Linear differential equations with constant coefficients

BETHANY FRALICK and REGINALD KOO

1. *Introduction*

We consider the second order homogeneous linear differential equation

$$ay'' + by' + cy = 0 \qquad \text{(H)}$$

with real coefficients a, b, c, and $a \neq 0$. The function $y = e^{mx}$ is a solution if, and only if, m satisfies the auxiliary equation $am^2 + bm + c = 0$. When the roots of this are the complex conjugates $m = p \pm iq$, then $y = e^{(p \pm iq)x}$ are complex solutions of (H). Nevertheless, real solutions are given by $y = c_1 e^{px} \cos qx + c_2 e^{px} \sin qx$.

In A level textbooks there are usually two methods of obtaining the solution when the auxiliary roots are complex. In one method the solution of the simple harmonic motion (S.H.M.) equation is obtained by quadratures, then a substitution reduces (H) to a S.H.M. equation: the auxiliary equation and complex exponentials $e^{(p \pm iq)x}$ are not used. In the alternative method the solution is obtained by using the auxiliary equation, and applying Euler's identity, namely $e^{ix} = \cos x + i \sin x$, to the linear combination $y = Ae^{(p+iq)x} + Be^{(p-iq)x}$. See, for instance, Plumpton-Tomkys [1] for both methods.

The thrust of this Article is to suggest that a more satisfactory method for obtaining real solutions in the case of complex auxiliary roots is to generalise. That is, consider (H) with complex coefficients, find all complex solutions, then extract the real solutions as a certain subset of the complex solutions. The set of complex solutions denoted by $\mathbb{S}_C$ is a 2-dimensional complex vector space. The set of real solutions denoted by $\mathbb{S}_R$ is a subset, though not a linear subspace, of $\mathbb{S}_C$. However $\mathbb{S}_R$ is a real vector space, and we will show that its dimension can be either 0, 1 or 2.

2. *The S.H.M. equation*

Simple harmonic motion is the simplest case of oscillatory motion studied in Mechanics. For instance, consider a particle moving in a straight line whose position relative to an origin O is given by $x = a \sin \omega t$. Then its acceleration is $\ddot{x} = -a\omega^2 \sin \omega t = -\omega^2 x$. Simple harmonic motion is motion in which acceleration is proportional to displacement from the origin, and directed towards the origin.

The equation of simple harmonic motion is $y'' + \omega^2 y = 0$, where $\omega > 0$. The solution is obtained as follows: here we do not employ the auxiliary equation. With $v = \dfrac{dy}{dx}$ we have $v\dfrac{dv}{dy} + \omega^2 y = 0$; hence $v\,dv + \omega^2 y\,dy = 0$. Integrating we obtain $v^2 + \omega^2 y^2 = a^2\omega^2$, where the integration constant is denoted by $a^2\omega^2$ for some $a > 0$. Hence $\dfrac{dy}{dx} = \pm\omega\sqrt{a^2 - y^2}$.

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Consider $\dfrac{dy}{dx} = +\omega\sqrt{a^2 - y^2}$. The function $h(x, y) = \omega\sqrt{a^2 - y^2}$ is defined on the strip $\Gamma = \{(x, y) : -a \leqslant y \leqslant a\}$. Hence solution curves $y = \phi(x)$ lie in Γ. There are constant solutions, namely $y = a$ and $y = -a$. Consider solutions in the open strip $-a < y < a$. A solution y is strictly increasing, and the inverse function satisfies $\dfrac{dx}{dy} = \dfrac{1}{\omega\sqrt{a^2 - y^2}}$. Hence $\omega x = \displaystyle\int \frac{dy}{\sqrt{a^2 - y^2}} = \sin^{-1}\left(\frac{y}{a}\right) + b$. For clarity, we denote by $\operatorname{Sin} x$ the restriction of the function $\sin x$ to the interval $I = \left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$. The solution obtained is $y = a \operatorname{Sin}(\omega x - b)$, with integration constant b. We note that $y = a \operatorname{Sin}(\omega x - b)$ is the restriction of $y = a \sin(\omega x - b)$ to the interval $J_b = \{x : \omega x - b \in I\}$. These functions are translations of the function $y = a \operatorname{Sin} x$ for $-\frac{\pi}{2\omega} < x < \frac{\pi}{2\omega}$.

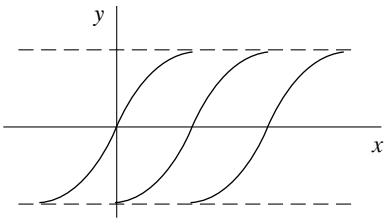


FIGURE 1: $y = a \operatorname{Sin}(\omega x - b) : b \in \mathbb{R}$

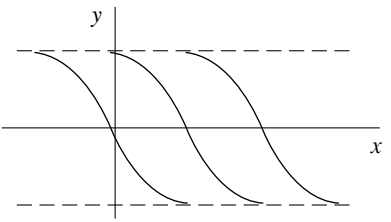


FIGURE 2: $y = -a \operatorname{Sin}(\omega x - b) : b \in \mathbb{R}$

Consider $\dfrac{dy}{dx} = -\omega\sqrt{a^2 - y^2}$. There are constant solutions $y = a$ and $y = -a$. Consider solutions in the open strip $-a < y < a$. A solution y is strictly decreasing, and the inverse function satisfies $\dfrac{dx}{dy} = -\dfrac{1}{\omega\sqrt{a^2 - y^2}}$. Hence

$$\omega x = -\int \frac{dy}{\sqrt{a^2 - y^2}} = -\sin^{-1}\left(\frac{y}{a}\right) + b'.$$

The solution is $y = -a \operatorname{Sin}(\omega x - b')$, with integration constant b' and domain $J_{b'} = \{x : \omega x - b' \in I\}$.

The functions $y = a \operatorname{Sin}(\omega x - b)$ and $y = -a \operatorname{Sin}(\omega x - b')$ may now be glued together at the maximum and minimum points. The joins must be at the points where $y' = 0$ in both equations, to ensure existence of y'' in the joined equation, defined on $\mathbb{R}$, $y = a \sin(\omega x + \varepsilon)$. Using the trigonometric addition formula we see that the solution of the S.H.M. equation $y'' + \omega^2 y = 0$ can be written in the form

$$y = A \cos \omega x + B \sin \omega x \tag{1}$$

where now A and B are the integration constants.

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We note that the solutions to the equation $\left(\frac{dy}{dx}\right)^2 = \omega^2(a^2 - y^2)$ properly contain solutions to the equation $y'' + \omega^2 y = 0$. Functions $y = a\,\mathrm{Sin}\,(\omega x - b)$, $y = -a\,\mathrm{Sin}\,(\omega x - b')$, $y = -a$ and $y = a$ can be suitably pieced together to give solutions of $\left(\frac{dy}{dx}\right)^2 = \omega^2(a^2 - y^2)$. For instance, the function $\phi(x)$ sketched below is a differentiable solution to the first-order equation $\left(\frac{dy}{dx}\right)^2 = \omega^2(a^2 - y^2)$, but $\phi(x)$ is not twice differentiable and does not satisfy the S.H.M. equation on $\mathbb{R}$.

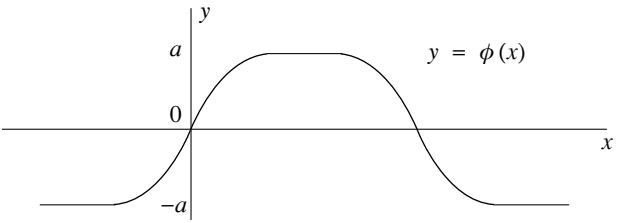


FIGURE 3

We now present the two methods for obtaining the solution of

$$ay'' + by' + cy = 0 \qquad \text{(H1)}$$

when the auxiliary roots are complex conjugates, that is, when $b^2 - 4ac < 0$.

3. *Solution of* (H1) *via the S.H.M. equation*

Denote the auxiliary roots for (H1) by $p \pm iq$, $q \neq 0$. Thus equation (H1) takes the form

$$y'' - 2py' + (p^2 + q^2)y = 0. \qquad (2)$$

Write $z = e^{-px}y$. Then $z' = e^{-px}(y' - py)$ and $z'' = e^{-px}(y'' - 2py' + p^2y)$. Therefore $z'' + q^2z = e^{-px}\{y'' - 2py' + (p^2 + q^2)y\}$; the equation (2) reduces to S.H.M. equation $z'' + q^2z = 0$. The solution is, by (1), $z = A\cos qx + B\sin qx$, and hence the solution to (H1) is $y = e^{px}(A\cos qx + B\sin qx)$.

4. *Solution of* (H1) *via complex exponentials*

When the auxiliary roots for (H1) are $p \pm iq$ then $y = Ae^{(p+iq)x} + Be^{(p-iq)x}$ is a solution for any complex A, B. Applying Euler's identity,

$$\begin{aligned} y &= Ae^{(p+iq)x} + Be^{(p-iq)x} \\ &= Ae^{px}(\cos qx + i\sin qx) + Be^{px}(\cos qx - i\sin qx) \\ &= (A + B)e^{px}\cos qx + i(A - B)e^{px}\sin qx. \end{aligned}$$

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For any real or complex C, D there exist A, B such that $A + B = C$, $i(A - B) = D$. Hence the real solutions are $y = Ce^{px}\cos qx + De^{px}\sin qx$ for real constants C, D.

5. *The linear differential equation with complex coefficients*

Consider the differential equation

$$ay'' + by' + cy = 0 \qquad \text{(HC)}$$

where a, b, c are real or complex coefficients.

The independent variable is denoted by x and is real, and the unknown $y = \phi(x)$ is real- or complex-valued. A solution of (HC) is a complex-valued function $\phi : J \to \mathbb{C}$ with domain some interval $J \subset \mathbb{R}$ such that ϕ has two derivatives in J and $a\phi''(x) + b\phi'(x) + c\phi(x) = 0$ for all $x \in J$. Since $\phi''(x) = -\frac{b}{a}\phi'(x) - \frac{c}{a}\phi(x)$, we see that a solution ϕ is infinitely differentiable. A family of complex functions $y = \phi(x, c_1, c_2)$ is said to constitute all solutions if a member of this family exists that is a solution that satisfies any given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_1$ where x_0 is real, and y_0, y_1 are complex. It is essential also to restrict to real coefficients a, b, c and seek real solutions. In this case the family of real functions $y = \phi(x, c_1, c_2)$ constitute all solutions if a member of this family exists that is a solution that satisfies any given initial conditions $y(x_0) = y_0$, $y'(x_0) = y_1$ where x_0 is real, and y_0, y_1 real.

Theorem: Consider

$$ay'' + by' + cy = 0 \qquad \text{(HC)}$$

where a, b, c are real or complex coefficients.

The function $y = e^{mx}$ is a complex-valued solution of (HC) if, and only if,

$$am^2 + bm + c = 0.$$

(i) If the roots are distinct, $m_1 \neq m_2$, then all complex solutions of (HC) are

$$y = c_1e^{m_1x} + c_2e^{m_2x}$$

where c_1, c_2 are *complex* integration constants.

(ii) If the roots are repeated, denoted m, then all complex solutions of (HC) are

$$y = c_1e^{mx} + c_2xe^{mx}$$

where c_1, c_2 are *complex* integration constants.

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6. *Real solutions of* (HC)

Geometrical vectors **a**, **b**, **c** in space are called linearly independent if the relation $\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = \mathbf{0}$ implies that the scalars α, β and γ are all zero. Linear independence is equivalent to three conditions (i) $\mathbf{a} \neq \mathbf{0}$, (ii) **b** is not a scalar multiple of **a**, and (iii) **c** is not a linear combination of **a**, **b**. These remarks apply in any vector space. Thus for a vector space of real or complex-valued functions defined on $\mathbb{R}$, functions f, g, h are linearly independent if (i) $f \neq 0$ (zero function), (ii) g is not a scalar multiple of f, and (iii) h is not a linear combination of f, g, where the scalars are real or complex numbers depending on the ground field of the vector space.

The set of solutions $\mathbb{S}_C$ of (HC) is a complex vector space of dimension 2. The subset $\mathbb{S}_R$ of real solutions is a real vector space.

Example 1: Find all real solutions of $y'' - (1 + i)y' + iy = 0$.

The quadratic $m^2 - (1 + i)m + i = (m - 1)(m - i)$ has zeros $m = 1, i$. All complex solutions are $y = Ce^x + De^{ix}$ for complex constants C, D. Write $C = a + ib, D = c + id$, then

$$y = (a + ib)e^x + (c + id)(\cos x + i \sin x)$$

$$= (ae^x + c \cos x - d \sin x) + i(be^x + d \cos x + c \sin x).$$

The solution y is real if $be^x + d \cos x + c \sin x \equiv 0$. The functions $u_1 = \cos x, u_2 = \sin x, u_3 = e^x$ are independent, since: $u_1 \neq 0$, u_2 is not a multiple of u_1, and u_3 is not a combination of u_1, u_2. Hence b, d and c are all zero. All real solutions (contained in the set of complex solutions) are given by $y = ae^x$ with real a. Thus $\mathbb{S}_R$ has dimension 1.

Example 2: Find all real solutions of $y'' - 3iy' - 2y = 0$.

The auxiliary quadratic $m^2 - 3im - 2 = (m - i)(m - 2i)$ has zeros $m = i, 2i$. All complex solutions are $y = Ce^{ix} + De^{2ix}$ with C, D complex constants. Write $C = a + ib, D = c + id$, then

$$y = (a + ib)(\cos x + i \sin x) + (c + id)(\cos 2x + i \sin 2x)$$

$$= (a \cos x - b \sin x + c \cos 2x - d \sin 2x)$$

$$+i(b \cos x + a \sin x + d \cos 2x + c \sin 2x).$$

If y is real, then $b\cos x + a\sin x + d\cos 2x + c\sin 2x \equiv 0$. To see independence, write

$$b\cos x + a\sin = r_1 \cos(x + \varepsilon_1) \text{ and } d\cos 2x + c\sin 2x = r_2 \cos(2x + \varepsilon_2).$$

Thus $r_1 \cos(x + \varepsilon_1) + r_2 \cos(2x + \varepsilon_2) \equiv 0$. Differentiating twice we have $-r_1 \cos(x + \varepsilon_1) - 4r_2 \cos(2x + \varepsilon_2) \equiv 0$. Adding we obtain $r_2 \cos(2x + \varepsilon_2) \equiv 0$. Hence $r_2 = 0$, and also $r_1 = 0$, thus $a = b = c = d = 0$. Hence there is only one real solution, $y = 0$, the trivial solution. Thus $\mathbb{S}_R$ has dimension 0.

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These examples serve to illustrate

Theorem: Consider

$$ay'' + by' + cy = 0 \qquad \text{(HC)}$$

where a, b, c are real or complex coefficients.

Let m_1, m_2 denote the roots of the equation $am^2 + bm + c = 0$.

(i) If $m_1 = p + iq, m_2 = p' + iq'$ are complex non-real, $q \neq 0, q' \neq 0$, and non-conjugate then the coefficients a, b, c are not all real, and $\mathbb{S}_R$ has dimension 0.

(ii) If $m_1 = p + iq$, $m_2 = p - iq$ are complex conjugates, $q \neq 0$, then the coefficients a, b, c are real, and $\mathbb{S}_R$ has dimension 2.

(iii) If m_1 is real, and $m_2 = p + iq$ is complex non-real, $q \neq 0$, then the coefficients a, b, c are not all real, and $\mathbb{S}_R$ has dimension 1.

(iv) If m_1, m_2 are real, then the coefficients a, b, c are real, and $\mathbb{S}_R$ has dimension 2.

Proof: (i) Suppose that $m_1 = p + iq, m_2 = p' + iq', q \neq 0, q' \neq 0$, are non-conjugate. We divide into cases, when $m_1 = m_2$ and $m_1 \neq m_2$.

(i.a) If $m_1 = m_2 = p + iq, q \neq 0$, then complex solutions are

$$y = (a + ib)e^{px}(\cos qx + i \sin qx) + (c + id)xe^{px}(\cos qx + i \sin qx).$$

If y is real then

$$a \sin qx + b \cos qx + cx \sin qx + dx \cos qx \equiv 0.$$

Write

$$a \sin qx + b \cos qx = r_1 \cos(qx + \varepsilon_1) \text{ and } c \sin qx + d \cos qx = r_2 \cos(qx + \varepsilon_2).$$

Then

$$r_1 \cos(qx + \varepsilon_1) + r_2 x \cos(qx + \varepsilon_2) \equiv 0.$$

Now $u_1 = \cos(qx + \varepsilon_1) \neq 0$ (since $q \neq 0$), and $u_2 = x \cos(qx + \varepsilon_2)$ is not a scalar multiple of u_1. Hence $r_1 = 0 = r_2$, so a, b, c, d are all zero. The real solutions are $y = 0$.

(i.b) If $m_1 \neq m_2$ then complex solutions are

$$y = (a + ib)e^{px}(\cos qx + i \sin qx) + (c + id)e^{p'x}(\cos q'x + i \sin q'x).$$

If y is real then

$$ae^{px} \sin qx + be^{px} \cos qx + ce^{p'x} \sin q'x + de^{p'x} \cos q'x \equiv 0. \qquad (3)$$

As m_1, m_2 are non-conjugate and $m_1 \neq m_2$, we take two cases.

(i.b.1) $p = p'$ and $q \neq \pm q', q \neq 0, q' \neq 0$. Then, from (3)

$$a \sin qx + b \cos qx + c \sin q'x + d \cos q'x \equiv 0.$$

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Write

$a \sin qx + b \cos qx = r_1 \cos(qx + \varepsilon_1)$ and $c \sin q'x + d \cos q'x = r_2 \cos(q'x + \varepsilon_2)$.

Thus

$$r_1 \cos(qx + \varepsilon_1) + r_2 \cos(q'x + \varepsilon_2) \equiv 0.$$

Differentiate twice: $-q^2 r_1 \cos(qx + \varepsilon_1) - q'^2 r_2 \cos(q'x + \varepsilon_2) \equiv 0$. Now $q'^2 r_1 \cos(qx + \varepsilon_1) + q'^2 r_2 \cos(q'x + \varepsilon_2) \equiv 0$. Adding, we obtain $r_1 (q'^2 - q^2) \cos(qx + \varepsilon_1) \equiv 0$. Therefore $r_1 = 0$, since $q' \neq \pm q$ and $q \neq 0$. It follows that also $r_2 = 0$. Thus $a = b = c = d = 0$. The real solutions are $y = 0$.

(i.b.2) $p \neq p'$ and $q \neq 0, q' \neq 0$. Then from (3)

$$e^{px} r_1 \cos(qx + \varepsilon_1) + e^{p'x} r_2 \cos(q'x + \varepsilon_2) \equiv 0.$$

We have $r_1 e^{\lambda x} \cos(qx + \varepsilon_1) + r_2 \cos(q'x + \varepsilon_2) \equiv 0$, where $\lambda = p - p' \neq 0$. Since $u_1 = \cos(q'x + \varepsilon_2) \neq 0$, and $u_2 = e^{\lambda x} \cos(qx + \varepsilon_1)$ is not a multiple of u_1, we have $r_1 = 0 = r_2$, thus $a = b = c = d = 0$. The real solutions are $y = 0$.

Thus $\mathbb{S}_R$ has dimension 0.

(ii) If $m_1, m_2 = p \pm iq, q \neq 0$, then complex solutions are

$$y = (a + ib) e^{px} (\cos qx + i \sin qx) + (c + id) e^{px} (\cos qx - i \sin qx).$$

If y is real then

$$e^{px} \{(a - c) \sin qx + (b + d) \cos qx\} \equiv 0.$$

Hence $a = c, b = -d$. Real solutions are $y = e^{px} (2a \cos qx + 2d \sin qx)$.

Thus $\mathbb{S}_R$ has dimension 2.

(iii) If m_1 is real and $m_2 = p + iq$ where $q \neq 0$, then complex solutions are

$$y = (a + ib) e^{m_1 x} + (c + id) e^{px} (\cos qx + i \sin qx).$$

If y is real then

$$be^{m_1 x} + e^{px} (d \cos qx + c \sin qx) \equiv 0.$$

Write $u_1 = \sin qx$, $u_2 = \cos qx$, $u_3 = e^{(m_1 - p)x}$. Then $u_1 \neq 0$, u_2 is not a multiple of u_1, and u_3 is not a combination of u_1, u_2. Thus u_1, u_2, u_3 are independent. Hence $b = c = d = 0$. Real solutions are $y = ae^{m_1 x}$.

Thus $\mathbb{S}_R$ has dimension 1.

(iv) Let m_1, m_2 be real. There are two cases.

(iv.a) If $m_1 \neq m_2$, then complex solutions are

$$y = (a + ib) e^{m_1 x} + (c + id) e^{m_2 x}.$$

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If y is real then

$$be^{m_1x} + de^{m_2x} \equiv 0.$$

Hence $b = d = 0$. Real solutions are $y = ae^{m_1x} + ce^{m_2x}$.

(iv.b) If $m_1 = m_2$, then complex solutions are

$$y = (a + ib)e^{m_1x} + (c + id)xe^{m_1x}.$$

If y is real then

$$be^{m_1x} + dxe^{m_1x} \equiv 0.$$

Hence $b = 0 = d$. Real solutions are $y = ae^{m_1x} + cxe^{m_1x}$.

Thus $\mathbb{S}_R$ has dimension 2.

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1. C. Plumpton and W. A. Tomkys: *Sixth form pure mathematics, Volume two*, Pergamon Press (1963).

10.1017/mag.2019.57

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