seem very natural but it is satisfied in the following general situations.

Proposition 4.2. *Let $k \geq 0$ be even and $m > 0$, an E_1-ring A satisfies Hypothesis 4.1 for $a \in \pi_{mk}A$ if:*

(1) *A is an E_2-ring for which π_*A is concentrated in even degrees; or*
(2) *A is an R-algebra for an E_2-ring R where π_*R is concentrated in even degrees and a is in the image of the map $\pi_*R \to \pi_*A$.*

Proof. Assume that A is as in part (2), and let $r \in \pi_{mk}R$ detect a through the map $\pi_*R \to \pi_*A$. We choose an E_2-ring map $g\colon \mathbb{S}[\sigma_{mk}] \to R$ that carries σ_{mk} to r; see Proposition 3.13. Forgetting through g, see (4.2), one obtains a $\mathbb{S}[\sigma_{mk}]$-algebra structure on A. Indeed, through this structure, σ_{mk} acts through a as desired.

If A is as in part (1), then A is an A-algebra and A satisfies the assumption in part (2). Therefore, A satisfies Hypothesis 4.1. □

For instance, the Morava K-theory spectrum $K(n)$ and all E_1 $MU_{(p)}$-algebra forms of $BP\langle n\rangle$ satisfy Hypothesis 4.1 with respect to their non-negative degree homotopy classes.

Note that we are not assuming any pre-existing non-trivial grading on A; in fact, this will allow us to view it as an m-graded spectrum concentrated in weight 0. Given $a \in \pi_{mk}A$, the following construction adjoins an m-root to a.

Construction 4.3. Assume Hypothesis 4.1. We consider $\mathbb{S}[\sigma_{mk}]$ as an m-graded E_2-ring and A as an m-graded $\mathbb{S}[\sigma_{mk}]$-algebra, both concentrated in weight 0, using the functor F from § 2.2; we omit F in our notation.

Owing to Proposition 3.5 (Remark 3.6 for $k = 0$), there is a map

$$\phi : \mathbb{S}[\sigma_{mk}] \to \mathbb{S}[\sigma_k]$$

of m-graded E_2-rings that carries σ_{mk} to σ_k^m in homotopy, where σ_k is of weight 1 and σ_{mk} is of weight 0. Note that we omit the functor D^m in our notation. Considering the corresponding extension of scalars functor $- \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}[\sigma_k]$ between the ∞-categories of m-graded $\mathbb{S}[\sigma_{mk}]$-algebras and m-graded $\mathbb{S}[\sigma_k]$-algebras (see (4.1)), we define the m-graded E_1 $\mathbb{S}[\sigma_k]$-algebra $A(\sqrt[m]{a})$ through

$$A(\sqrt[m]{a}) := A \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}[\sigma_k].$$

This comes equipped with a map $A \to A(\sqrt[m]{a})$ of m-graded E_1 $\mathbb{S}[\sigma_{mk}]$-algebras; see (4.3).

Since $\pi_*(\mathbb{S}[\sigma_k])$ is free as a $\pi_*(\mathbb{S}[\sigma_{mk}])$-module, one obtains an isomorphism of rings

$$\pi_*A(\sqrt[m]{a}) \cong \pi_*(A)[z]/(z^m - a).$$

Therefore, we say $A(\sqrt[m]{a})$ *is obtained from A by adjoining an m-root to a.*

When A is p-local, observe that we have an equivalence of m-graded E_1 $\mathbb{S}[\sigma_k]$-algebras

$$A(\sqrt[m]{a}) \simeq A \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k],$$

where $\mathbb{S}_{(p)}[\sigma_i]$ denotes the p-localization of $\mathbb{S}[\sigma_i]$.

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It follows that the weight pieces of $A(\sqrt[m]{a})$ are given by the following:

$$A(\sqrt[m]{a})_i \simeq \Sigma^{ik} A \tag{4.4}$$

for each $0 \leq i < m$.

Note that $A(\sqrt[m]{a})$ might possibly depend on the $\mathbb{S}[\sigma_{mk}]$-algebra structure chosen on A. Therefore, every time we apply Construction 4.3, we fix an $\mathbb{S}[\sigma_{mk}]$-algebra structure on A.

Remark 4.4. If A is an E_3-ring with even homotopy, we may start with an E_2-map $\mathbb{S}[\sigma_{mk}] \to A$ for a given $a \in \pi_{mk}A$ with $k \geq 0$. By extending scalars, we obtain an E_2-ring map $A \wedge \mathbb{S}[\sigma_{mk}] \to A$ that equips A with the structure of an $A \wedge \mathbb{S}[\sigma_{mk}]$-algebra. Through this, $A(\sqrt[m]{a})$ is weakly equivalent as an m-graded $\mathbb{S}[\sigma_k]$-algebra to

$$A \wedge_{A \wedge \mathbb{S}[\sigma_{mk}]} A \wedge \mathbb{S}[\sigma_k].$$

In particular, $A(\sqrt[m]{a})$ admits the structure of an m-graded $A \wedge \mathbb{S}[\sigma_k]$-algebra.

Remark 4.5. In general, we do not expect the root adjunction $A \to A(\sqrt[m]{a})$ to satisfy a universal property. On the other hand, if A is an E_3-ring, $A(\sqrt[m]{a})$ is an A-algebra and the map $\pi_* A \to \pi_* A(\sqrt[m]{a})$ is étale, then it follows by [HP25, Theorem 1.10] that there is a bijection between homotopy classes of A-algebra maps $A(\sqrt[m]{a}) \to B$ and $\pi_* A$-algebra maps $\pi_* A(\sqrt[m]{a}) \to \pi_* B$ for any étale A-algebra B.

For the following, we fix an E_2-map $\mathbb{S}_{(p)}[\sigma_{2(p-1)}] \to \ell$ carrying $\sigma_{2(p-1)}$ to v_1.

Theorem 4.6. *There is an equivalence*

$$ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$$

of E_1 ℓ_p-algebras.

Proof. By Remark 4.4 above, $\ell_p(\sqrt[p-1]{v_1})$ is an ℓ_p-algebra. Let L_p denote the non-connective p-completed Adams summand. The E_1 L_p-algebra

$$\ell_p(\sqrt[p-1]{v_1}) \wedge_{\ell_p} L_p$$

is an étale E_1 L_p-algebra in the sense of Hesselholt and Pstragowski [HP25], and there is an isomorphism of $\pi_*(L_p)$-algebras

$$\pi_*(\ell_p(\sqrt[p-1]{v_1}) \wedge_{\ell_p} L_p) \cong \pi_*(KU_p).$$

It follows by [HP25, Theorem 1.10] that there is an equivalence of L_p-algebras

$$\ell_p(\sqrt[p-1]{v_1}) \wedge_{\ell_p} L_p \simeq KU_p.$$

Through this, $\ell_p(\sqrt[p-1]{v_1})$ serves as the connective cover of KU_p in E_1 ℓ_p-algebras. Hence, there is an equivalence of E_1 ℓ_p-algebras $ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$. □

In Theorem 9.3, we show that the Morava E-theory spectrum E_n is given by $\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} \widehat{E(n)}(\sqrt[p^n-1]{v_n})$ as an E_1-ring, where $\widehat{E(n)}$ is the $K(n)$-localized Johnson–Wilson spectrum.

Remark 4.7. In certain cases, it is possible to equip $A(\sqrt[m]{a})$ with the structure of an E_n-algebra for $n > 1$. For this, one may use the graded E_∞ MU-algebra $MU[\sigma_k]$ [HW22, Construction 2.6.1], where $k > 0$ is even. Indeed, $MU[\sigma_k]$ is the free graded E_1 MU-algebra over $\Sigma^k MU$. There is a map of graded E_∞-rings

$$MU[\sigma_{2(p^n-1)}] := L_{p^n-1} R_{p^n-1} MU[\sigma_2] \to MU[\sigma_2]$$

and Proposition 3.13 provides maps $\mathbb{S}[\sigma_k] \to MU[\sigma_k]$ of graded E_2-rings.

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It follows by [HW22, Remark 2.1.2] that a form of $BP\langle n\rangle$ admits the structure of an E_3 $MU[\sigma_{2(p^n-1)}]$-algebra, where $\sigma_{2(p^n-1)}$ acts through $v_n \in \pi_* BP\langle n\rangle$. We obtain an equivalence of p^n-1-graded E_1 $\mathbb{S}[\sigma_2]$-algebras

$$BP\langle n\rangle(\sqrt[p^n-1]{v_n}) := BP\langle n\rangle \wedge_{\mathbb{S}[\sigma_{2(p^n-1)}]} \mathbb{S}[\sigma_2] \simeq BP\langle n\rangle \wedge_{MU[\sigma_{2(p^n-1)}]} MU[\sigma_2].$$

This equips $BP\langle n\rangle(\sqrt[p^n-1]{v_n})$ with the structure of a p^n-1-graded E_3 $MU[\sigma_2]$-algebra.

4.3 The weight-0 piece of THH

Here, we prove our first result regarding the THH of the ring spectra obtained via root adjunction. Namely, we show that $\mathrm{THH}(A(\sqrt[m]{a}))$ contains $\mathrm{THH}(A)$ as a summand whenever A is p-local and $p \nmid m$.

It follows by [AMM+22, Example A.10] that for an m-graded E_1-ring Y, the m-grading on $\mathrm{THH}(Y)$ is obtained by applying the cyclic bar construction $b_\bullet(Y)$ of Y in the ∞-category of m-graded spectra. In simplicial level s and weight i, the m-graded cyclic bar construction of Y is given by the following:

$$b_s(Y)_i \simeq \bigvee_{k_0+\cdots+k_s=i\in\mathbb{Z}/m} Y_{k_0} \wedge \cdots \wedge Y_{k_s}.$$

Owing to [AMM+22, Corollary A.15], we have the following equality:

$$\mathrm{THH}(D(Y)) \simeq D(\mathrm{THH}(Y)), \tag{4.5}$$

where the functor $D(-)$ provides the underlying spectrum as usual; we often omit D in our notation. Furthermore, $\mathrm{THH}(Y)$ is an S^1-equivariant m-graded spectrum in a canonical way and the equivalence above is an equivalence of S^1-equivariant spectra.

Construction 4.8. Let R be an E_2-ring and S an E_1 R-algebra. For us, this will mean that the pair $(R,S) \in \mathrm{RMod}^{(2)}(\mathrm{Sp})$, where

$$\mathrm{RMod}^{(2)}(\mathcal{C}) = \mathrm{Alg}(\mathrm{RMod}(\mathcal{C}))$$

for an arbitrary symmetric monoidal ∞-category $\mathcal{C}$. Here, $\mathrm{RMod}(\mathcal{C})$ is the ∞-category of pairs (A,M), where A is an E_1-algebra and $M \in \mathrm{RMod}_A(\mathcal{C})$. Thus, objects in $\mathrm{RMod}^{(2)}(\mathcal{C})$ may be identified with pairs (A,M), where A is an E_2-algebra and M is an E_1 A-algebra in $\mathcal{C}$.

We remark that, in general, $\mathrm{RMod}^{(2)}(\mathcal{C})$ may be written as $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$, where $\mathcal{O}$ is the *tensor product* of operads

$$\mathcal{O} := \mathrm{RMod} \times E_1.$$

This tensor product of operads, studied in depth in [Lur17, § 2.2.5], is symmetric and satisfies the following universal property at the level of algebra objects:

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{E_1}(\mathrm{Alg}_{\mathrm{RMod}}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathrm{RMod}}(\mathrm{Alg}_{E_1}(\mathcal{C})).$$

Hence, applying the discussion to $\mathcal{C} = \mathrm{Sp}$ and R and S as above, we may view S as a right R-module in E_1-algebras.

Since THH is a symmetric monoidal functor from E_1-rings to spectra [NS18, § IV.2], we deduce that $\mathrm{THH}(S)$ is a right $\mathrm{THH}(R)$-module.

Proposition 4.9. *Let F be an m-graded E_1 E-algebra and $E \to F'$ a map of m-graded E_2-rings. There is a natural equivalence of m-graded right $\mathrm{THH}(F')$-modules in S^1-equivariant spectra,*

$$\mathrm{THH}(F \wedge_E F') \simeq \mathrm{THH}(F) \wedge_{\mathrm{THH}(E)} \mathrm{THH}(F'),$$

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whose underlying ungraded equivalence is that of right $\mathrm{THH}(F')$*-modules in cyclotomic spectra. If* E *and* F *are concentrated in weight 0, then we have the following:*

$$\mathrm{THH}(F \wedge_E F')_i \simeq \mathrm{THH}(F) \wedge_{\mathrm{THH}(E)} (\mathrm{THH}(F')_i).$$

Proof. Let us recall that the functor

$$\mathrm{THH} : \mathrm{Alg}_{\mathrm{Sp}} \to \mathrm{CycSp}$$

is symmetric monoidal. Furthermore, it commutes with sifted colimits; indeed, this can be seen from the fact that it can be decomposed into a composition of functors comprised of taking tensor products and realizations of simplicial objects, both of which commute with sifted colimits. Thus, there will be a natural equivalence

$$\begin{aligned}\mathrm{THH}(F \wedge_E F') &\simeq \mathrm{THH}(|| \operatorname{Bar}_\bullet(F, E, F')||) \\ &\simeq || \operatorname{Bar}_\bullet(\mathrm{THH}(F)_\bullet, \mathrm{THH}(E)_\bullet, \mathrm{THH}(F')_\bullet)|| \\ &\simeq \mathrm{THH}(F) \wedge_{\mathrm{THH}(E)} \mathrm{THH}(F').\end{aligned}$$

This allows us to deduce that THH preserves the sifted colimit given by the double-sided bar construction; this can be computed at the level of underlying spectra by the bilinear pairing

$${}_F\,\mathrm{BMod}_E \times_E \mathrm{BMod}_{F'} \to {}_F\,\mathrm{BMod}_{F'}$$

corresponding to the relative tensor product. Furthermore, as THH preserves the sifted colimits corresponding to this relative tensor product, the above equivalence is compatible with right $\mathrm{THH}(F')$ module structures. The analogous claims all hold when accounting for additional gradings, by recalling that THH promotes to a sifted colimit preserving symmetric monoidal functor from algebras in graded spectra to S^1-equivariant objects in graded spectra. In particular, if E and F are concentrated in weight 0, we deduce the equivalence

$$\mathrm{THH}(F \wedge_E F')_i \simeq \mathrm{THH}(F) \wedge_{\mathrm{THH}(E)} (\mathrm{THH}(F')_i)$$

of graded $\mathrm{THH}(F)$-modules. □

Remark 4.10. The $m = 1$ case of the proposition above provides the non-graded case.

One may consider $\mathbb{S}[\sigma_k]$ as an E_1-ring obtained by adjoining an m-root to $\mathbb{S}[\sigma_{mk}]$. Proposition 4.12 identifies the weight-0 piece of $\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])$. Before Proposition 4.12, we state and prove a well-known fact.

Lemma 4.11. *Let* $\varphi\colon M \to N$ *be a map between bounded-below spectra. Then* φ *is an equivalence if and only if* $H\mathbb{Z} \wedge \varphi$ *is an equivalence. If, furthermore,* M *and* N *are* p*-local, then* φ *is an equivalence if and only if* $H\mathbb{Z}_{(p)} \wedge \varphi$ *is an equivalence.*

Proof. Let K be the fiber of φ and let i be the smallest i such that $\pi_i K \neq 0$. Owing to the Tor spectral sequence of [EKM+97, Theorem IV.4.1], we have $\pi_i(H\mathbb{Z} \wedge K) = \pi_i K$. Therefore, if $H\mathbb{Z} \wedge K \simeq 0$, then $K \simeq 0$ and φ is an equivalence.

If M and N are p-local, then φ is an equivalence if and only if $\mathbb{S}_{(p)} \wedge \varphi$ is an equivalence. It follows by the previous result that φ is an equivalence if and only if $H\mathbb{Z} \wedge \mathbb{S}_{(p)} \wedge \varphi \simeq H\mathbb{Z}_{(p)} \wedge \varphi$ is an equivalence. □

For the following, let $k \geq 0$ be even and let $m > 1$. Furthermore, fix a map of m-graded E_2-rings $\mathbb{S}_{(p)}[\sigma_{mk}] \to \mathbb{S}_{(p)}[\sigma_k]$ provided by Proposition 3.5 (Remark 3.6 for $k = 0$).

Proposition 4.12. *In the situation above, assume that* $p \nmid m$*. The induced map*

$$\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}])_0 \xrightarrow{\simeq} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])_0$$

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is an equivalence of E_1-rings. Since $\mathbb{S}_{(p)}[\sigma_{mk}]$ is concentrated in weight 0, we obtain the following chain of equivalences:

$$D(\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}])) \simeq \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}])_0 \xrightarrow{\simeq} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])_0$$

of E_1-rings using Lemma 2.3.

Proof. It suffices to prove that the map

$$H\mathbb{Z}_{(p)} \wedge \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}])_0 \to H\mathbb{Z}_{(p)} \wedge \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])_0$$

is an equivalence; see Lemma 4.11.

By the base-change formula for THH, this is equivalent to the following map,

$$\mathrm{THH}^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_{mk}]) \to \mathrm{THH}^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_k]),$$

being an equivalence in weight 0. Here, $H\mathbb{Z}_{(p)}[\sigma_k]$ denotes the free $H\mathbb{Z}_{(p)}$-algebra on $\mathbb{S}^k_{(p)}$ given by $H\mathbb{Z}_{(p)} \wedge \mathbb{S}_{(p)}[\sigma_k]$.

The map $H\mathbb{Z}_{(p)}[\sigma_{mk}] \to H\mathbb{Z}_{(p)}[\sigma_k]$ induces a map

$$\phi^r \colon E^r \to F^r$$

from the Bökstedt spectral sequence computing $\mathrm{THH}^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_{mk}])$ to the Bökstedt spectral sequence computing $\mathrm{THH}^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_k])$. Since the weight grading on the THH of an m-graded ring spectrum comes from a weight grading on the corresponding cyclic bar construction, the Bökstedt spectral sequence computing THH of an m-graded ring spectrum admits an m-grading, i.e. it splits into m summands in a canonical way. Therefore, in our situation, it is sufficient to show that ϕ^2 is an isomorphism on weight 0.

We have

$$\phi^2 \colon \mathbb{Z}_{(p)}[\sigma_{mk}] \otimes \Lambda_{\mathbb{Z}_{(p)}}(d(\sigma_{mk})) \to \mathbb{Z}_{(p)}[\sigma_k] \otimes \Lambda_{\mathbb{Z}_{(p)}}(d(\sigma_k)), \tag{4.6}$$

where d denotes the Connes operator. The degrees of the classes above are given by the following:

$$\deg(\sigma_{mk}) = (0, mk), \quad \deg(d(\sigma_{mk})) = (1, mk),$$
$$\deg(\sigma_k) = (0, k), \quad \deg(d(\sigma_k)) = (1, k).$$

Furthermore, σ_{mk} and $d(\sigma_{mk})$ are in weight 0 and σ_k and $d(\sigma_k)$ are in weight 1. In particular, all of E^2 is weight 0 and the weight-0 piece of F^2 is the $\mathbb{Z}_{(p)}$-module generated by the classes σ_k^{im} and $\sigma_k^{(i+1)m-1}d(\sigma_k)$ over $i \geq 0$.

Since $\phi^2(\sigma_{mk}) = \sigma_k^m$, we obtain that

$$\phi^2(d(\sigma_{mk})) = d(\phi^2(\sigma_{mk})) = d(\sigma_k^m) = m\sigma_k^{m-1}d(\sigma_k).$$

Therefore, we have

$$\phi^2(\sigma_{mk}^i) = \sigma_k^{im} \quad \text{and} \quad \phi^2(\sigma_{mk}^i d(\sigma_{mk})) = m\sigma_k^{(i+1)m-1}d(\sigma_k).$$

Since $p \nmid m$, we have that m is a unit. Using this, one observes that ϕ^2 is an isomorphism after restricting and corestricting to weight 0 as desired. □

In the situation of Hypothesis 4.1, $A \to A(\sqrt[m]{a})$ is a map of m-graded E_1-rings and A is concentrated in weight 0. Therefore, there is a map

$$\mathrm{THH}(A) \to \mathrm{THH}(A(\sqrt[m]{a}))_0, \tag{4.7}$$

where A above denotes the underlying E_1-ring of A. This map is S^1-equivariant since it is the restriction of the S^1-equivariant m-graded map $\mathrm{THH}(A) \to \mathrm{THH}(A(\sqrt[m]{a}))$ along $0 \to \mathbb{Z}/m$ as $\mathrm{THH}(A)$ is concentrated in weight 0.

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Theorem 4.13. *Assume Hypothesis 4.1 and that A is p-local and $p \nmid m$. Then (4.7) is an equivalence.*

Proof. Recall that $A(\sqrt[m]{a})$ is given by

$$A \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k],$$

where A and $\mathbb{S}_{(p)}[\sigma_{mk}]$ are concentrated in weight 0. Owing to Proposition 4.9, we have

$$\mathrm{THH}(A(\sqrt[m]{a}))_0 \simeq \mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}])} (\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])_0)$$

and it follows by Proposition 4.12 that the map

$$\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}]) \xrightarrow{\simeq} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])_0$$

is an equivalence. This identifies $\mathrm{THH}(A(\sqrt[m]{a}))_0$ with $\mathrm{THH}(A)$ as desired. □

Remark 4.14. It should be possible to relax the hypothesis on A that it is p-local by assuming that m is invertible in $\pi_0 A$. Our results assuming A is p-local and $p \nmid m$ should be true under this weaker assumption as well, but we do not pursue this here. We would like to thank the anonymous referee for pointing this out.

5. Adjoining roots and algebraic K-theory

We now prove Theorem 1.1 from the introduction. For the rest of this section, assume Hypothesis 4.1. We established that $A(\sqrt[m]{a})$ is an m-graded ring spectrum and therefore $\mathrm{THH}(A(\sqrt[m]{a}))$ is an S^1-equivariant m-graded spectrum; see (4.5). One might define $\mathrm{TC}^-(A(\sqrt[m]{a}))$ as an m-graded spectrum given by

$$\mathrm{TC}^-(A(\sqrt[m]{a}))_i \simeq \mathrm{THH}(A(\sqrt[m]{a}))_i^{hS^1}.$$

Since m is finite, the underlying spectrum of an m-graded spectrum, provided by the functor D, is given by a finite coproduct which is equivalent to the corresponding finite product. In particular, D commutes with all limits and colimits. Because of this, we have

$$D(\mathrm{TC}^-(A(\sqrt[m]{a}))) \simeq \mathrm{TC}^-(D(A(\sqrt[m]{a})))$$

and, therefore, we often omit D in our notation.

Similarly, $\mathrm{TP}(A(\sqrt[m]{a}))$ and $(\mathrm{THH}(A(\sqrt[m]{a}))^{tC_p})^{hS^1}$ admit the structure of m-graded spectra and these constructions commute with the functor D as above. Combining this with Theorem 4.13, we obtain the following result.

Theorem 5.1. *Assume Hypothesis 4.1, that A is p-local and $p \nmid m$. The maps*

$$\mathrm{TC}^-(A) \xrightarrow{\simeq} \mathrm{TC}^-(A(\sqrt[m]{a}))_0,$$
$$\mathrm{TP}(A) \xrightarrow{\simeq} \mathrm{TP}(A(\sqrt[m]{a}))_0,$$
$$(\mathrm{THH}(A)^{tC_p})^{hS^1} \xrightarrow{\simeq} ((\mathrm{THH}(A(\sqrt[m]{a}))^{tC_p})^{hS^1})_0$$

induced by $A \to A(\sqrt[m]{a})$ are all equivalences.

When A is connective and p-local, $A(\sqrt[m]{a})$ is also connective and p-local. By [NS18, Corollary 1.5] (and the discussion afterwards), the topological cyclic homology of $A(\sqrt[m]{a})$ is defined via the following fiber sequence:

$$\mathrm{TC}(A(\sqrt[m]{a})) \to \mathrm{THH}(A(\sqrt[m]{a}))^{hS^1} \xrightarrow{\varphi_p^{hS^1} - can} (\mathrm{THH}(A(\sqrt[m]{a}))^{tC_p})^{hS^1}. \tag{5.1}$$

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As mentioned previously, the middle term and the third term above admit canonical splittings into m-cofactors. Furthermore, *can* respects this splitting since it only depends on the S^1-equivariant structure of $\mathrm{THH}(A(\sqrt[m]{a}))$.

However, $\mathrm{TC}(A(\sqrt[m]{a}))$ do not necessarily split into m-cofactors. This is due to the fact that the Frobenius map does not necessarily respect the grading. Indeed, the Frobenius is given by maps

$$\varphi_p\colon\ \mathrm{THH}(A(\sqrt[m]{a}))_i \to \mathrm{THH}(A(\sqrt[m]{a}))_{ip}^{tC_p},$$

see [AMM+22, Corollary A.9]. On the other hand, we obtain the following splitting of $\mathrm{TC}(A(\sqrt[m]{a}))$.

Construction 5.2. Assume Hypothesis 4.1 and that A is connective and p-local where $p\nmid m$. In this situation, p is a non-zero divisor in $\mathbb{Z}/m$. Therefore, the Frobenius map on $\mathrm{THH}(A(\sqrt[m]{a}))$ carries pieces of non-zero weight to non-zero weight pieces. Moreover, φ_p carries zero weight to zero weight. Therefore, the map $\varphi_p - can$ splits as a coproduct of their restriction to zero weight and their restriction to non-zero weight. In particular, the fiber sequence in (5.1) admits a splitting as follows:

$$\begin{aligned}\mathrm{TC}(A(\sqrt[m]{a}))_0 \vee \mathrm{TC}(A(\sqrt[m]{a}))_1 &\to \mathrm{THH}(A(\sqrt[m]{a}))_0^{hS^1} \vee \mathrm{THH}(A(\sqrt[m]{a}))_{>0}^{hS^1}\\ &\xrightarrow{(\varphi_p)_0 - can_0 \vee (\varphi_p)_{>0} - can_{>0}} (\mathrm{THH}(A(\sqrt[m]{a}))_0^{tC_p})^{hS^1} \vee (\mathrm{THH}(A(\sqrt[m]{a}))^{tC_p})_{>0}^{hS^1}.\end{aligned}$$

Here, $(-)_{>0}$ denotes restriction to weight $\neq 0$. We have

$$\mathrm{TC}(A(\sqrt[m]{a})) \simeq \mathrm{TC}(A(\sqrt[m]{a}))_0 \vee \mathrm{TC}(A(\sqrt[m]{a}))_1, \tag{5.2}$$

where $\mathrm{TC}(A(\sqrt[m]{a}))_0$ denotes the fiber of the map $(\varphi_p)_0 - can_0$ and $\mathrm{TC}(A(\sqrt[m]{a}))_1$ denotes the fiber of the map $(\varphi_p)_{>0} - can_{>0}$.

Remark 5.3. There are interesting cases where one obtains further splittings of the topological cyclic homology spectrum $\mathrm{TC}(A(\sqrt[m]{a}))$. For instance, if $p=1$ in $\mathbb{Z}/m$, one obtains that $\mathrm{TC}(A(\sqrt[m]{a}))$ splits into m-summands. This happens to be the case when $m=p-1$ or when p is odd and $m=2$. We exploit this in Construction 7.3 to obtain a splitting of $\mathrm{TC}(ku_p)$ into $p-1$ summands. Moreover, if $m=p^n-1$, then one obtains an underlying $p-1$-grading of $\mathrm{THH}(A(\sqrt[m]{a}))$ by Kan extending through $\mathbb{Z}/(p^n-1)\to\mathbb{Z}/(p-1)$. This provides a $p-1$-grading for $\mathrm{TC}(A(\sqrt[m]{a}))$.

Theorem 5.4. *Assume Hypothesis 4.1 with $p\nmid m$ and that A is p-local and connective. Under the equivalence (5.2), the canonical map*

$$\mathrm{TC}(A) \to \mathrm{TC}(A(\sqrt[m]{a}))$$

is equivalent to the inclusion of the first wedge summand.

Proof. Since A is concentrated in weight 0, the map $\mathrm{TC}(A)\to\mathrm{TC}(A(\sqrt[m]{a}))$ factors through the map

$$\mathrm{TC}(A) \to \mathrm{TC}(A(\sqrt[m]{a}))_0 \tag{5.3}$$

induced by the canonical map $\mathrm{THH}(A)\to\mathrm{THH}(A(\sqrt[m]{a}))_0$. The map $\mathrm{THH}(A)\to\mathrm{THH}(A(\sqrt[m]{a}))_0$ of cyclotomic spectra is an equivalence due to Theorem 4.13. Considering the construction of $\mathrm{TC}(A(\sqrt[m]{a}))_0$, one observes that this equivalence induces an equivalence between the fiber sequences defining $\mathrm{TC}(A)$ and $\mathrm{TC}(A(\sqrt[m]{a}))_0$. In other words, (5.3) is an equivalence as desired. □

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Finally, we obtain the desired splitting for $K(A(\sqrt[m]{a}))$.

Theorem 5.5 (Theorem 1.1). *Assume Hypothesis 4.1 with $p \nmid m$ and $k > 0$. Furthermore, assume that A is p-local and connective. In this situation, the following map:*

$$K(A) \to K(A(\sqrt[m]{a}))$$

is the inclusion of a wedge summand.

Proof. Since $|a| = mk$ and $k > 0$, we have

$$\pi_0 A(\sqrt[m]{a}) = \pi_0 A. \tag{5.4}$$

We start by constructing a map of m-graded E_1-algebras

$$A(\sqrt[m]{a}) \to H\pi_0 A \tag{5.5}$$

that induces an isomorphism on π_0, where $H\pi_0 A$ is concentrated in weight 0. Weight-0 Postnikov truncation [HW22, Lemma B.0.6] provides a map of graded E_2-rings $\mathbb{S}[\sigma_k] \to \mathbb{S}$ that we consider as a map of m-graded E_2-rings by left Kan extending through $\mathbb{Z} \to \mathbb{Z}/m$.

This provides a map of m-graded E_1-rings

$$A(\sqrt[m]{a}) \simeq A \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}[\sigma_k] \to A \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}$$

(see (4.3)), where the right-hand side is concentrated in weight 0. Postcomposing with the ordinary Postnikov truncation, we obtain (5.5).

Owing to the Dundas–Goodwillie–McCarthy theorem [DGM13], there is the following pullback square provided by the map (5.5).

$$\begin{array}{ccc} K(A(\sqrt[m]{a})) & \longrightarrow & \mathrm{TC}(A(\sqrt[m]{a})) \simeq \mathrm{TC}(A) \vee \mathrm{TC}(A(\sqrt[m]{a}))_1 \\ \downarrow & & \downarrow \\ K(H\pi_0 A) & \longrightarrow & \mathrm{TC}(H\pi_0 A) \end{array}$$

The equivalence in the upper-right corner follows by Construction 5.2 and Theorem 5.4.

The map $A(\sqrt[m]{a}) \to H\pi_0 A$ induces a map of m-graded spectra

$$f\colon \mathrm{THH}(A(\sqrt[m]{a})) \to \mathrm{THH}(H\pi_0 A).$$

Since $H\pi_0 A$ is concentrated in weight 0, $\mathrm{THH}(H\pi_0 A)$ is also concentrated in weight 0. Therefore, the map f is trivial on $\mathrm{THH}(A(\sqrt[m]{a}))_{>0}$. This shows that the right vertical map above induces the trivial map on $\mathrm{TC}(A(\sqrt[m]{a}))_1$. Using this, we obtain that the pullback square above splits as a coproduct of the following pullback squares.

$$\begin{array}{ccc} K(A) & \longrightarrow & \mathrm{TC}(A) \\ \downarrow & & \downarrow \\ K(H\pi_0 A) & \longrightarrow & \mathrm{TC}(H\pi_0 A) \end{array}$$

$$\begin{array}{ccc} \mathrm{TC}(A(\sqrt[m]{a}))_1 & \xrightarrow{\simeq} & \mathrm{TC}(A(\sqrt[m]{a}))_1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

By the naturality of the pullback squares provided by the Dundas–Goodwillie–McCarthy theorem and Theorem 5.4, we see that the inclusion of the first cofactor in the coproduct of pullback

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squares above is indeed induced by the map $A \to A(\sqrt[m]{a})$. This shows that

$$K(A(\sqrt[m]{a})) \simeq K(A) \vee \mathrm{TC}(A(\sqrt[m]{a}))_1$$

as desired, where the inclusion of the first cofactor above is given by $K(A) \to K(A(\sqrt[m]{a}))$. □

Using the purity theorem for algebraic K-theory, we obtain the following for non-connective A.

COROLLARY 5.6. *Assume Hypothesis 4.1 with $p \nmid m$ and $k > 0$. If A is p-local, then the map*

$$L_{T(i)}K(A) \to L_{T(i)}K(A(\sqrt[m]{a}))$$

is the inclusion of a wedge summand for every $i \geq 2$. In particular, if A is of height larger than 0 and A satisfies the redshift conjecture, then $A(\sqrt[m]{a})$ also satisfies the redshift conjecture.

Proof. Let cA denote the connective cover of A in $\mathbb{S}[\sigma_{mk}]$-algebras. We consider the following commuting diagram of m-graded E_1-rings.

$$\begin{array}{ccc} cA & \longrightarrow & A \\ \downarrow & & \downarrow \\ (cA)(\sqrt[m]{a}) & \longrightarrow & A(\sqrt[m]{a}) \end{array}$$

Every spectrum with bounded-above homotopy is $T(i)$-locally trivial for every $i \geq 1$. Taking fibers, one obtains that the horizontal arrows above are $T(i)$-equivalence for every $i \geq 1$; see (4.4).

It follows by [LMM+24, Corollary E] that the horizontal maps above induce $T(i)$-equivalences in algebraic K-theory for every $i \geq 2$. The result follows by applying Theorem 5.5 to the left vertical map. □

6. A variant of logarithmic THH

Here, we introduce our definition of $\log \mathrm{THH}$ and identify $\mathrm{THH}(A(\sqrt[m]{a}))$ using $\mathrm{THH}(A)$ and $\log \mathrm{THH}$ of A whenever A is p-local and $p \nmid m$. Through our definition, $\log \mathrm{THH}$ admits a canonical structure of a cyclotomic spectrum; in upcoming work, Devalapurkar and the third author develop a very general notion of logarithmic structures for E_2-algebras and a corresponding theory of $\log \mathrm{THH}$ which subsumes the definition we use here. In particular, this will recover the variant due to Rognes, which is defined by way of the replete bar construction; cf. [Rog09, RSS15]

Our definition of $\log \mathrm{THH}$ starts with a definition of $\log \mathrm{THH}$ of the free algebra $\mathbb{S}[\sigma_k]$, where $k \geq 0$ is even as before. We consider σ_k to be in weight 1.

For a graded E_n-ring spectrum E, we denote the *weight connective cover* of E by $E_{\geq 0}$. Indeed, the weight connective cover is obtained by restricting and then left Kan extending through the inclusion $\mathbb{N} \to \mathbb{Z}$. The counit of this adjunction provides a map $E_{\geq 0} \to E$ of graded E_n-algebras.

CONSTRUCTION 6.1. Analogous to Construction 3.3, let $\mathbb{S}[\sigma_k^{\pm 1}] := \mathrm{sh}^k(\mathbb{S}[t^{\pm 1}])$. The graded E_∞-map $\mathbb{S}[t] \to \mathbb{S}[t^{\pm 1}]$ provides a graded E_2-map $\mathbb{S}[\sigma_k] \to \mathbb{S}[\sigma_k^{\pm 1}]$. Furthermore, by the definition of the shearing functor, $\mathbb{S}[\sigma_k^{\pm 1}]$ is indeed given by ϕ^k of Variant 3.2; in particular, $\mathbb{S}[\sigma_{mk}^{\pm 1}]$ is the restriction of $\mathbb{S}[\sigma_k^{\pm 1}]$ along $\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z}$. Applying the adjunction $L_m \dashv R_m$ induced by $\cdot m$ to the map $\mathbb{S}[\sigma_{mk}] \to \mathbb{S}[\sigma_k]$, one observes that $\mathbb{S}[\sigma_{mk}]$ is also the restriction of $\mathbb{S}[\sigma_k]$ along $\cdot m$. Therefore,

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the counit of $L_m \dashv R_m$ provides a commutative diagram of graded E_2-rings.

$$\begin{array}{ccc} \mathbb{S}[\sigma_{mk}] & \longrightarrow & \mathbb{S}[\sigma_{mk}^{\pm 1}] \\ \downarrow & & \downarrow \\ \mathbb{S}[\sigma_k] & \longrightarrow & \mathbb{S}[\sigma_k^{\pm 1}] \end{array}$$

Remark 6.2. One may also take weight connective covers in the ∞-category of graded S^1-equivariant spectra by using the left Kan extension/restriction adjunction induced by $\mathbb{N}^{ds} \to \mathbb{Z}$. This provides a map

$$\mathrm{THH}(\mathbb{S}[\sigma_k]) \to \mathrm{THH}(\mathbb{S}[\sigma_k^{\pm 1}])_{\geq 0}$$

of graded E_1-algebras in S^1-equivariant spectra factoring the map $\mathrm{THH}(\mathbb{S}[\sigma_k]) \to \mathrm{THH}(\mathbb{S}[\sigma_k^{\pm 1}])$.

The following is analogous to the description of the replete bar construction of commutative $\mathcal{J}$-space monoids generated by a single element; cf. [Rog09, Proposition 3.21], [SS19, § 8.5] and [RSS15, §§ 6 and 7].

Definition 6.3. Let $k \geq 0$ be even. The log THH of $\mathbb{S}[\sigma_k]$ with respect to $\sigma_k \in \pi_k \mathbb{S}[\sigma_k]$ is the weight connective cover of the THH of $\mathbb{S}[\sigma_k{}^{\pm 1}]$. In other words, it is the S^1-equivariant E_1-algebra

$$\mathrm{THH}(\mathbb{S}[\sigma_k] \,|\, \sigma_k) := \mathrm{THH}(\mathbb{S}[\sigma_k^{\pm 1}])_{\geq 0}.$$

Similarly, the p-local counterpart is defined as follows:

$$\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \,|\, \sigma_k) := \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k^{\pm 1}])_{\geq 0}.$$

The following example provides a justification for this definition of log THH by showing that its $H\mathbb{Z}$-homology provides what should be the logarithmic Hochschild homology of the free algebra $\mathbb{Z}[\sigma_k]$; cf. [KN22, Example 10.3].

Example 6.4. Considering $H\mathbb{Z}$ as a graded E_∞-algebra concentrated in weight 0, we deduce that

$$H\mathbb{Z} \wedge (\mathrm{THH}(\mathbb{S}[\sigma_k^{\pm 1}])_{\geq 0}) \simeq (H\mathbb{Z} \wedge \mathrm{THH}(\mathbb{S}[\sigma_k^{\pm 1}]))_{\geq 0}.$$

Therefore, $H\mathbb{Z}_* \,\mathrm{THH}(\mathbb{S}[\sigma_k] \,|\, \sigma_k)$ is given by the weight connective cover of

$$\mathrm{THH}_*^{H\mathbb{Z}}(H\mathbb{Z}[\sigma_k^{\pm 1}]) \cong \mathbb{Z}[\sigma_k^{\pm 1}] \otimes \Lambda(d\sigma_k), \tag{6.1}$$

where $d\sigma_k$ is of weight 1 and degree $k+1$ and σ_k is of weight 1 and degree k. The isomorphism above follows by the usual Bökstedt spectral sequence considerations applied together with the Hochschild–Kostant–Rosenberg (HKR) theorem. Taking the weight connective cover of (6.1), we obtain

$$H\mathbb{Z}_* \,\mathrm{THH}(\mathbb{S}[\sigma_k] \,|\, \sigma_k) \cong \mathbb{Z}[\sigma_k] \otimes \Lambda(d \log \sigma_k),$$

where $d \log \sigma_k$ is of weight 0 and homotopical degree 1 and corresponds to $(d\sigma_k)/\sigma_k$. Furthermore, the map

$$H\mathbb{Z}_* \,\mathrm{THH}(\mathbb{S}[\sigma_k]) \to H\mathbb{Z}_* \,\mathrm{THH}(\mathbb{S}[\sigma_k] \,|\, \sigma_k)$$

carries $d\sigma_k$ to $d\sigma_k = \sigma_k d \log \sigma_k$.

Recall from Construction 4.8 that when A is a $\mathbb{S}[\sigma_k]$-algebra, $\mathrm{THH}(A)$ admits the structure of a right $\mathrm{THH}(\mathbb{S}[\sigma_k])$-module. We use this structure in the following definition. Recall that Proposition 4.2 provides various cases of interest where the assumptions on A in the following definition are satisfied.

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Definition 6.5. Let A be an E_1 $\mathbb{S}[\sigma_k]$-algebra and assume that the unit map $\mathbb{S}[\sigma_k] \to A$ carries $\sigma_k \in \pi_k \mathbb{S}[\sigma_k]$ to $a \in \pi_k A$ with even $k \geq 0$. We define the log THH of A with respect to a as the following S^1-equivariant spectrum:

$$\mathrm{THH}(A \mid a) := \mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_k])} \mathrm{THH}(\mathbb{S}[\sigma_k] \mid \sigma_k).$$

If A is assumed to be p-local, we use the following equivalent definition:

$$\mathrm{THH}(A \mid a) := \mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k).$$

The definition of log THH we provide above is analogous to the definitions used in [Rog09, SS19, RSS15].

Remark 6.6. We remark that $\mathrm{THH}(\mathbb{S}[\sigma_k] \mid \sigma_k)$ should be a cyclotomic spectrum as the Frobenius maps of THH multiply the weight by p and this should provide $\mathrm{THH}(A \mid a)$ above with the structure of a cyclotomic spectrum. However, since we don't explicitly need this for our application, displaying this will take us too far afield, and so we leave the details to the future work of Devalapurkar and the third author.

Since the definition of log THH is given by the extension of scalars functor

$$- \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_k])} \mathrm{THH}(\mathbb{S}[\sigma_k] \mid \sigma_k) \colon \mathrm{RMod}_{\mathrm{THH}(\mathbb{S}[\sigma_k])} \to \mathrm{RMod}_{\mathrm{THH}(\mathbb{S}[\sigma_k] \mid \sigma_k)}$$

corresponding to the E_1-algebra map $\mathrm{THH}(\mathbb{S}[\sigma_k]) \to \mathrm{THH}(\mathbb{S}[\sigma_k] \mid \sigma_k)$, we deduce that $\mathrm{THH}(A \mid a)$ is equipped with the structure of a right $\mathrm{THH}(\mathbb{S}[\sigma_k] \mid \sigma_k)$-module. Furthermore, the unit of the adjunction given by the extension of scalars functor above and the corresponding forgetful functor provides a map

$$\mathrm{THH}(A) \to \mathrm{THH}(A \mid a) \tag{6.2}$$

of right $\mathrm{THH}(\mathbb{S}[\sigma_k])$-modules; see (4.3).

Remark 6.7. Using $MU[\sigma_k]$ mentioned in Remark 4.7, it is possible to equip log THH with the structure of an E_n-algebra for $n > 0$ in favorable cases. For instance, for the E_3 $MU[\sigma_{2(p^n-1)}]$-algebra form of $BP\langle n \rangle$ constructed in [HW22], $\mathrm{THH}(BP\langle n \rangle \mid v_n)$ admits the structure of an E_1-ring. Indeed, using the map of E_2-rings $\mathrm{THH}(MU[\sigma_{2(p^n-1)}]) \to \mathrm{THH}(BP\langle n \rangle)$, we obtain an E_1-ring

$$\mathrm{THH}(BP\langle n \rangle) \wedge_{\mathrm{THH}(MU[\sigma_{2(p^n-1)}])} \mathrm{THH}(MU[\sigma_{2(p^n-1)}^{\pm 1}])_{\geq 0},$$

equivalent to $\mathrm{THH}(BP\langle n \rangle \mid v_n)$. This equivalence follows by the following chain of equivalences:

$$\begin{aligned}
&\mathrm{THH}(BP\langle n \rangle) \wedge_{\mathrm{THH}(MU[\sigma_{2(p^n-1)}])} \mathrm{THH}(MU[\sigma_{2(p^n-1)}^{\pm 1}])_{\geq 0} \\
&\quad \simeq \mathrm{THH}(BP\langle n \rangle) \wedge_{\mathrm{THH}(MU) \wedge \mathrm{THH}(\mathbb{S}[\sigma_{2(p^n-1)}])} \mathrm{THH}(MU) \wedge \mathrm{THH}(\mathbb{S}[\sigma_{2(p^n-1)}^{\pm 1}])_{\geq 0} \\
&\quad \simeq \mathrm{THH}(BP\langle n \rangle) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_{2(p^n-1)}])} \mathrm{THH}(\mathbb{S}[\sigma_{2(p^n-1)}]),
\end{aligned}$$

obtained from the equivalence of E_2 MU-algebras $MU[\sigma_{2(p^n-1)}] \simeq MU \wedge \mathbb{S}[\sigma_{2(p^n-1)}]$ mentioned in Remark 4.7.

Furthermore, Hahn and Yuan [HY20, 1.11 and 1.12] show that there is an E_∞-map $MU[\sigma_2] \to ku_p$ for $p = 2$ and claim that their methods provide such a map for odd primes too. In this situation, $\mathrm{THH}(ku_p \mid u_2)$ is equipped with the structure of an E_∞-ring (by arguing as above) where u_2 denotes the Bott element. Note that log THH of ku_p with respect to u_2 is also constructed as an E_∞-ring in [RSS18].

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Remark 6.8. In work in progress, Devalapurkar and the second author show that, in a general context, for every E_2-ring with even homotopy, log THH, as in our definition, may be equipped with a canonical E_1-algebra structure in cyclotomic spectra.

To study the log THH of E_1-rings obtained via root adjunctions, we use the following constructions.

Construction 6.9. Using Construction 6.1 and the weight connective cover adjunction mentioned in Remark 6.2, we obtain the following commuting diagram of graded E_1-algebras.

$$\begin{array}{ccccc} \mathbb{S}[\sigma_{mk}] & \longrightarrow & \mathrm{THH}(\mathbb{S}[\sigma_{mk}]) & \longrightarrow & \mathrm{THH}(\mathbb{S}[\sigma_{mk}] \,|\, \sigma_{mk}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{S}[\sigma_k] & \longrightarrow & \mathrm{THH}(\mathbb{S}[\sigma_k]) & \longrightarrow & \mathrm{THH}(\mathbb{S}[\sigma_k] \,|\, \sigma_k) \end{array} \tag{6.3}$$

Construction 6.10. Assume Hypothesis 4.1. By Proposition 4.9, there is an equivalence

$$\mathrm{THH}(A(\sqrt[m]{a})) \simeq \mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_{mk}])} \mathrm{THH}(\mathbb{S}[\sigma_k]).$$

This equips $\mathrm{THH}(A(\sqrt[m]{a}))$ with the structure of a right $\mathrm{THH}(\mathbb{S}[\sigma_k])$-module in m-graded spectra. Considering the map

$$\mathrm{THH}(\mathbb{S}[\sigma_k]) \to \mathrm{THH}(\mathbb{S}[\sigma_k] \,|\, \mathbb{S}[\sigma_k])$$

as a map of m-graded E_1-ring spectra, Definition 6.5 may be employed at the level of m-graded spectra. This shows that $\mathrm{THH}(A(\sqrt[m]{a}) \,|\, \sqrt[m]{a})$ admits a canonical structure of a right $\mathrm{THH}(\mathbb{S}[\sigma_k] \,|\, \sigma_k)$-module in m-graded spectra. Furthermore, the map $\mathrm{THH}(A(\sqrt[m]{a})) \to \mathrm{THH}(A(\sqrt[m]{a}) \,|\, \sqrt[m]{a})$ is a map of m-graded $\mathrm{THH}(\mathbb{S}[\sigma_k])$-modules.

6.1 Logarithmic THH-étale root adjunctions

Here, our goal is to show that when A is p-local and $p \nmid m$, root adjunction is log-THH-étale. In other words, we show that there is an equivalence of m-graded spectra

$$\mathrm{THH}(A \,|\, a) \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k] \simeq \mathrm{THH}(A(\sqrt[m]{a}) \,|\, \sqrt[m]{a}).$$

Remark 6.11. A notion of log-THH-étalenes is already defined in [RSS18]. In the language of Rognes, Sagave and Schlichtkrull [RSS18], log-THH-étaleness of $A \to A(\sqrt[m]{a})$ would be expressed by an equivalence

$$\mathrm{THH}(A(\sqrt[m]{a}) \,|\, \sqrt[m]{a}) \simeq A(\sqrt[m]{a}) \wedge_A \mathrm{THH}(A \,|\, a). \tag{6.4}$$

Since we only assume A to be E_1, $\mathrm{THH}(A \,|\, a)$ may not admit an A-module structure and, therefore, the right-hand side above may not be defined in our generality. On the other hand, if one starts with an E_3-algebra A with even homotopy, the log THH of A may be given an A-module structure and we obtain that $A \to A(\sqrt[m]{a})$ is log-THH-étale in the sense of (6.4) whenever A is p-local and $p \nmid m$.

Proposition 6.12. *For $k \geq 0$, the spectra $\mathrm{THH}(\mathbb{S}[\sigma_k] \,|\, \sigma_k)$ and $\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \,|\, \sigma_k)$ are connective in homotopy.*

Proof. This follows by the fact that the weight connective part of the cyclic bar construction on $\mathbb{S}[\sigma_k^{\pm 1}]$ ($\mathbb{S}_{(p)}[\sigma_k^{\pm 1}]$) is connective in homotopy in each simplicial degree. □

We start by proving a log-THH-étaleness result for the p-localized free E_1-algebra $\mathbb{S}_{(p)}[\sigma_{mk}]$.

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Proposition 6.13. *Let $k \geq 0$ be even and let $m > 0$ with $p \nmid m$. In this situation, there is an equivalence of left $\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk})$-modules in m-graded spectra:*

$$\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk}) \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k] \simeq \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k),$$

where the left $\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk})$-module structure on the right-hand side is provided by Construction 6.9.

Proof. We start by constructing the desired map. First, there is a composite map of m-graded E_1-ring spectra

$$\mathbb{S}_{(p)}[\sigma_k] \to \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k]) \to \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k), \tag{6.5}$$

which is, in particular, a map of left $\mathbb{S}_{(p)}[\sigma_{mk}]$-modules in m-graded spectra by forgetting structure through the m-graded E_1-ring map $\mathbb{S}_{(p)}[\sigma_{mk}] \to \mathbb{S}_{(p)}[\sigma_k]$. Using the extension of scalars functor induced by the map

$$\mathbb{S}_{(p)}[\sigma_{mk}] \to \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk}) \tag{6.6}$$

of m-graded E_1-algebras, we obtain the desired map:

$$f\colon\ \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk}) \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k] \to \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k)$$

of left $\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk})$-modules in m-graded spectra from (6.5). Here, we used the fact that the left $\mathbb{S}_{(p)}[\sigma_{mk}]$-module structure on $\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k)$ used in (6.5) is compatible with the one obtained by forgetting the canonical left $\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk})$-module structure on $\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k)$ through (6.6); this follows by the p-local version of (6.3).

What remains is to show that f is an equivalence. Since f is a map between p-local connective spectra (Proposition 6.12), it is sufficient to show that $H\mathbb{Z}_{(p)} \wedge f$ is an equivalence; see Lemma 4.11.

By inspection on the two-sided bar construction defining relative smash products, one obtains that

$$\begin{aligned} &H\mathbb{Z}_{(p)} \wedge \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk}) \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k] \\ &\quad \simeq (H\mathbb{Z}_{(p)} \wedge \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk})) \wedge_{H\mathbb{Z}_{(p)}[\sigma_{mk}]} H\mathbb{Z}_{(p)}[\sigma_k]. \end{aligned}$$

Using the base-change formula for THH, we obtain that $H\mathbb{Z}_{(p)} \wedge f$ is given by the canonical map

$$H\mathbb{Z}_{(p)} \wedge f\colon\ \mathrm{THH}^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_{mk}^{\pm 1}])_{\geq 0} \wedge_{H\mathbb{Z}_{(p)}[\sigma_{mk}]} H\mathbb{Z}_{(p)}[\sigma_k] \to \mathrm{THH}^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_k^{\pm 1}])_{\geq 0}.$$

To prove that $H\mathbb{Z}_{(p)} \wedge f$ is an equivalence, we argue as in the proof of Proposition 4.12. The map of Bökstedt spectral sequences computing the map

$$\mathrm{THH}_*^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_{mk}^{\pm 1}]) \to \mathrm{THH}_*^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_k^{\pm 1}]) \tag{6.7}$$

is given on the second page, due to the HKR theorem, by the ring map

$$\phi\colon \mathbb{Z}_{(p)}[\sigma_{mk}^{\pm 1}] \otimes \Lambda_{\mathbb{Z}_{(p)}}(d(\sigma_{mk})) \to \mathbb{Z}_{(p)}[\sigma_k^{\pm 1}] \otimes \Lambda_{\mathbb{Z}_{(p)}}(d(\sigma_k)) \tag{6.8}$$

satisfying

$$\phi(\sigma_{mk}) = \sigma_k^m \quad \text{and} \quad \phi(d(\sigma_{mk})) = m\sigma_k^{m-1} d(\sigma_k),$$

where m is a unit as $p \nmid m$. In particular,

$$\phi(\sigma_{mk}^{-1} d(\sigma_{mk})) = \sigma_k^{-m} m \sigma_k^{m-1} d(\sigma_k) = m\sigma_k^{-1} d(\sigma_k). \tag{6.9}$$

Here, σ_{mk} and σ_k are in degrees $(0, mk)$ and $(0, k)$, respectively, and $d(\sigma_{mk})$ and $d(\sigma_k)$ are in degrees $(1, mk)$ and $(1, k)$, respectively. Furthermore, σ_{mk} and $d(\sigma_{mk})$ are of weight m and

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σ_k and $d(\sigma_k)$ are of weight 1. In particular, both Bökstedt spectral sequences degenerate on the second page and the map ϕ provides the map (6.7).

Taking connective covers in weight direction and identifying $\sigma_{mk}^{-1}d(\sigma_{mk})$ as $d\log\sigma_{mk}$ and $\sigma_k^{-1}d(\sigma_k)$ as $d\log\sigma_k$, we obtain that the map

$$\mathrm{THH}_*^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_{mk}^{\pm 1}])_{\geq 0} \to \mathrm{THH}_*^{H\mathbb{Z}_{(p)}}(H\mathbb{Z}_{(p)}[\sigma_k^{\pm 1}])_{\geq 0}$$

is given by a map

$$\mathbb{Z}_{(p)}[\sigma_{mk}] \otimes \Lambda_{\mathbb{Z}_{(p)}}(d\log\sigma_{mk}) \to \mathbb{Z}_{(p)}[\sigma_k] \otimes \Lambda_{\mathbb{Z}_{(p)}}(d\log\sigma_k)$$

that carries σ_{mk} to σ_k^m and $d\log\sigma_{mk}$ to $d\log\sigma_k$ up to a unit due to (6.9) as $p\nmid m$.

Upon extending scalars with respect to the map $\mathbb{Z}_{(p)}[\sigma_{mk}] \to \mathbb{Z}_{(p)}[\sigma_k]$, this map becomes an isomorphism. In other words, $\pi_*(H\mathbb{Z}_{(p)} \wedge f)$ is an isomorphism and, therefore, f is an equivalence. □

The following theorem provides the log-THH-étaleness of root adjunction in ring spectra.

Theorem 6.14. *Assume Hypothesis 4.1 with $p\nmid m$ and that A is p-local. In this situation, there is an equivalence of m-graded spectra*

$$\mathrm{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a}) \simeq \mathrm{THH}(A \mid a) \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k].$$

In other words, as an m-graded spectrum, $\mathrm{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a})$ is given by

$$\mathrm{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a})_i \simeq \Sigma^{ik}\,\mathrm{THH}(A \mid a)$$

for every $0 \leq i < m$.

Proof. We have the following chain of equivalences:

$$\begin{aligned}
&\mathrm{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a}) \simeq \mathrm{THH}(A(\sqrt[m]{a})) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k)\\
&\quad\simeq (\mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}])} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k])} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k)\\
&\quad\simeq \mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}])} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk}) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}]\mid\sigma_{mk})} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k)\\
&\quad\simeq \mathrm{THH}(A \mid a) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}]\mid\sigma_{mk})} \mathrm{THH}(\mathbb{S}_{(p)}[\sigma_k] \mid \sigma_k)\\
&\quad\simeq \mathrm{THH}(A \mid a) \wedge_{\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}]\mid\sigma_{mk})} (\mathrm{THH}(\mathbb{S}_{(p)}[\sigma_{mk}] \mid \sigma_{mk}) \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k])\\
&\quad\simeq \mathrm{THH}(A \mid a) \wedge_{\mathbb{S}_{(p)}[\sigma_{mk}]} \mathbb{S}_{(p)}[\sigma_k].
\end{aligned}$$

The first and fourth equivalences follow by the definition of log THH, the second equivalence follows by our definition of root adjunction and Proposition 4.9 and the fifth equivalence follows by Proposition 6.13. □

Remark 6.15. In [RSS18], the authors show that $\ell \to ku_{(p)}$ is log-THH-étale. This compares to our result above, since we show that $ku_p \simeq \ell_p(\sqrt[p-1]{v_1})$ in Theorem 4.6.

6.2 Relating THH and logarithmic THH

The goal of this section is to show that there is a fiber sequence

$$\mathrm{THH}(A) \to \mathrm{THH}(A \mid a) \to \Sigma\,\mathrm{THH}(A/a)$$

under our usual assumptions. The E_1-ring A/a above is described in the following construction, which is analogous to [RSS15, Lemmas 6.14 and 6.15].

Construction 6.16. Let A be an $\mathbb{S}[\sigma_k]$-algebra where σ_k acts through $a \in \pi_k A$, where $k \geq 0$ is even. The weight-0 Postnikov section of $\mathbb{S}[\sigma_k]$ provides a map $\mathbb{S}[\sigma_k] \to \mathbb{S}$ of E_2-rings

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[HW22, B.0.6]. Considering the extension of scalars functor $-\wedge_{\mathbb{S}[\sigma_k]}\mathbb{S}$ from the ∞-category of E_1 $\mathbb{S}[\sigma_k]$-algebras to the ∞-category of E_1 $\mathbb{S}$-algebras, one equips

$$A/a := A\wedge_{\mathbb{S}[\sigma_k]}\mathbb{S}$$

with the structure of an E_1-ring spectrum. Since $\mathbb{S}$ is the cofiber of the map $\Sigma^k\mathbb{S}[\sigma_k]\xrightarrow{\cdot\sigma_k}\mathbb{S}[\sigma_k]$, A/a is indeed the cofiber of the map $\Sigma^k A\xrightarrow{\cdot a} A$.

Considering $\mathbb{S}[\sigma_k]$ as a graded E_2-ring, we have

$$\mathrm{THH}(\mathbb{S}[\sigma_k])_0\simeq\mathbb{S}.$$

This can be observed by inspection of the cyclic bar construction on $\mathbb{S}[\sigma_k]$ or by computing the $H\mathbb{Z}$-homology of the left-hand side above. This is used in the statement of the following proposition.

Remark 6.17. The following proposition is analogous to [RSS15, Proposition 6.11]. We remark that, unlike there, we do not take S^1-equivariance into account, which leads to a simpler proof.

Proposition 6.18. *The cofiber of the map*

$$\mathrm{THH}(\mathbb{S}[\sigma_k])\to\mathrm{THH}(\mathbb{S}[\sigma_k]\,|\,\sigma_k)$$

is given by $\Sigma\mathbb{S}$ *concentrated in weight* 0 *as a left* $\mathrm{THH}(\mathbb{S}[\sigma_k])$*-module in graded spectra. Here, the left* $\mathrm{THH}(\mathbb{S}[\sigma_k])$*-module structure on* $\mathbb{S}$ *is given by the weight-Postnikov truncation map of graded* E_1*-rings*

$$\mathrm{THH}(\mathbb{S}[\sigma_k])\to\mathrm{THH}(\mathbb{S}[\sigma_k])_0\simeq\mathbb{S}.$$

Proof. Let M be the cofiber of the map f below in left $\mathrm{THH}(\mathbb{S}[\sigma_k])$-modules in graded spectra:

$$\mathrm{THH}(\mathbb{S}[\sigma_k])\xrightarrow{f}\mathrm{THH}(\mathbb{S}[\sigma_k],\,|\,\sigma_k)\to M.$$

We start by computing $H\mathbb{Z}_*M$. The map

$$H\mathbb{Z}_*f\colon\ \mathrm{THH}_*^{H\mathbb{Z}}(H\mathbb{Z}[\sigma_k])\to\mathrm{THH}_*^{H\mathbb{Z}}(H\mathbb{Z}[\sigma_k^{\pm1}])_{\geq0}$$

is given by the ring map

$$\mathbb{Z}[\sigma_k]\otimes\Lambda(d(\sigma_k))\to\mathbb{Z}[\sigma_k]\otimes\Lambda(d\log\sigma_k)$$

that carries σ_k to σ_k and $d(\sigma_k)$ to $\sigma_k d\log\sigma_k$; this follows by the Böksedt spectral sequences in (4.6) and (6.8). This map is injective and the only class that is not in the image is $d\log\sigma_k$. We obtain an equivalence of graded $H\mathbb{Z}$-modules

$$H\mathbb{Z}\wedge M\simeq\Sigma H\mathbb{Z},\tag{6.10}$$

where the right-hand side is concentrated in weight 0.

Since M is connective in homotopy (by Proposition 6.12), we deduce that it is concentrated in weight 0 by the equivalence above. Again, since M is connective, the Tor spectral sequence shows that the lowest-degree non-trivial homotopy group of $H\mathbb{Z}\wedge M$ is given by the lowest-degree non-trivial homotopy group of M. By this, and (6.10), there is a map of graded spectra $\Sigma\mathbb{S}\to M$ (with $\Sigma\mathbb{S}$ placed in weight 0), which when smashed with $H\mathbb{Z}$ gives the degree-1 class in $H\mathbb{Z}$ homology. This map

$$\Sigma\mathbb{S}\xrightarrow{\simeq}M$$

is an equivalence (by Lemma 4.11) because one gets an equivalence after smashing with $H\mathbb{Z}$.

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We need to improve this to an equivalence of left $\mathrm{THH}(\mathbb{S}[\sigma_k])$-modules in graded spectra. Since M is a left $\mathrm{THH}(\mathbb{S}[\sigma_k])$-module in graded spectra, there is a map (by adjunction)

$$\Sigma\,\mathrm{THH}(\mathbb{S}[\sigma_k]) \to M$$

of left $\mathrm{THH}(\mathbb{S}[\sigma_k])$-modules in graded spectra carrying 1 to 1 in homotopy. Taking weight-0 Postnikov sections [HW22, B.0.6], we obtain an equivalence of left $\mathrm{THH}(\mathbb{S}[\sigma_k])$-modules in graded spectra

$$\Sigma\,\mathrm{THH}(\mathbb{S}[\sigma_k])_0 \xrightarrow{\simeq} M.$$

This map is an equivalence because it carries 1 to 1 in homotopy by construction and since both sides are equivalent as spectra to $\Sigma\mathbb{S}$. □

We are ready to provide the cofiber sequence relating THH to $\log\mathrm{THH}$.

Theorem 6.19. *Let A be an $\mathbb{S}[\sigma_k]$-algebra where σ_k acts through $a \in \pi_k A$ with even $k \geq 0$. In this situation, there is a cofiber sequence of spectra*

$$\mathrm{THH}(A) \to \mathrm{THH}(A \mid a) \to \Sigma\,\mathrm{THH}(A/a).$$

The corresponding cofiber sequence for $\mathrm{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a})$ is a cofiber sequence of m-graded spectra.

Proof. Proposition 6.18 provides the following cofiber sequence of left $\mathrm{THH}(\mathbb{S}[\sigma_k])$-modules in graded spectra:

$$\mathrm{THH}(\mathbb{S}[\sigma_k]) \to \mathrm{THH}(\mathbb{S}[\sigma_k] \mid \sigma_k) \to \Sigma\mathbb{S}.$$

Applying the functor $\mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_k])} -$ to this cofiber sequence, we obtain the following cofiber sequence:

$$\mathrm{THH}(A) \to \mathrm{THH}(A \mid a) \to \mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_k])} \Sigma\mathbb{S}. \tag{6.11}$$

What is left is to identify the cofiber above as $\mathrm{THH}(A/a)$. We have

$$\mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_k])} \Sigma\mathbb{S} \simeq \Sigma\,\mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_k])} \mathbb{S}. \tag{6.12}$$

Here, $\mathbb{S}$ on the right-hand side denotes the de-suspension of $\Sigma\,\mathrm{THH}(\mathbb{S}[\sigma_k])_0$ as a left $\mathrm{THH}(\mathbb{S}[\sigma_k])$-module in graded spectra; see Proposition 6.18. This is $\mathrm{THH}(\mathbb{S}[\sigma_k])_0$, which admits the structure of a graded E_1-ring spectrum equipped with a map $\mathrm{THH}(\mathbb{S}[\sigma_k]) \to \mathrm{THH}(\mathbb{S}[\sigma_k])_0$ of graded E_1-ring spectra given by the relevant weight-0 Postnikov section map. Indeed, owing to the universal property of Postnikov sections, this weight-0 Postnikov section map factors the map of graded E_1-rings $\mathrm{THH}(\mathbb{S}[\sigma_k]) \to \mathrm{THH}(\mathbb{S})$ induced by the weight-0 Postnikov section map $\mathbb{S}[\sigma_k] \to \mathbb{S}$; i.e. we have a factorization of this map of graded E_1-algebras as

$$\mathrm{THH}(\mathbb{S}[\sigma_k]) \to \mathrm{THH}(\mathbb{S}[\sigma_k])_0 \xrightarrow{\simeq} \mathrm{THH}(\mathbb{S}).$$

The second map above is an equivalence as its domain and codomain are equivalent to $\mathbb{S}$ as spectra and it carries the unit to the unit by construction. In particular, we can replace $\mathbb{S}$ on the right-hand side of (6.12) with $\mathrm{THH}(\mathbb{S})$. This provides the first equivalence below:

$$\begin{aligned}\Sigma\,\mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_k])} \mathbb{S} &\simeq \Sigma\,\mathrm{THH}(A) \wedge_{\mathrm{THH}(\mathbb{S}[\sigma_k])} \mathrm{THH}(\mathbb{S}) \\ &\simeq \Sigma\,\mathrm{THH}(A \wedge_{\mathbb{S}[\sigma_k]} \mathbb{S}) \\ &\simeq \Sigma\,\mathrm{THH}(A/a). \end{aligned} \tag{6.13}$$

The second equivalence follows by Proposition 4.9 and the third equivalence follows by our description of the E_1-algebra A/a in Construction 6.16. Equations (6.12) and (6.13) identify the cofiber in (6.11) as $\mathrm{THH}(A/a)$, providing the cofiber sequence claimed in the theorem.

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The statement regarding the cofiber sequence in m-graded spectra for $A(\sqrt[m]{a})$ follows by utilizing the same arguments. □

Remark 6.20. The above localization sequence is of fundamental importance in the theory of log THH. A proof of the above localization sequence for general E_2 log structures, using more general methods, will be supplied by Devalpurkar and Moulinos (*Logarithmic structures for E_2-algebras*, personal communication, 2022).

6.3 THH after root adjunction

Here, we identify $\mathrm{THH}(A(\sqrt[m]{a}))$ in terms of $\mathrm{THH}(A)$ and $\mathrm{THH}(A \mid a)$.

Theorem 6.21 (Theorem 1.3). *Assume Hypothesis 4.1 with $p \nmid m$ and that A is p-local. In this situation, the m-graded spectrum $\mathrm{THH}(A(\sqrt[m]{a}))$ is given by*

$$\mathrm{THH}(A(\sqrt[m]{a}))_0 \simeq \mathrm{THH}(A)$$

and

$$\mathrm{THH}(A(\sqrt[m]{a}))_i \simeq \Sigma^{ik}\,\mathrm{THH}(A \mid a) \quad \text{for } 0 < i < m.$$

In particular, there is an equivalence of spectra:

$$\mathrm{THH}(A(\sqrt[m]{a})) \simeq \mathrm{THH}(A) \vee \left(\bigvee_{0<i<m} \Sigma^{ik}\,\mathrm{THH}(A \mid a) \right).$$

Proof. The identification of $\mathrm{THH}(A(\sqrt[m]{a}))_0$ is provided by Proposition 4.13. Therefore, it is sufficient to provide the identification of $\mathrm{THH}(A(\sqrt[m]{a}))_i$ for $i \neq 0$.

Owing to Theorem 6.14,

$$\mathrm{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a})_i \simeq \Sigma^{ik}\,\mathrm{THH}(A \mid a). \tag{6.14}$$

Therefore, it is sufficient to show that

$$\mathrm{THH}(A(\sqrt[m]{a}))_i \simeq \mathrm{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a})_i \tag{6.15}$$

whenever $i \neq 0$. This follows once we show that the cofiber of the map

$$\mathrm{THH}(A(\sqrt[m]{a})) \to \mathrm{THH}(A(\sqrt[m]{a}) \mid \sqrt[m]{a}) \tag{6.16}$$

of m-graded spectra is concentrated in weight 0. From Theorem 6.19, the cofiber of this map is given by

$$\mathrm{THH}(A(\sqrt[m]{a})/\sqrt[m]{a}),$$

where $A(\sqrt[m]{a})/\sqrt[m]{a}$ is defined to be $A(\sqrt[m]{a}) \wedge_{\mathbb{S}[\sigma_k]} \mathbb{S}$. Therefore, we have

$$A(\sqrt[m]{a})/\sqrt[m]{a} := A(\sqrt[m]{a}) \wedge_{\mathbb{S}[\sigma_k]} \mathbb{S} \simeq A \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}[\sigma_k] \wedge_{\mathbb{S}[\sigma_k]} \mathbb{S} \simeq A \wedge_{\mathbb{S}[\sigma_{mk}]} \mathbb{S}.$$

Since A, $\mathbb{S}[\sigma_{mk}]$ and $\mathbb{S}$ are concentrated in weight 0, we obtain that $A(\sqrt[m]{a})/\sqrt[m]{a}$ and therefore $\mathrm{THH}(A(\sqrt[m]{a})/\sqrt[m]{a})$ are also concentrated in weight 0. This proves that the cofiber of (6.16) is concentrated in weight 0, which proves (6.15), and this, together with (6.14), proves the theorem. □

7. Algebraic K-theory of complex and real topological K-theories

Here, we start by showing that $K(ku_p)$ splits into $p-1$ non-trivial summands. Afterwards, we show that ku_p may be constructed from ko_p via root adjunction. We use this to obtain an explicit

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description of the $V(1)$-homotopy of $K(ko_p)$ from the first author's computation of $V(1)_*K(ku_p)$ (see [Aus10]).

7.1 Adams' splitting result for 2-vector bundles

Recall that $\pi_* ku_p \cong \mathbb{Z}_p[u]$ and $\pi_* \ell_p \cong \mathbb{Z}_p[v_1]$, where $|u| = 2$ and $|v_1| = 2p-2$. The E_∞-map $\ell_p \to ku_p$ carries v_1 to u^{p-1} in homotopy. For the rest of this section, we fix a map $\mathbb{S}[\sigma_{2(p-1)}] \to \ell_p$ of E_2-algebras carrying $\sigma_{2(p-1)}$ to v_1 and perform root adjunction using this map. Recall from Theorem 4.6 that there is an equivalence of E_1 ℓ_p-algebras

$$\ell_p(\sqrt[p-1]{v_1}) \simeq ku_p.$$

This equips ku_p with the structure of a $p-1$-graded E_1 ℓ_p-algebra, which further equips $\mathrm{THH}(ku_p)$ with the structure of a $p-1$-graded S^1-equivariant spectrum.

Let $p > 3$ be a prime and let $V(1)$ denote the type-2 finite spectrum used in [Aus05]; $V(1)$ is a homotopy ring spectrum.

There is another grading on $V(1)_* \mathrm{THH}(ku_p)$ that the first author calls the δ-grading [Aus05]. The group $\Delta := \mathbb{Z}/(p-1)$ acts on the E_∞-ring ku_p through Adams operations. Let $\delta \in \Delta$ be a chosen generator and let $\alpha \in \mathbb{F}_p^\times$ satisfy $\pi_*(\mathbb{S}/p \wedge \delta)(u) = \alpha u$, where

$$\pi_*(\mathbb{S}/p \wedge \delta)\colon \pi_*(\mathbb{S}/p \wedge ku_p) \to \pi_*(\mathbb{S}/p \wedge ku_p) \cong \mathbb{F}_p[u].$$

We say u^i has δ-weight i as $\pi_*(\mathbb{S}/p \wedge \delta)(u^i) = \alpha^i u^i$. Similarly, one says $x \in V(1)_* \mathrm{THH}(ku_p)$ has δ-weight i if the self map of $V(1)_* \mathrm{THH}(ku_p)$ induced by δ carries x to $\alpha^i x$. One defines δ-weight in a similar way on other invariants of ku_p (see [Aus05, Definition 8.2]).

Proposition 7.1. *The group $V(1)_* \mathrm{THH}(ku_p)_i$ is given by the classes of δ-weight i in $V(1)_* \mathrm{THH}(ku_p)$.*

Proof. Since $H\mathbb{F}_p \wedge ku_p$ is a $p-1$-graded E_1 $H\mathbb{F}_p$-algebra, there is a $p-1$-grading on $\mathrm{HH}^{\mathbb{F}_p}_*(H\mathbb{F}_{p_*} ku_p)$. By inspection on the Hochschild complex, one observes that the δ-weight grading on $\mathrm{HH}^{\mathbb{F}_p}_*(H\mathbb{F}_{p_*} ku_p)$ agrees with the weight grading. In particular, the δ-weight grading and the weight grading agree on the second page of the Bökstedt spectral sequence computing $\mathrm{HH}^{\mathbb{F}_p}(H\mathbb{F}_p \wedge ku_p)$. From [Aus05, Proof of Theorem 9.15], this shows that the δ-weight grading and the weight grading agree on $\mathrm{HH}^{\mathbb{F}_p}_*(H\mathbb{F}_p \wedge ku_p)$. Furthermore, there is a basis of $\mathrm{HH}^{\mathbb{F}_p}_*(H\mathbb{F}_p \wedge ku_p)$ as an $\mathbb{F}_p$-module where δ-weight is defined for each basis element. Therefore, the $H\mathbb{F}_p$-module $\mathrm{HH}^{H\mathbb{F}_p}(H\mathbb{F}_p \wedge ku_p)$ splits as a coproduct of suspensions of $H\mathbb{F}_p$ in such a way that the map $\mathrm{HH}^{H\mathbb{F}_p}(H\mathbb{F}_p \wedge \delta)$ is given by the respective multiplication map corresponding to the δ-weight on each cofactor. Using this, one observes that the δ-weight and the weight grading agree on $H_*(V(1) \wedge \mathrm{THH}(ku_p); \mathbb{F}_p)$.

The Hurewicz map

$$V(1)_* \mathrm{THH}(ku_p) \to H_*(V(1) \wedge \mathrm{THH}(ku_p); \mathbb{F}_p)$$

is injective and this map preserves both gradings. From this, we deduce that the weight grading and the δ-weight grading agree on $V(1)_* \mathrm{THH}(ku_p)$. □

In general, THH of m-graded ring spectra may not result in an m-graded cyclotomic spectrum as the Frobenius map does not preserve the grading; it multiplies the grading by p. On the other hand, for ku_p, $\mathrm{THH}(ku_p)$ is $p-1$-graded and $p = 1$ in $\mathbb{Z}/(p-1)$. In particular, the Frobenius map preserves the grading and one obtains that $\mathrm{THH}(ku)$ is a $p-1$-graded cyclotomic spectrum.

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Proposition 7.2. *The S^1-equivariant structure on $\mathrm{THH}(ku_p)_i$ lifts to a cyclotomic structure for which there is an equivalence*

$$\mathrm{THH}(ku) \simeq \prod_{i\in\mathbb{Z}/(p-1)} \mathrm{THH}(ku)_i$$

of cyclotomic spectra.

Proof. The monoid $\mathbb{Z}/(p-1)$ satisfies the conditions in [AMM+22, Lemma A.7] needed to endow $\mathrm{THH}(ku)$ with an L_p twisted cyclotomic structure [AMM+22, Corollary A.9]. However, since $p \cong 1 \mod p-1$, this ends up being the identity functor on $\mathbb{Z}/(p-1)$-graded spectra. Thus, one obtains a sequence of S^1-equivariant maps

$$\mathrm{THH}(ku)_i \to \mathrm{THH}(ku)_i^{tC_p}$$

for each $i \in \mathbb{Z}/(p-1)$, which is precisely the relevant additional piece of structure needed to view this as a cyclotomic object. □

Construction 7.3. Here, we construct a splitting of $K(ku_p)$ using Proposition 7.2. Since the product mentioned in Proposition 7.2 is a finite product, it is at the same time a coproduct. In particular, it commutes with all limits and colimits. Therefore, the fiber sequence defining $\mathrm{TC}(ku_p)$ splits into a product of fiber sequences

$$\mathrm{TC}(ku_p)_i \to \mathrm{THH}(ku_p)_i^{hS^1} \xrightarrow{(\varphi_p)_i - can_i} (\mathrm{THH}(ku_p)_i^{tC_p})^{hS^1}.$$

Hence, there is a splitting of $\mathrm{TC}(ku_p)$:

$$\mathrm{TC}(ku_p) \simeq \prod_{i\in\mathbb{Z}/(p-1)} \mathrm{TC}(ku_p)_i,$$

where $\mathrm{TC}(ku_p)_i := \mathrm{TC}(\mathrm{THH}(ku_p)_i)$. Arguing as in the proof of Theorem 5.5, one obtains a map $ku_p \to H\mathbb{Z}_p$ of $p-1$-graded E_1-rings where $H\mathbb{Z}_p$ is concentrated in weight 0. Therefore, the induced map $\mathrm{THH}(ku_p) \to \mathrm{THH}(\mathbb{Z}_p)$ of $p-1$-graded spectra is trivial in non-zero weight. By inspection on the product splitting of the fiber sequence defining $\mathrm{TC}(ku_p)$, we consider $\mathrm{TC}(ku_p) \to \mathrm{TC}(\mathbb{Z}_p)$ as a map of $p-1$-graded spectra, where $\mathrm{TC}(\mathbb{Z}_p)$ is concentrated in weight 0. Again, as in the proof of Theorem 5.5, this splits the pullback square (from the Dundas–Goodwillie–McCarthy theorem) relating $\mathrm{TC}(ku_p)$ to $K(ku_p)$, resulting in a splitting of $K(ku_p)$ that we denote by

$$K(ku_p) \simeq \bigvee_{i\in\mathbb{Z}/(p-1)} K(ku_p)_i.$$

Here, $K(ku_p)_0 \simeq K(\ell_p)$ due to Theorem 5.5.

To understand the resulting splitting of $K(ku_p)$, we identify the $V(1)$-homotopy of each weight piece. The computation of $V(1)_*K(ku_p)$ is due to the first author [Aus10, Theorem 8.1] and these groups are given below:

$$\begin{aligned} V(1)_*K(ku_p) \cong\ & \mathbb{F}_p[b] \otimes \Lambda(\lambda_1, a_1) \oplus \mathbb{F}_p[b] \otimes \mathbb{F}_p\{\partial\lambda_1, \partial b, \partial a_1, \partial\lambda_1 a_1\} \\ & \oplus \mathbb{F}_p[b] \otimes \Lambda(a_1) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} \\ & \oplus \mathbb{F}_p[b] \otimes \Lambda(\lambda_1) \otimes \mathbb{F}_p\{\sigma_n, \lambda_2 t^{p^2-p} \mid 1 \le n \le p-2\} \\ & \oplus \mathbb{F}_p\{s\}. \end{aligned} \tag{7.1}$$

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Here, $|b| = 2p+2$, $|\partial| = -1$, $|\lambda_1| = 2p-1$, $|a_1| = 2p+3$, $|\sigma_n| = 2n+1$, $|t| = -2$, $|\lambda_2| = 2p^2-1$ and $|s| = 2p-3$. We assign weights to these classes in a way that turns $V(1)_*K(ku_p)$ into a $p-1$-graded abelian group. The weights of σ_n, b, a_1, ∂, λ_1, t, λ_2 and s are given by n, 1, 1, 0, 0, 0, 0 and 0, respectively. Classes denoted by tensor products or products above have the canonical degrees and weights. Furthermore, the isomorphism above is that of $\mathbb{F}_p[b]$-modules and $b^{p-1} = -v_2$.

Theorem 7.4. *For the equivalence of spectra*

$$K(ku_p) \simeq \bigvee_{i \in \mathbb{Z}/(p-1)} K(ku_p)_i$$

provided by Construction 7.3, there is an equivalence

$$K(ku_p)_0 \simeq K(\ell_p)$$

and there are isomorphisms

$$V(1)_*(K(ku_p)_i) \cong (V(1)_*K(ku_p))_i$$

for each $i \in \mathbb{Z}/(p-1)$, where the right-hand side denotes the weight-i piece of the $p-1$-grading on $V(1)_*K(ku_p)$ described above.

Proof. The identification of $K(ku_p)_0$ is given in Construction 7.3. This provides the identification of $V(1)_*K(ku_p)_0$ as $(V(1)_*K(ku_p))_0$, since this is precisely the image of the map

$$V(1)_*K(\ell_p) \to V(1)_*K(ku_p);$$

see [Aus05, Theorem 10.2]. The identification of $V(1)_*(K(ku_p)_i)$ for $i \neq 0$ follows by noting, from Proposition 7.1, that it is sufficient to keep track of the contribution of δ-weight i classes in $V(1)_*\,\mathrm{THH}(ku_p)$ to $V(1)_*\,\mathrm{TC}(ku_p)$. This follows by inspection of [Aus10, §§ 5 and 7]. □

7.2 Algebraic K-theory of real K-theory

Let $p > 3$. Using Theorem 5.5, the splitting of $K(ku_p)$ discussed above and our root adjunction formalism, we obtain a straightforward computation of $V(1)_*K(ko_p)$ from our knowledge of $V(1)_*K(ku_p)$ from [Aus10]. Here, ko_p denotes the connective cover of the p-completed real topological K-theory spectrum KO_p. We have $\pi_*KO_p \cong \mathbb{Z}_p[\alpha^{\pm 1}]$ with $|\alpha| = 4$.

There is a subgroup C_2 of $\Delta \cong \mathbb{Z}/(p-1)$ such that $KO_p \simeq KU_p^{hC_2}$. Through this, the induced map $KO_p \to KU_p$ carries α to u^2 up to a unit that we are going to omit. Since $L \simeq (KU_p)^{h\Delta}$, we obtain a sequence of E_∞-maps

$$L_p \to KO_p \to KU_p,$$

where the first map carries v_1 to $\alpha^{p-1/2}$ in homotopy.

Theorem 7.5. *For $p > 3$, there is an equivalence*

$$ko_p \simeq \ell_p\big(\sqrt[(p-1)/2]{v_1}\big)$$

of E_1 ℓ_p-algebras.

Proof. This follows as in the proof of Theorem 4.6 by noting that $p \nmid p - 1/2$. □

Furthermore, ku_p may also be obtained from ko_p via root adjunction; for this root adjunction, we use the $\mathbb{S}_p[\sigma_4]$-algebra structure on ko_p provided by Theorem 7.5. To identify the resulting 2-graded E_1-ring structure on ku_p, we use the symmetric monoidal functor

$$D' \colon \mathrm{Fun}(\mathbb{Z}/(p-1), \mathrm{Sp}) \to \mathrm{Fun}(\mathbb{Z}/2, \mathrm{Sp})$$

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given by left Kan extension through the canonical map $\mathbb{Z}/(p-1) \to \mathbb{Z}/2$.

PROPOSITION 7.6. *For $p > 3$, there is an equivalence*

$$ko_p(\sqrt[2]{\alpha}) \simeq D'(ku_p)$$

of 2-graded E_1-algebras, where D' is defined above and the $p-1$-grading on ku_p is given by Theorem 4.6.

Proof. From Theorem 7.5, ko_p is an $\mathbb{S}[\sigma_4]$-algebra given by

$$\ell_p \wedge_{\mathbb{S}[\sigma_{2(p-1)}]} \mathbb{S}[\sigma_4].$$

To adjoin a root to ko_p using this structure, we use the sequence of maps

$$\mathbb{S}[\sigma_{2(p-1)}] \to \mathbb{S}[\sigma_4] \to D'(\mathbb{S}[\sigma_2])$$

of 2-graded E_2-ring spectra, where $\mathbb{S}[\sigma_{2(p-1)}]$ and $\mathbb{S}[\sigma_4]$ are concentrated in weight 0 and $\mathbb{S}[\sigma_2]$ above is given its canonical $p-1$-grading so that σ_2 in $D'(\mathbb{S}[\sigma_2])$ lies in weight 1.

We obtain the following equivalences of 2-graded E_1-rings:

$$ko_p(\sqrt[2]{\alpha}) \simeq \ell_p \wedge_{\mathbb{S}[\sigma_{2(p-1)}]} \mathbb{S}[\sigma_4] \wedge_{\mathbb{S}[\sigma_4]} D'(\mathbb{S}[\sigma_2]) \simeq \ell_p \wedge_{\mathbb{S}[\sigma_{2(p-1)}]} D'(\mathbb{S}[\sigma_2]). \tag{7.2}$$

The functor D' is a left adjoint and it is symmetric monoidal. Therefore, it commutes with the two-sided bar construction defining relative smash products. This provides the second equivalence in the following equivalences of 2-graded E_1-algebras:

$$D'(ku_p) \simeq D'(\ell_p \wedge_{\mathbb{S}[\sigma_{2(p-1)}]} \mathbb{S}[\sigma_2]) \simeq D'(\ell_p) \wedge_{D'(\mathbb{S}[\sigma_{2(p-1)}])} D'(\mathbb{S}[\sigma_2]). \tag{7.3}$$

The first equivalence above follows by Theorem 4.6 and the relative smash product in the middle is taken in $p-1$-graded spectra. Since ℓ_p and $\mathbb{S}[\sigma_{2(p-1)}]$ are concentrated in weight 0, the right-hand side above is equivalent to the right-hand side of (7.2); this follows by Lemma 2.3. In other words, (7.2) and (7.3) agree. □

Recall that the spectra $K(ku_p)_i$ are given in Construction 7.3 and the groups $V(1)_*K(ku_p)_i$ are identified in Theorem 7.4.

THEOREM 7.7. *For $p > 3$, there is an equivalence of spectra*

$$K(ko_p) \simeq \bigvee_{0 \le i < (p-1)/2} K(ku_p)_{2i}.$$

Therefore, we have

$$V(1)_*K(ko_p) \cong \bigoplus_{0 \le i < (p-1)/2} V(1)_*K(ku_p)_{2i}$$

and $V(1)_*K(ko_p)$, as an abelian group, is given by

$$\begin{aligned} V(1)_*K(ko_p) \cong\ & \mathbb{F}_p[b^2] \otimes \Lambda(\lambda_1, ba_1) \oplus \mathbb{F}_p[b^2] \otimes \mathbb{F}_p\{\partial\lambda_1, b\partial b, b\partial a_1, b\partial\lambda_1 a_1\} \\ & \oplus \mathbb{F}_p[b^2] \otimes \Lambda(ba_1) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} \\ & \oplus \mathbb{F}_p[b^2] \otimes \Lambda(\lambda_1) \otimes \mathbb{F}_p\{b^{\epsilon(n)}\sigma_n, \lambda_2 t^{p^2-p} \mid 1 \le n \le p-2\} \\ & \oplus \mathbb{F}_p\{s\}, \end{aligned}$$

where $\epsilon(n) = 1$ if n is odd and $\epsilon(n) = 0$ if n is even. Here, the class denoted by $(b^2)^{(p-1)/2}$ is $-v_2$.

As a consequence, we have an isomorphism of abelian groups

$$T(2)_*K(ko) \cong T(2)_*K(\ell_p)[b^2]/((b^2)^{(p-1)/2} + v_2).$$

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Proof. We start by identifying $\mathrm{THH}(ko_p)$ as a cyclotomic spectrum. We have the following chain of equivalences:

$$\begin{aligned}\mathrm{THH}(ko_p) &\simeq \mathrm{THH}(ko_p(\sqrt[2]{\alpha}))_0\\ &\simeq \mathrm{THH}(D'(ku_p))_0\\ &\simeq (D'(\mathrm{THH}(ku_p)))_0\\ &\simeq \prod_{0\le i<(p-1)/2} \mathrm{THH}(ku_p)_{2i}.\end{aligned}$$

The first equivalence above follows by Theorem 4.13, the second equivalence follows by Proposition 7.6 and the third equivalence is a consequence of [AMM+22, Corollary A.15]. The last equivalence above follows by the description of D' as a left Kan extension; see § 2.2. Indeed, this shows that the following composite map of cyclotomic spectra is an equivalence:

$$\mathrm{THH}(ko_p) \to \mathrm{THH}(ku_p) \simeq \prod_{0\le i<p-1} \mathrm{THH}(ku_p)_i \to \prod_{0\le i<(p-1)/2} \mathrm{THH}(ku_p)_{2i}.$$

Here, the equivalence in the middle follows by Proposition 7.2. The last map above is the canonical projection.

The composite equivalence of cyclotomic spectra above shows that

$$\mathrm{TC}(\mathrm{THH}(ko_p)) \simeq \mathrm{TC}\left(\prod_{0\le i<(p-1)/2} \mathrm{THH}(ku_p)_{2i}\right) \simeq \prod_{0\le i<(p-1)/2} \mathrm{TC}(ku_p)_{2i}.$$

Considering the Dundas–Goodwillie–McCarthy theorem with respect to the composite $ko_p \to ku_p \to H\mathbb{Z}_p$, we obtain that the splitting of the pullback square relating $K(ku_p)$ with $\mathrm{TC}(ku_p)$ (mentioned in Construction 7.3) provides a splitting for $K(ko_p)$ given by

$$K(ko_p) \simeq \prod_{0\le i<(p-1)/2} K(ku_p)_{2i}. \tag{7.4}$$

The first and second statements in the theorem follow from this splitting. The third statement follows by this, and by inspection of (7.1).

For the last statement, note that $T(2)_*K(ko) \cong T(2)_*K(ko_p)$ due to the purity of algebraic K-theory and [LMM+24, Lemma 2.2 (vi)]. It follows, by Theorem 7.4, that the map

$$T(2)_*K(ku_p) \xrightarrow{\cdot b^i} T(2)_*K(ku_p)$$

carries $T(2)_*K(ku_p)_0$ to $T(2)_*K(ku_p)_i$ for $i < p-1$, where the map above multiplies by b^i. Using this fact, together with [Aus10, Proposition 1.2 (b)], provides isomorphisms

$$T(2)_*K(\ell_p) \cong T(2)_*K(ku_p)_0 \xrightarrow{\cong} T(2)_*K(ku_p)_i$$

given by $\cdot b^i$ for $i < p-1$. This, together with (7.4), provides the desired identification of $T(2)_*K(ko) \cong T(2)_*K(ko_p)$ as $T(2)_*K(\ell_p)[b^2]/((b^2)^{(p-1)/2} + v_2)$. □

8. Root adjunction and Lubin–Tate spectra

Recall that in [HW22], Hahn and Wilson prove that there are E_3 MU-algebra forms of $BP\langle n\rangle$. Furthermore, their constructions provide an E_3 $MU[\sigma_{2(p^n-1)}]$-algebra form of $BP\langle n\rangle$, where $\sigma_{2(p^n-1)}$ acts through v_n (see [HW22, Remark 2.1.2]).

To relate particular forms of $BP\langle n\rangle$ to Lubin–Tate spectra, we use the spherical Witt vectors constructed by Lurie [Lur18, Example 5.2.7]. For a given discrete perfect $\mathbb{F}_p$-algebra B_0, this

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provides an E_∞-ring $\mathbb{S}_{W(B_0)}$ that is flat over $\mathbb{S}$ in the sense of [Lur17, Definition 7.2.2.10]. Therefore, it follows by [Lur17, Proposition 7.2.2.13] and [Mao23, Proposition 2.7] that

$$\pi_n(\mathbb{S}_{W(B_0)} \wedge F) \cong W(B_0) \otimes \pi_n F \tag{8.1}$$

for every spectrum F. We would like to thank Jeremy Hahn for showing us the proof of the following proposition.

Proposition 8.1. *Fix an E_3 MU-algebra form of $BP\langle n\rangle$. Then $\mathbb{S}_{W(B_0)} \wedge BP\langle n\rangle$ satisfies the redshift conjecture for every discrete perfect $\mathbb{F}_p$-algebra B_0.*

Proof. There is an equivalence of E_2-algebras in S^1-equivariant spectra

$$\mathbb{S}_{W(B_0)} \wedge \mathrm{THH}^{MU}(BP\langle n\rangle) \simeq \mathrm{THH}^{\mathbb{S}_{W(B_0)} \wedge MU}(\mathbb{S}_{W(B_0)} \wedge BP\langle n\rangle).$$

Therefore, it follows by (8.1) that we have

$$\mathrm{THH}_*^{\mathbb{S}_{W(B_0)} \wedge MU}(\mathbb{S}_{W(B_0)} \wedge BP\langle n\rangle) \cong W(B_0) \otimes \mathrm{THH}_*^{MU}(BP\langle n\rangle) \tag{8.2}$$

and this is a polynomial algebra over $W(B_0)[v_1, \dots, v_n]$ due to [HW22, Theorem 2.5.4], where one of the generators is denoted by $\sigma^2 v_{n+1}$. The rest of the argument follows as in the proofs of [HW22, Theorems 2.5.4 and 5.0.1, Corollary 5.0.2] by considering

$$\pi_*(\mathrm{THH}^{\mathbb{S}_{W(B_0)} \wedge MU}(\mathbb{S}_{W(B_0)} \wedge BP\langle n\rangle)^{hS^1}) \tag{8.3}$$

instead of $\pi_*(\mathrm{THH}^{MU}(BP\langle n\rangle)^{hS^1})$.

Namely, we obtain from [HW22, Theorem 2.5.4] that (8.2) is concentrated in even degrees. Therefore, the corresponding homotopy fixed-point spectral sequence degenerates on the second page, providing the $W(B_0)[v_1, \dots, v_n]$-algebra (8.3) as

$$W(B_0) \otimes \mathrm{THH}_*^{MU}(BP\langle n\rangle)[\![t]\!]$$

since (8.2) is polynomial. Using the map from $\pi_*(\mathrm{THH}^{MU}(BP\langle n\rangle)^{hS^1})$ to (8.3), we deduce from [HW22, Theorem 5.0.1] that v_{n+1} in (8.3) is represented by the class $t\sigma^2 v_{n+1}$.

Considering the action of $v_0, \dots, v_{n+1}$ on (8.3) described above, one observes that

$$L_{T(n+1)} \mathrm{THH}^{\mathbb{S}_{W(B_0)} \wedge MU}(\mathbb{S}_{W(B_0)} \wedge BP\langle n\rangle)^{hS^1} \not\simeq *.$$

Using this and the E_2-map

$$L_{T(n+1)} K(\mathbb{S}_{W(B_0)} \wedge BP\langle n\rangle) \to L_{T(n+1)} \mathrm{THH}^{\mathbb{S}_{W(B_0)} \wedge MU}(\mathbb{S}_{W(B_0)} \wedge BP\langle n\rangle)^{hS^1},$$

we deduce that $L_{T(n+1)} K(\mathbb{S}_{W(B_0)} \wedge BP\langle n\rangle) \not\simeq *$ as desired. □

Construction 8.2. Fix an E_3 $MU[\sigma_{2(p^n-1)}]$-algebra form of $BP\langle n\rangle$ where $\sigma_{2(p^n-1)}$ acts through v_n and let k be a perfect field of characteristic p. Recall that in this situation, degree p^n-1-root adjunction to v_n provides a p^n-1-graded E_3 $MU[\sigma_2]$-algebra; see Remark 4.7. We consider the E_3 MU-algebra

$$E := (L_{K(n)}(\mathbb{S}_{W(k)} \wedge BP\langle n\rangle))(\sqrt[p^n-1]{v_n}).$$

It follows by [Hov97, Theorem 1.5.4] and [Hov95, Theorem 1.9] that

$$\pi_* E \cong W(k)[|u_1, \dots, u_{n-1}|][u^{\pm 1}],$$

where $|u_i| = 0$ and $|u| = -2$. Furthermore, the resulting E_3-map $MU \to BP\langle n\rangle \to E$ provides a formal group law over $\pi_* E$.

For a given perfect $\mathbb{F}_p$-algebra B_0 and a height-n formal group law Γ over B_0, we let $E_{(B_0,\Gamma)}$ denote the corresponding Lubin–Tate spectrum. By Lurie's generalization [Lur18, § 5] of the Goerss–Hopkins–Miller theorem [Rez98, GH04], $E_{(B_0,\Gamma)}$ is an E_∞-ring.

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Proposition 8.3. *In the setting of Construction 8.2, E is equivalent to $E_{(k,\Gamma)}$ as an E_1-ring for some height-n formal group law Γ over k.*

Proof. By construction, the map $\pi_* MU_{(p)} \to \pi_* E$ carries v_i to $u_i u^{-(p^i-1)}$ for $0 < i < n$ and v_n to $u^{-(p^n-1)}$. Therefore, the corresponding formal group law on $\pi_0 E$ is the universal deformation of the resulting height-n formal group law Γ over k. This follows by [Lur10, Theorem 5 in Lecture 21]. Alternatively, one can directly check the conditions given in [LT66, Proposition 1.1]. It follows from the Hopkins–Miller theorem that there is an equivalence of E_1-rings $E \simeq E_{(k,\Gamma)}$ (see [Rez98, Theorem 7.1]). □

Burklund et al. are going to use the following example in their construction of a counter-example to the telescope conjecture.

Example 8.4. Let k be a perfect algebraic extension of $\mathbb{F}_p$. We know from [Ram23, Corollary 4.31] that the E_1-algebra structure on $E_{(k,\Gamma)}$ lifts to a unique E_d-algebra structure for every $1 \leq d \leq \infty$. Since E in Construction 8.2 is an E_3-ring, it follows from Proposition 8.3 that there is an E_3-equivalence

$$E \simeq E_{(k,\Gamma)}$$

for Γ as in Proposition 8.3. Through this equivalence, we may equip $E_{(k,\Gamma)}$ with the structure of an E_3 MU-algebra. In particular, we obtain a map of E_3 MU-algebras

$$BP\langle n\rangle \to E_{(k,\Gamma)}.$$

Furthermore, $E_{(k,\Gamma)}$ is a $\mathbb{Z}/(p^n-1)$-graded E_3 $MU[\sigma_2]$-algebra where the weight-1 class σ_2 acts through u^{-1}.

Theorem 8.5. *In the setting of Construction 8.2 and Proposition 8.3, the canonical map*

$$L_{T(n+1)} K(\mathbb{S}_{W(k)} \wedge BP\langle n\rangle) \to L_{T(n+1)} K(E_{(k,\Gamma)})$$

is the inclusion of a non-trivial wedge summand.

Proof. It follows by Corollary 5.6 and Proposition 8.3 that

$$L_{T(n+1)} K(L_{K(n)}(\mathbb{S}_{W(k)} \wedge BP\langle n\rangle)) \to L_{T(n+1)} K(E_{(k,\Gamma)})$$

is the inclusion of a wedge summand. Since

$$\mathbb{S}_{W(k)} \wedge BP\langle n\rangle \to L_{K(n)}(\mathbb{S}_{W(k)} \wedge BP\langle n\rangle)$$

is a $T(n) \vee T(n+1)$-equivalence, the result follows by [LMM+24, Purity Theorem]. □

We finally prove the following theorem.

Theorem 8.6 (Yuan [Yua21]). *For every perfect $\mathbb{F}_p$-algebra B_0 and height-n formal group law Γ over B_0, the Lubin–Tate spectrum $E_{(B_0,\Gamma)}$ satisfies the redshift conjecture.*

Proof. There is an E_∞-map $E_{(B_0,\Gamma)} \to E_{(k,\Gamma')}$ for some algebraically closed field k of characteristic p and Γ' is the corresponding height-n formal group law on k. Since there is an induced E_∞-map $K(E_{(B_0,\Gamma)}) \to K(E_{(k,\Gamma')})$, it suffices to prove the redshift conjecture for $E_{(k,\Gamma')}$.

Since k is algebraically closed, there is a unique formal group law of height n over k (see [Laz55, Theorem IV]). Using Proposition 8.3, we deduce that $E_{(k,\Gamma')} \simeq E$ for E as in Construction 8.2. Combining Proposition 8.1 and Theorem 8.5, we obtain that $E_{(k,\Gamma')}$ satisfies the redshift conjecture as desired. □

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We remark that it should be possible to prove the redshift conjecture for all E_1 MU-algebra forms of $BP\langle n\rangle$ by constructing maps $BP\langle n\rangle \to E_{(k,\Gamma)}$ through root adjunction.

9. Algebraic K-theory and THH of Morava E-theories

In this section, we work with a particular form of Lubin–Tate spectra. This is the Morava E-theory spectrum E_n, and E_n is central in the Ausoni–Rognes program for the computation of $K(\mathbb{S})$. When we say Morava E-theory E_n, we mean the Lubin–Tate spectrum corresponding to the height-n Honda formal group. This formal group is characterized by its p-series

$$[p]_n(x) = x^{p^n},$$

and admits a canonical form over $\mathbb{F}_{p^n}$, in the sense that all of its endomorphisms are defined over this field. In this section, we prove a splitting result for the algebraic K-theory of the Morava E-theory E_n and the corresponding two-periodic Morava K-theory. Furthermore, we show that the THH of E_n may be obtained from the THH of the $K(n)$-localized Johnson–Wilson spectrum through base change.

9.1 An identification of Morava E-theory

Here, we provide an alternate description of E_n in terms of its fixed points and spherical Witt vectors. We have

$$\pi_* E_n \cong W(\mathbb{F}_{p^n})[|u_1, \dots, u_{n-1}|][u^{\pm 1}],$$

where $|u_i| = 0$ and $|u| = -2$.

Proposition 9.1. *The map*

$$\pi_* \mathbb{S}_{W(\mathbb{F}_{p^n})} \cong W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \pi_* \mathbb{S}_p \to \pi_* E_n$$

obtained via the map $\pi_* \mathbb{S}_p \to \pi_* E_n$ and the canonical $W(\mathbb{F}_{p^n})$-module structure on $\pi_* E_n$ lifts to a map of E_∞ $\mathbb{S}_p$-algebras

$$\mathbb{S}_{W(\mathbb{F}_{p^n})} \to E_n.$$

Proof. This is a consequence of Lurie's theory of thickenings of relatively perfect morphisms [Lur18, § 5.2]. Indeed, $\mathbb{S}_p \to \mathbb{S}_{W(\mathbb{F}_{p^n})}$ is an $\mathbb{S}_p$-thickening of $H\mathbb{F}_p \to H\mathbb{F}_{p^n}$ in the sense of [Lur18, Definition 5.2.1]; see [Lur18, Example 5.2.7].

In particular, this implies that the space of E_∞ $\mathbb{S}_p$-algebra maps from $\mathbb{S}_{W(\mathbb{F}_{p^n})}$ to the connective cover cE_n of E_n is given by the set of $\mathbb{F}_p$-algebra maps

$$\hom_{\mathbb{F}_p - \mathcal{A}\mathrm{lg}}(\mathbb{F}_{p^n}, \mathbb{F}_{p^n}[|u_1, \dots, u_{n-1}|]), \tag{9.1}$$

where this correspondence is given by the functor $\pi_0(-)/p$.

Let $f\colon \mathbb{S}_{W(\mathbb{F}_{p^n})} \to cE_n$ be the map of E_∞ $\mathbb{S}_p$-algebras corresponding to the canonical $\mathbb{F}_p$-algebra map in (9.1); in particular, $\pi_0(f)/p$ is the canonical map in (9.1). We first show that $\pi_0 f$ is given by the canonical inclusion

$$W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})[|u_1, \dots, u_{n-1}|].$$

Since $W(\mathbb{F}_p)$ is the ring of integers of the unique unramified extension $\mathbb{Q}_p[\mu_{p^n-1}]$ of $\mathbb{Q}_p$ of degree d, $W(\mathbb{F}_{p^n})$ is generated as a $\mathbb{Z}_p$-algebra by a primitive $p^n - 1$ root of the unit. Since the roots of the unit of $\pi_0 cE_n$ are all in the image of the canonical inclusion $W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})[|u_1, \dots, u_{n-1}|]$, one observes that the map $\pi_0 f$ has to factor through the canonical inclusion $W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})[|u_1, \dots, u_{n-1}|]$. Furthermore, there is a unique ring map

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$W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})$ that lifts the identity map on $\mathbb{F}_{p^n}$. This shows that $\pi_0 f$ is given by the canonical inclusion $W(\mathbb{F}_{p^n}) \to W(\mathbb{F}_{p^n})[|u_1, \dots, u_{n-1}|]$. Since $\pi_* f$ is a map of $\pi_*\mathbb{S}_p$-modules, it follows that the composition of f with the map $cE_n \to E_n$ provides the map claimed in the proposition. □

Let *Gal* denote the Galois group $\mathrm{Gal}(\mathbb{F}_{p^n}, \mathbb{F}_p)$. From the Goerss–Hopkins–Miller theorem, there is an action of *Gal* on the E_∞-algebra E_n for which

$$\pi_* E_n^{\mathrm{hGal}} \cong \mathbb{Z}_p[|u_1, \dots, u_{n-1}|][u^{\pm 1}],$$

where the degrees of the generators are as in $\pi_* E_n$.

Proposition 9.2. *There is an equivalence of E_∞-$\mathbb{S}_p$-algebras*

$$\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} E_n^{\mathrm{hGal}} \simeq E_n.$$

Proof. This equivalence is given by the following composite map of E_∞ $\mathbb{S}_p$-algebras:

$$\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} E_n^{\mathrm{hGal}} \to E_n \wedge_{\mathbb{S}_p} E_n \to E_n, \tag{9.2}$$

where the first map is induced by the map provided by Proposition 9.1 and the second map is given by the multiplication map of E_n. Owing to the flatness of $\mathbb{S}_{W(\mathbb{F}_{p^n})}$, this map induces the canonical map

$$W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[|u_1, \dots, u_{n-1}|][u^{\pm 1}] \to W(\mathbb{F}_{p^n})[|u_1, \dots, u_{n-1}|][u^{\pm 1}] \tag{9.3}$$

at the level of homotopy groups [Lur17, 7.2.2.13]. Since $W(\mathbb{F}_{p^n})$ is a free $\mathbb{Z}_p$-module of finite rank, the functor $W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} -$ is given by taking an n-fold product of $-$. In particular, the functor $W(\mathbb{F}_{p^n}) \otimes_{\mathbb{Z}_p} -$ commutes with completions. This shows that (9.3) is an isomorphism, as desired. □

9.2 Algebraic *K*-theory of Morava *E*-theories

There is a finite subgroup $\mathbb{F}_{p^n}^\times$ of the Morava stabilizer group such that $K = \mathbb{F}_{p^n}^\times \rtimes Gal$ acts on the E_∞-algebra E_n. Furthermore,

$$E_n^{hK} \simeq \widehat{E(n)},$$

where $\widehat{E(n)}$ denotes the $K(n)$-localization of the Johnson–Wilson spectrum $E(n)$; see [Rog08, § 5.4.7]. We have

$$\pi_* E(n) \cong \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}][v_n^{\pm 1}] \quad \text{and} \quad \pi_* \widehat{E(n)} \cong \mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}][v_n^{\pm 1}]^\wedge_I,$$

where I denotes the ideal $(p, v_1, \dots, v_{n-1})$. Since $\widehat{E(n)}$ is given by E_n^{hK}, there is a K-equivariant map of E_∞-algebras $\widehat{E(n)} \to E_n$. In particular, this provides a map

$$\widehat{E(n)} \to E_n^{\mathrm{hGal}}$$

of E_∞-algebras. This map carries v_n to $u^{-(p^n-1)}$ and v_i to $u_i u^{-(p^i-1)}$ for $0 < i < n$.

For the following, we fix an E_2-map $\mathbb{S}[\sigma_{2(p^n-1)}] \to \widehat{E(n)}$ to adjoin roots. Recall from Remark 4.4 that in this situation, $\widehat{E(n)}(\sqrt[p^n-1]{v_n})$ is an $\widehat{E(n)}$-algebra.

Theorem 9.3. *There are equivalences of E_1 $\widehat{E(n)}$-algebras*

$$E_n^{\mathrm{hGal}} \simeq \widehat{E(n)}(\sqrt[p^n-1]{v_n}),$$
$$E_n \simeq \mathbb{S}_{W(\mathbb{F}_q)} \wedge_{\mathbb{S}_p} \widehat{E(n)}(\sqrt[p^n-1]{v_n}),$$

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where the class u^{-1} corresponds to $\sqrt[p^n-1]{v_n}$ at the level of homotopy groups for both of these equivalences.

In particular, E_n^{hGal} and E_n are p^n-1-graded E_1 $\widehat{E(n)}$-algebras with

$$(E_n^{\mathrm{hGal}})_i \simeq \Sigma^{2i}\widehat{E(n)}$$

and

$$(E_n)_i \simeq \Sigma^{2i}\mathbb{S}_{W(\mathbb{F}_q)} \wedge_{\mathbb{S}_p} \widehat{E(n)}$$

for every $0 \leq i < p^n - 1$.

Proof. By the description of $\pi_*(\widehat{E(n)}) \to \pi_*(E_n^{\mathrm{hGal}})$ given above, this map is an order p^n-1 root adjunction to v_n followed by a change of variables on v_i (with u_i) for $i < n$. In particular, we have

$$\pi_*(\widehat{E(n)}(\sqrt[p^n-1]{v_n})) \cong \pi_* E_n^{\mathrm{hGal}}.$$

Furthermore, the map of rings

$$\pi_*\widehat{E(n)} \to \pi_*(\widehat{E(n)}(\sqrt[p^n-1]{v_n})) \cong (\pi_*\widehat{E(n)})[z]/(z^{p^n-1} - v_n)$$

is a map of étale rings as v_n and p^n-1 are invertible in $\pi_*\widehat{E(n)}$. Through [HP25, Theorem 1.10], we obtain the first equivalence in the theorem. The second equivalence follows by the first equivalence and Proposition 9.2. The statement on graded ring structures follows by the fact that root adjunction results in m-graded ring spectra; see Construction 4.3. □

We are ready to prove our result on the K-theory of Morava E-theories. For this, we use the following composite map:

$$E(n) \to \widehat{E(n)} \to \widehat{E(n)}(\sqrt[p^n-1]{v_n}) \xrightarrow{\simeq} E_n^{\mathrm{hGal}} \tag{9.4}$$

of E_1-rings, where the last map above is given by Theorem 9.3. Using Proposition 9.2, we obtain the following composite:

$$\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge E(n) \to \mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge \widehat{E(n)} \to \mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge E_n^{\mathrm{hGal}} \to \mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} E_n^{\mathrm{hGal}} \xrightarrow{\simeq} E_n. \tag{9.5}$$

Theorem 9.4. *The maps*

$$K(E(n)) \to K(E_n^{\mathrm{hGal}}),$$
$$K(\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge E(n)) \to K(E_n)$$

induced by those above are inclusions of wedge summands after $T(n+1)$-localization.

Proof. The first map in (9.4) is a $T(n) \vee T(n+1)$-equivalence and hence induces a $T(n+1)$-equivalence in algebraic K-theory [LMM+24, Purity Theorem]. Therefore, the first equivalence in the theorem follows by applying Corollary 5.6 to $\widehat{E(n)}(\sqrt[p^n-1]{v_n})$.

Similarly, the first and third maps in (9.5) induce $T(n+1)$-equivalences in algebraic K-theory. The second equivalence in the theorem follows by observing that there is an equivalence of E_1-algebras

$$\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge (\widehat{E(n)}(\sqrt[p^n-1]{v_n})) \simeq (\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge \widehat{E(n)})(\sqrt[p^n-1]{v_n})$$

and then applying Corollary 5.6. □

9.3 Two-periodic Morava K-theories

We obtain analogous results for two-periodic Morava K-theories. Taking a quotient with respect to a regular sequence in $\pi_*\mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} \widehat{E(n)}$, one obtains an $\widehat{E(n)}$-algebra $K(n)$ (see [Laz03,

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Ang08, HW18]). Here, $K(n)$ is the Morava K-theory spectrum with coefficients in $\mathbb{F}_{p^n}$. We have

$$\pi_* K(n) \cong \mathbb{F}_{p^n}[v_n^{\pm 1}].$$

Using the $\widehat{E(n)}$-algebra structure on $K(n)$, we adjoin roots and define the two-periodic Morava K-theory as follows:

$$K_n := K(n)(\sqrt[p^n-1]{v_n}).$$

In this case

$$\pi_* K_n \cong \mathbb{F}_{p^n}[u^{\pm 1}],$$

where $|u| = -2$. Together with Theorem 9.3, this provides the following commuting diagram of E_1-rings, which justifies our definition of K_n.

$$\begin{array}{ccc} \mathbb{S}_{W(\mathbb{F}_{p^n})} \wedge_{\mathbb{S}_p} \widehat{E(n)} & \longrightarrow & E_n \\ \downarrow & & \downarrow \\ K(n) & \longrightarrow & K_n \end{array}$$

In particular, K_n is a $p^n - 1$-graded E_1-ring in a non-trivial way and we obtain the following from Corollary 5.6.

Theorem 9.5. *The map*

$$K(K(n)) \to K(K_n)$$

is the inclusion of a wedge summand after $T(i)$-localization for every $i \geq 2$.

Corollary 9.6. *If $K(n)$ satisfies the redshift conjecture, then so does K_n.*

The $V(1)$-homotopy of $K(k(1))$ is computed by Ausoni and Rognes in [AR12] for $p > 3$. In particular, their computation shows that $K(1)$ satisfies the redshift conjecture. We obtain the following.

Corollary 9.7. *The two-periodic Morava K-theory K_1 of height 1 satisfies the redshift conjecture for $p > 3$.*

9.4 THH descent for Morava *E*-theories

Theorem 6.19 identifies THH of various periodic ring spectra with their log THH. For instance, the Morava E-theory spectrum E_n is periodic with unit u in degree -2. Since $E_n/(u^{-1}) \simeq 0$, the canonical map

$$\mathrm{THH}(E_n) \xrightarrow{\simeq} \mathrm{THH}(E_n \,|\, u^{-1})$$

is an equivalence. Using this, together with our result on log-THH-étaleness of root adjunction, we show that $\mathrm{THH}(E_n)$ may be obtained from $\mathrm{THH}(\widehat{E(n)})$ via base change up to p-completion. Such base-change formulas and their relationship with Galois descent problems for THH were studied by Mathew [Mat17].

Theorem 9.8. *The canonical map*

$$\mathrm{THH}(\widehat{E(n)}) \wedge_{\widehat{E(n)}} E_n^{\mathrm{hGal}} \xrightarrow{\simeq} \mathrm{THH}(E_n^{\mathrm{hGal}})$$

is an equivalence.

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Proof. Recall from Theorem 9.3 that there is an equivalence of $\widehat{E(n)}$-algebras

$$E_n^{\mathrm{hGal}} \simeq \widehat{E(n)}(\sqrt[p^n-1]{v_n}).$$

Therefore, it follows by Construction 4.3 that

$$\begin{aligned}\mathrm{THH}(\widehat{E(n)}) \wedge_{\widehat{E(n)}} E_n^{\mathrm{hGal}} &\simeq \mathrm{THH}(\widehat{E(n)}) \wedge_{\widehat{E(n)}} \widehat{E(n)} \wedge_{\mathbb{S}_{(p)}[\sigma_{2(p^n-1)}]} \mathbb{S}_{(p)}[\sigma_2] \\ &\simeq \mathrm{THH}(\widehat{E(n)}) \wedge_{\mathbb{S}_{(p)}[\sigma_{2(p^n-1)}]} \mathbb{S}_{(p)}[\sigma_2]. \end{aligned} \tag{9.6}$$

Since v_n is a unit in $\widehat{E(n)}$ and u^{-1} is a unit in E_n^{hGal}, Theorem 6.19 provides the equivalences

$$\mathrm{THH}(\widehat{E(n)}) \xrightarrow{\simeq} \mathrm{THH}(\widehat{E(n)} \,|\, v_n) \quad \text{and} \quad \mathrm{THH}(E_n^{\mathrm{hGal}}) \xrightarrow{\simeq} \mathrm{THH}(E_n^{\mathrm{hGal}} \,|\, u^{-1}).$$

Using these equivalences, together with Theorem 6.14, we obtain that the following canonical map is an equivalence:

$$\mathrm{THH}(\widehat{E(n)}) \wedge_{\mathbb{S}_{(p)}[\sigma_{2(p^n-1)}]} \mathbb{S}_{(p)}[\sigma_2] \xrightarrow{\simeq} \mathrm{THH}(E_n^{\mathrm{hGal}}).$$

This, together with (9.6), provides the desired result. □

Theorem 9.9. *The canonical map*

$$\mathrm{THH}(\widehat{E(n)}) \wedge_{\widehat{E(n)}} E_n \xrightarrow{\simeq_p} \mathrm{THH}(E_n)$$

is an equivalence after p-completion.

Proof. The first equivalence below follows by Proposition 9.2 and the second follows by Theorem 9.8:

$$\begin{aligned}\mathrm{THH}(\widehat{E(n)}) \wedge_{\widehat{E(n)}} E_n &\simeq \mathrm{THH}(\widehat{E(n)}) \wedge_{\widehat{E(n)}} (E_n^{\mathrm{hGal}} \wedge_{\mathbb{S}_p} \mathbb{S}_{W(\mathbb{F}_{p^n})}) \\ &\simeq \mathrm{THH}(E_n^{\mathrm{hGal}}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(\mathbb{F}_{p^n})}.\end{aligned}$$

Therefore, it is sufficient to show that the canonical map

$$\mathrm{THH}(E_n^{\mathrm{hGal}}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(\mathbb{F}_{p^n})} \to \mathrm{THH}(E_n^{\mathrm{hGal}} \wedge_{\mathbb{S}_p} \mathbb{S}_{W(\mathbb{F}_{p^n})})$$

is an equivalence after p-completion. This follows by the following canonical diagram of $\mathbb{S}/p$-equivalences.

$$\begin{array}{ccc}
\mathrm{THH}(E_n^{\mathrm{hGal}}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(\mathbb{F}_{p^n})} & \longrightarrow & \mathrm{THH}(E_n^{\mathrm{hGal}} \wedge_{\mathbb{S}_p} \mathbb{S}_{W(\mathbb{F}_{p^n})}) \\
\downarrow \simeq_p & & \downarrow \simeq_p \\
\mathrm{THH}^{\mathbb{S}_p}(E_n^{\mathrm{hGal}}) \wedge_{\mathbb{S}_p} \mathbb{S}_{W(\mathbb{F}_{p^n})} & \longrightarrow & \mathrm{THH}^{\mathbb{S}_p}(E_n^{\mathrm{hGal}} \wedge_{\mathbb{S}_p} \mathbb{S}_{W(\mathbb{F}_{p^n})}) \\
\downarrow \simeq_p & \nearrow \simeq & \\
\mathrm{THH}^{\mathbb{S}_p}(E_n^{\mathrm{hGal}}) \wedge_{\mathbb{S}_p} \mathrm{THH}^{\mathbb{S}_p}(\mathbb{S}_{W(\mathbb{F}_{p^n})}) & &
\end{array}$$

The right-hand vertical map and the upper-left vertical map are $\mathbb{S}/p$-equivalences due to [Mao23, Lemma 5.20]. The fact that the lower-left vertical map is an $\mathbb{S}/p$-equivalence follows by [Mao23, proof of Lemma 5.20] and the fact that the composite $\mathbb{S}_{W(\mathbb{F}_{p^n})} \to \mathrm{THH}(\mathbb{S}_{W(\mathbb{F}_{p^n})}) \to \mathbb{S}_{W(\mathbb{F}_{p^n})}$ is an equivalence. This shows that the upper horizontal map is an $\mathbb{S}/p$-equivalence, proving the theorem. □

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Conflicts of interest

None.

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