Pappus in two dimensions, Mamikon in three

It pays students meeting a geometrical theorem to see if it has useful analogues in other dimensions.

In *The Collection* [1] Pappus laments the decline of Alexandrian mathematics since the golden age of Archimedes and Apollonius but says that the lean years have at least yielded *this*, and he quotes (without proof) what contemporary students know as centroid theorems. (Paul Guldin published proofs in 1640, hence the frequent attribution to him.) One deals with the surface area of a solid of revolution. The other, which concerns us here, deals with its volume:

> The volume of a solid of revolution is the product of the area of cross-section in the plane containing, and to one side of, the axis of rotation and the circumference of the path its centroid follows.

The two-dimensional analogue would read:

> The *area* of a *region* of revolution is the product of the *length* of cross-section in a *line* containing, and to one side of, the *centre* of rotation, and the circumference of the path its centroid follows.

The 'region of revolution' is simply an annulus (Figure 1). The centroid is the midpoint of segment L, length $l = R - r$, the circumference of its path $2\pi\rho = 2\pi\left(\frac{R + r}{2}\right)$. The analogue is not useful as a means of computation since we can find the area more simply by subtracting the area of the inner circle from that of the outer, but considering the annular lamina and integrating along the axis of rotation provides one way to prove the three-dimensional theorem.

Compare Figure 2(a). We have a chord of the outer circle, tangent to the inner, of half-length t. As a schoolboy, Mamikon Mnatsakanian was required to prove that the area of the annulus is πt^2. Recognising the right triangle OMC (Figure 2(b)), this he readily did. But what intrigued him was that, in completing a circuit of the figure, the segment T sweeps out the required area, which is therefore that of the circle in Figure 2(c) [2, p. 6].

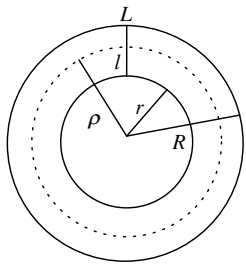


FIGURE 1

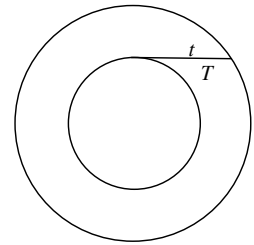


FIGURE 2(a)

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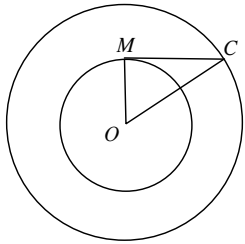


FIGURE 2(b)

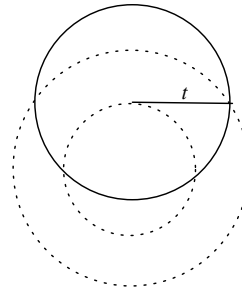


FIGURE 2(c)

He generalised this insight as Mamikon's sweeping-tangent method. [2, chapter 1], [3] cite many problems to which the method provides a short, elementary route to the solution. A classic example is the determination of the area defined by a tractrix, its vertical tangent and horizontal asymptote. Germans call this curve the *Hundkurve*. The image is that of a dog on a lead by which it is persuaded to depart from its intended direction and follow the owner. Hugo Steinhaus' model [4, p. 250], and that used in [2, pp. 13-14] and [5], is a toy cart pulled by a child (Figure 3).

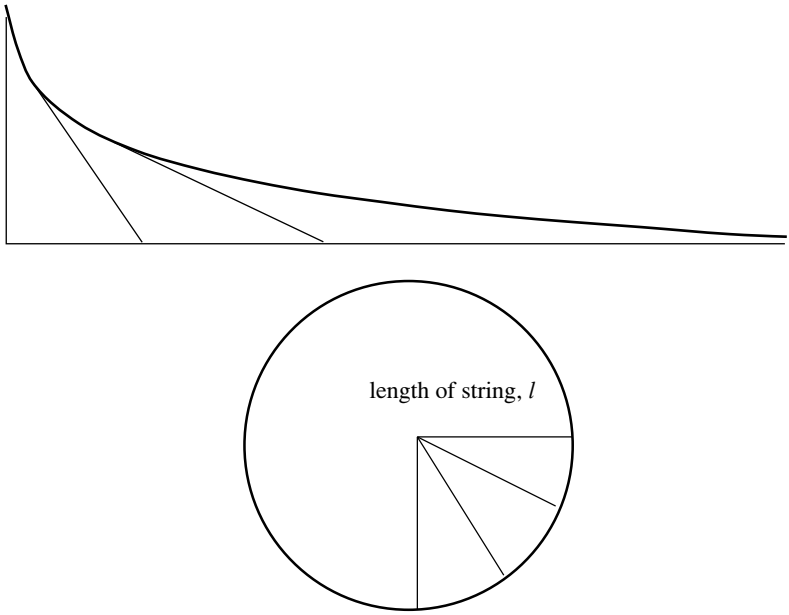


FIGURE 3

The string is tangent to the curve at every point. Mamikon's observation is that it sweeps out a series of infinitesimal sectors of radius l, in the course of which it makes a quarter-turn. The required area is therefore $\frac{\pi l^2}{4}$. The

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method comes into its own in situations where the tangents are not equal in length, but the reader will do better to explore these applications in [2], with its one thousand (sic) beautiful colour illustrations, than settle for my pale imitations.

We shall stick with the original case identified in Figures 2(a), 2(b), 2(c) and seek a three-dimensional analogue. (Note Mamikon's own three-dimensional extension. In [2, pp. 12-13] he shows how his method handles space curves. Note also that Pappus' theorem can itself be extended to cover the case where the centroid follows a space curve [6].) As in the Pappus case, we can treat Figure 2(a)/(b) as a lamina and integrate. However, this is only useful if the tangents all lie in the vertical plane containing MC. This is the case for a convex solid through which an axial, cylindrical hole has been bored. Corresponding to Figure 2(c) in the two-dimensional case, we generate a convex solid without a hole. But since in general we need to apply Pappus' theorem to the new solid, there is nothing to be gained.

An exception is the sphere. In three dimensions Figure 2(a), 2(b) becomes the horizontal projection of the solid and MC, DE vertical sections through it (Figure 4). MC is a semicircle, radius t, and the corresponding solid is generated by rotating this element a whole turn about the diameter through M. From the vertical section through to do DE we see that $t = \sqrt{R^2 - r^2}$:

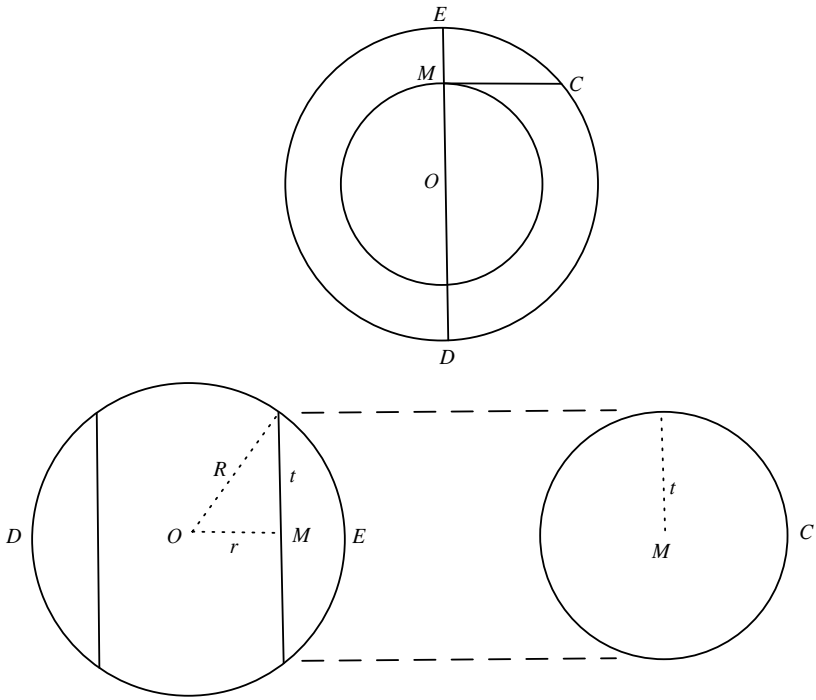


FIGURE 4

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The solid is therefore a sphere, volume $\frac{4}{3}\pi t^3 = \frac{4}{3}\pi\left(R^2 - r^2\right)^{3/2}$. In other words, if the bored sphere sits on a table, its volume is that of a complete sphere of the same height. Hence the title of the puzzle based on this observation, the 'napkin-ring problem' [7, pp. 54-55]. The questioner asks: What is the volume of a napkin-ring of height h? Martin Gardner's shrewd puzzler reasons that, since he has not been given the diameter of the hole, he is free to shrink it to zero, producing a sphere of diameter h.

I am pleased to see that Wikipedia now honours Mamikon by devoting the entry 'Visual calculus' to his method.

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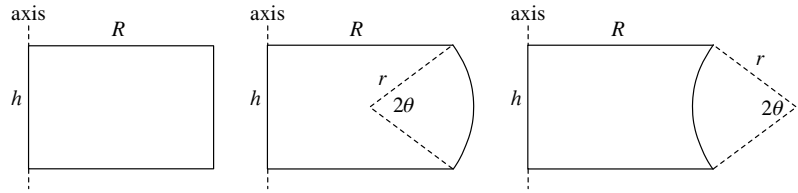
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PAUL STEPHENSON
Böhmerstraße 66, 45144 Essen, Germany
e-mail: *pstephenson1@me.com*

An area conundrum

Three solids of revolution are formed by rotating the areas in Figure 1 about the axes indicated. These form the basis of an interesting classroom discussion: which of solids (a), (b) and (c) has the greatest surface area? This is equivalent to asking: which of the solids has the greatest curved surface area?



(a) cylinder (b) outer curved edge (c) inner curved edge

FIGURE 1

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