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A note on surface geometry with special reference to twist

By P. G. LOWE
University of Strathclyde, Glasgow

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Abstract. Classical differential geometry defines two symmetrical fundamental forms, the first dealing with length measurement, the second essentially with curvature. In applications, particularly in the mechanics of plates and shells, certain unsymmetrical features of the fundamental forms are of interest and are here investigated.

1. *Introduction.* The geometry of surfaces, treated from the classical differential standpoint, begins by defining two quadratic forms. The so called first fundamental form has coefficients which in the older literature (5) are denoted by E, F, G; in the more modern literature (7) the equivalent quantities are denoted by the surface metric tensor, $a_{\alpha\beta}$. Essentially the first fundamental form, I, enables *lengths* to be measured on the surface. If the surface coordinates are denoted by u, v, then

$$\left.\begin{aligned} I &\equiv E\,du^2 + 2F\,du\,dv + G\,dv^2, \\ &\equiv a_{\alpha\beta}du^\alpha du^\beta. \end{aligned}\right\} \tag{1}$$

In order to describe the *shape* of the surface, a second fundamental form, II, is required. The coefficients of this quadratic form are most often denoted by L, M, N (or $b_{\alpha\beta}$), and

$$\left.\begin{aligned} II &\equiv L\,du^2 + 2M\,du\,dv + N\,dv^2, \\ &\equiv b_{\alpha\beta}du^\alpha du^\beta. \end{aligned}\right\} \tag{2}$$

The tensors $a_{\alpha\beta}$, $b_{\alpha\beta}$ are always symmetric, independently of the choice of surface coordinates; the choice of coordinates is arbitrary. In what follows tensors will not be used. Instead, second order matrices such as

$$A = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} L & M \\ M & N \end{bmatrix},$$

will be used. Note that these matrices too are symmetric, even when the surface coordinates u, v, are curvilinear and non-orthogonal.

The question investigated below is what are the (physical) components of the (2×2) matrix $\mathbf{k}$, which will be called the *curvature matrix*? Once $\mathbf{k}$ is known for some initial choice of coordinates u, v, at a given point, then the (normal) curvature *and twist* along any other direction through this point are defined. In the standard treatments there

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are theorems relating the new curvature to the values associated with u, v, for example Euler's Theorem (4), but statements relating to the twists appear to be lacking.

2. *Geometrical preliminaries.* Choose a system of (two) coordinates (u, v) on the surface. In general the net of u, v (space) curves will be curvilinear, and individual u, v curves will intersect, but not necessarily orthogonally. Let O be an origin, and a typical point on the surface $P(u, v)$ be at radius vector $\mathbf{r}(u, v)$ from the origin.

Then

$$\mathbf{r} = \mathbf{r}(u, v)$$

and a neighbouring point $Q(u+du, v+dv)$ will be distance ds from P such that

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

Here $E \equiv \mathbf{r}_1 \cdot \mathbf{r}_1$, $F \equiv \mathbf{r}_1 \cdot \mathbf{r}_2$, $G \equiv \mathbf{r}_2 \cdot \mathbf{r}_2$ and the notation

$$(\,)_1 = \frac{\partial(\,)}{\partial u}, \quad (\,)_2 = \frac{\partial(\,)}{\partial v}$$

has been adopted. The vectors $\mathbf{r}_1, \mathbf{r}_2$ are *tangent* vectors along the u, v directions respectively. Hence the *normal* vector $\mathbf{n}$ can be found from

$$\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|}. \tag{3}$$

As so defined, $\mathbf{n}$ is a *unit* normal vector, $|\mathbf{r}_1 \times \mathbf{r}_2| = \sqrt{(EG - F^2)} \equiv H$, and $\mathbf{r}_1, \mathbf{r}_2, \mathbf{n}$ form a right handed triad of vectors.

An important pair of relations then follow:

$$\mathbf{n}_1 = \left(\frac{FM - GL}{H^2}\right)\mathbf{r}_1 + \left(\frac{FL - EM}{H^2}\right)\mathbf{r}_2, \tag{4}$$

and the corresponding expression for $\mathbf{n}_2$. This fundamental pair of relations is due to Weingarten (6). The classical treatment now sections the surface at point P by a plane containing the normal $\mathbf{n}$; then in the neighbourhood of P the line of section is circular, with curvature κ_n given by

$$\kappa_n = \frac{II}{I}. \tag{5}$$

When the sectioning plane is taken tangential to the u line then $dv = 0$ and

$$\kappa_n \equiv \kappa_{uu} = \frac{L}{E}, \tag{6}$$

together with the companion relation, $\kappa_{vv} = N/G$. The curvature is negative when $\mathbf{n}$ as defined by (3) is directed away from the centre of curvature.

Apart from investigating principal directions, namely those directions for which κ_n is extremal, at this point classical differential geometry moves off to consider properties of geodesics and hence leaves the curvature scene. However, from the point of view of *use* of these results, there is still left unexplored the question of other components associated with κ_n.

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The important relation is (4), and the corresponding expression for $\mathbf{n}_2$.

Consider first how to recover the curvature κ_{uu} given by (6), but by use of (4) rather than (5). As an alternative to the classical definition of normal curvature given by (5), we may proceed by noting that the *curvature* is given by the arc rate of change of $\mathbf{n}$ in the $\mathbf{r}_1$ direction.

Recalling that $\mathbf{r}_1$ and $\mathbf{r}_2$ are not orthogonal, then the arc rate of change of $\mathbf{n}$ parallel to the $\mathbf{r}_1$ direction is given by

$$-\kappa_{uu} = \left(\frac{FM-GL}{H^2}\right)\sqrt{E}\times 1\times\frac{1}{\sqrt{E}}+\left(\frac{FL-EM}{H^2}\right)\sqrt{G}\times\frac{F}{\sqrt{(EG)}}\times\frac{1}{\sqrt{E}},$$

$$= -\frac{L}{E}.$$

Hence the classical curvature term is recovered. There remains for investigation the arc rate at which $\mathbf{n}$ changes *normal* to $\mathbf{r}_1$. This quantity is what will be termed the *twist*, and denoted by κ_{uv}. Note that this change of $\mathbf{n}$ *normal* to $\mathbf{r}_1$ must lie in the tangent plane since $\mathbf{n}$ is a *unit* normal, and hence any change can have only $\mathbf{r}_1$ and $\mathbf{r}_2$ components, and no $\mathbf{n}$ component.

Thus, from (4)

$$\kappa_{uv} = -\left(\frac{FL-EM}{H^2}\right)\sqrt{G}\times\frac{H}{\sqrt{EG}}\times\frac{1}{\sqrt{E}},$$

$$= \left(\frac{EM-FL}{EH}\right). \tag{7}$$

The sign convention for κ_{uv} used here is such that $\mathbf{n}$, looked at along $\mathbf{r}_1$ increasing, *rotates* according to a right hand screw rule.

By a similar analysis, but with reference to $\mathbf{r}_2$, there are obtained the two remaining curvature matrix ($\mathbf{k}$) components,

$$\kappa_{vv} = \frac{N}{G},$$

and (8)

$$\kappa_{vu} = \left(\frac{GM-FN}{GH}\right).$$

Then the curvature matrix $\mathbf{k}$ introduced earlier can be written for the choice of u, v coordinates as

$$\mathbf{k} \equiv \begin{bmatrix} \kappa_{uu} & \kappa_{uv} \\ \kappa_{vu} & \kappa_{vv} \end{bmatrix}, \tag{9}$$

$$= \begin{bmatrix} \dfrac{L}{E} & \dfrac{EM-FL}{EH} \\ \dfrac{GM-FN}{GH} & \dfrac{N}{G} \end{bmatrix}. \tag{10}$$

Note that in general this matrix is *not* symmetric. Indeed it is symmetric only when $F = 0$, namely when the coordinates u, v are orthogonal. Then $\kappa_{uv} = \kappa_{vu} = M/\sqrt{(EG)}$.

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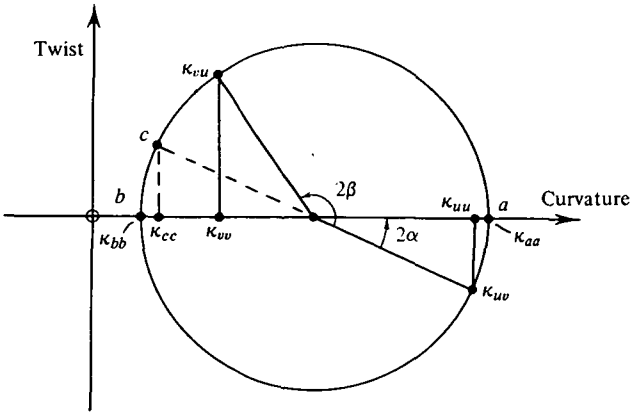


Fig. 1.

A convenient representation of the matrix **k** is by means of a circle diagram, analogous to the Mohr circle of stress (3, 2). The appropriate diagram is given in Fig. 1.

This representation summarises the curvature/twist information for all directions through the given point. Implied too on the circle diagram are the relations such as Euler's theorem (4),

$$\kappa_{uu} = \kappa_{aa}\cos^2\alpha + \kappa_{bb}\sin^2\alpha,$$

and Dupin's theorem (4),

$$\kappa_{aa} + \kappa_{bb} = \kappa_{uu} + \kappa_{cc}.$$

Here α is the (counter-clockwise) angle between the u direction and the principal direction a, and β is the angle between the tangent vectors $\mathbf{r}_1, \mathbf{r}_2$.

It seems strange that the circle construction is not referred to in any of the standard works on differential geometry, though the use of the circle construction in the context of *shallow* (nearly flat) surfaces is referred to in some mechanics texts (1), p. 11.

3. *An example.* Consider the hyperboloid of revolution

$$\frac{x^2+y^2}{a^2} - \frac{z^2}{b^2} = 1. \tag{11}$$

Let

$$\Omega^2 \equiv b^2 + z^2\left(1 + \left(\frac{a}{b}\right)^2\right),$$

then the principal curvatures can be shown to be given by

$$\kappa_{aa} = -\frac{b}{\Omega a}, \quad \kappa_{bb} = +\frac{ab}{\Omega^3}.$$

Here curvature is positive if the centre of curvature lies in the positive **n** direction.

Since these are principal curvatures, the associated twists are zero and

$$(\mathbf{k})_{a,b} = \begin{bmatrix} -\dfrac{b}{\Omega a} & 0 \\ 0 & +\dfrac{ab}{\Omega^3} \end{bmatrix}. \tag{12}$$

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The coordinate directions associated are latitude circles, and longitude meridians, treating z as the polar axis. These lines of curvature are here referred to as 'a' and 'b' coordinate lines (Fig. 2).

Suppose however it is decided to choose non-orthogonal coordinates on the hyperboloid to take advantage of the real (straight) asymptotic lines on the surface. Now use as coordinates a polar angle (θ) and a length (s) measured along the appropriate asymptotic line through a given point P (Fig. 2). Then it is straightforward to show that

$$\tan\phi = \frac{b}{a}, \quad \Omega^2\cos^2\phi = a^2\sin^2\phi + s^2\cos^2\phi,$$

$$E = a^2 + s^2\cos^2\phi,$$

$$F = a\cos\phi,$$

$$G = 1,$$

$$L = -\tan\phi\frac{E}{\Omega},$$

$$M = -\frac{a\sin\phi}{\Omega},$$

$$N = 0.$$

Hence

$$\kappa_{\theta\theta} = \frac{L}{E} = -\frac{\tan\phi}{\Omega} = -\frac{b}{\Omega a},$$

$$\kappa_{\theta s} = \frac{EM - FL}{EH} = 0,$$

$$\kappa_{ss} = \frac{N}{G} = 0,$$

$$\kappa_{s\theta} = \frac{GM - FN}{GH} = -\frac{a\tan\phi}{\Omega^2}.$$

Hence finally,

$$(\mathbf{k})_{\theta,s} = \begin{bmatrix} -\dfrac{\tan\phi}{\Omega} & 0 \\ -\dfrac{a\tan\phi}{\Omega^2} & 0 \end{bmatrix}.$$

This information can be represented on the circle diagram (Fig. 3) from which it can be found that

$$\tan\mu = \frac{\Omega}{a},$$

$$R = \frac{\sin\phi E}{2\Omega^3\cos^3\phi}.$$

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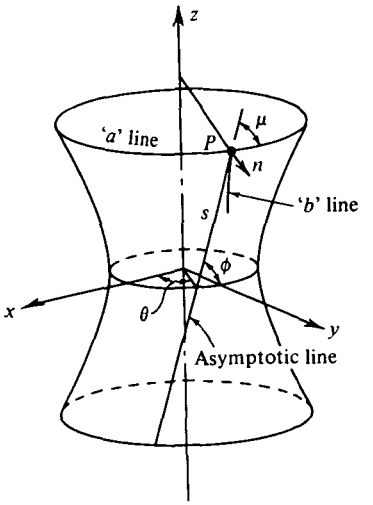


Fig. 2

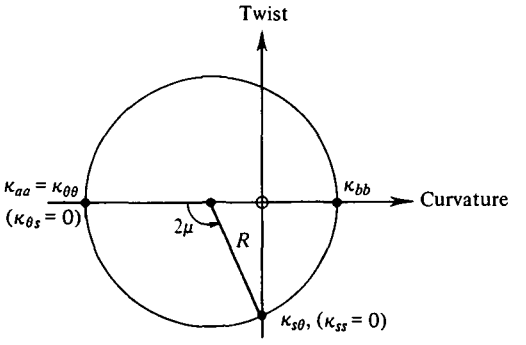


Fig. 3

Thus

$$\kappa_{bb} = 2R - |\kappa_{\theta\theta}|,$$
$$= \frac{\sin\phi E}{\Omega^3 \cos^3\phi} - \frac{\tan\phi}{\Omega},$$
$$= \frac{ab}{\Omega^3},$$

so confirming the values found by initial reference to the principal directions as coordinate directions.

4. *Conclusions.* The classical treatment of surface curvature is curiously incomplete. Interest is focussed on the normal curvature while the associated twist is not evaluated. In applications to mechanics of plate and shell structures, frequently it is not convenient or easy to avoid discussion of twist components and a systematic elementary evaluation of twist components in terms of components of the first and second fundamental forms is called for. This has been supplied in the present discussion.

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Postscript. Exceptionally symmetry of the curvature matrix may occur even though $F \neq 0$, namely the coordinates are not orthogonal. If $L = N = 0$, then symmetry of the matrix occurs. An example is the hyperbolic paraboloid in Monge form, $z = cxy$, where x, y, z are cartesian coordinates (28 September 1979).

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