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Stabilized C^*-algebras constructed from symbolic dynamical systems

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Abstract. We construct stabilized C^*-algebras from subshifts by using the dynamical property of the symbolic dynamical systems. We prove that the construction is dynamical and the C^*-algebras are isomorphic to the tensor product C^*-algebras between the algebra of all compact operators on a separable Hilbert space and the C^*-algebras constructed from creation operators on sub-Fock spaces associated with the subshifts. We also prove that the gauge actions on the stabilized C^*-algebras are invariant for topological conjugacy as two-sided subshifts under some conditions. Hence, if two subshifts are topologically conjugate as two-sided subshifts, the associated stabilized C^*-algebras are isomorphic so that their K-groups are isomorphic.

1. *Introduction*

In [**Ma**], the author introduced and studied a class of C^*-algebras associated with subshifts as a generalization of the Cuntz–Krieger algebras $\mathcal{O}_A$ for the topological Markov shifts. Many structural properties of the Cuntz–Krieger algebras have been generalized to the C^*-algebras of the class. For a subshift (Λ, σ), we denote by $\mathcal{O}_\Lambda$ the C^*-algebra associated with it. Then the C^*-algebra is generated by n-partial isometries $S_1, \ldots, S_n$ with $\sum_{j=1}^n S_j S_j^* = 1$. It has a universal property subject to some operator relations of the generating partial isometries [**Ma4**]. It is also a simple C^*-algebra under a certain irreducibility condition for (Λ, σ) [**Ma, Ma3**]. The C^*-algebra has a canonical action α of the one-dimensional torus group $\mathbb{T}$ that is defined by $\alpha_t(S_j) = e^{2\pi\sqrt{-1}t} S_j$ for $t \in \mathbb{T}(= \mathbb{R}/\mathbb{Z})$, $j = 1, \ldots, n$. The action α is called the gauge action and played a crucial role in analysing the structure of the C^*-algebra $\mathcal{O}_\Lambda$ as in [**Ma**]. The behaviour is closely related to a dynamical property for the subshift (Λ, σ) (cf. [**MWY**]). When we study a subshift as a topological dynamical system, we need to prove that the associated C^*-algebra object is invariant under topological conjugacy. For a matrix A with entries in $\{0, 1\}$, Cuntz and Krieger in [**CK**], proved that the isomorphism type of the stabilized Cuntz–Krieger algebra $\mathcal{O}_A \otimes \mathcal{K}$ with gauge action is invariant under topological conjugacy

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of a Markov shift $(\Lambda_A\sigma)$ as a two-sided subshift, where $\mathcal{K}$ is the C^*-algebra of all compact operators on a separable, infinite-dimensional Hilbert space. In [**Ma4**] (cf. [**Ma**]), it has been shown that the isomorphism type of the C^*-dynamical system $(\mathcal{O}_\Lambda, \alpha)$ is invariant under topological conjugacy of the associated one-sided subshift.

In this paper, we generalize the above Cuntz and Krieger's result for Markov shifts to subshifts in general. We will prove that the conjugacy class of the stabilized C^*-dynamical system $(\mathcal{O}_\Lambda \otimes \mathcal{K}, \alpha \otimes \mathrm{id})$ is invariant for topological conjugacy as a two-sided subshift under some conditions. Our proof is based on ideas of Cuntz's method for full shifts in [**C**] and Cuntz and Krieger's method for topological Markov shifts in [**CK**] (cf. [**C2, Ev**]). In the proof, we will introduce the notion of a dimensionally block covering set for a finite or countably infinite σ-invariant subset H of Λ. It is an analogue for a dimensionally covering set of a sofic shift defined by Krieger in [**Kr3**]. We will prove that the notion of dimensionally block covering set is invariant for topological conjugacy by using M. Nasu's factorization theorem of conjugacy [**N**]. We consider Boolean algebras of sets consisting of suitable subsets of $W^u(H)$ on which uniformly finite-dimensional homeomorphisms work well. We will show that the unstable manifolds $W^u(H)$ for H with uniformly finite-dimensional homeomorphisms and the Boolean algebra $\mathfrak{B}(H)$ are invariant for topological conjugacy of two-sided subshifts. As seen in the Cuntz and Krieger's proof of [**CK**, §3], we will know that the C^*-algebra $\mathcal{O}_\Lambda \otimes \mathcal{K}$ forms of the crossed product of an AF-algebra constructed from uniformly finite-dimensional homeomorphisms on $W^u(H)$ by an automorphisms of the shift. Since we know that the gauge action corresponds to the dual action of the shift, we will conclude that the stabilized C^*-algebra $\mathcal{O}_\Lambda \otimes \mathcal{K}$ with gauge action is an invariant of the conjugacy class of the subshift as a two-sided subshift. In particular, we see that the isomorphism class of the stabilized fixed point algebra $\mathcal{F}_\Lambda (= \mathcal{O}_\Lambda^\alpha) \otimes \mathcal{K}$ under gauge action α is invariant for topological conjugacy of two-sided subshift (Λ, σ). As a corollary, we obtain that the following K-theoretic groups for these C^*-algebras are topological conjugacy invariants for two-sided subshifts:

$$K_0(\mathcal{O}_\Lambda), \quad K_1(\mathcal{O}_\Lambda), \quad Ext(\mathcal{O}_\Lambda), \quad K_0(\mathcal{F}_\Lambda).$$

In the above groups, $K_0(\mathcal{F}_\Lambda)$ corresponds to the dimension group for subshift (Λ, σ) (cf. [**Kr, Kr3, Le, Ma2, Ma3**]) and $Ext(\mathcal{O}_\Lambda)$ corresponds to the Bowen–Franks group for subshift (Λ, σ) (cf. [**BF, Ma5**]).

2. *Subshifts and the associated C^*-algebras*

In the rest of this paper, a finite set $\Sigma = \{1, 2, \ldots, n\}$ is fixed.

Let $\Sigma^{\mathbb{Z}}$ and $\Sigma^{\mathbb{N}}$ be, respectively, the infinite product spaces $\prod_{i=-\infty}^{\infty} \Sigma_i$ and $\prod_{i=1}^{\infty} \Sigma_i$ where $\Sigma_i = \Sigma$, endowed with the product topology. The transformation σ on $\Sigma^{\mathbb{Z}}, \Sigma^{\mathbb{N}}$ given by $(\sigma(x))_i = x_{i+1}$, $i \in \mathbb{Z}, \mathbb{N}$ is called the (full) shift. Let Λ be a shift invariant closed subset of $\Sigma^{\mathbb{Z}}$, i.e. $\sigma(\Lambda) = \Lambda$. The topological dynamical system $(\Lambda, \sigma|_\Lambda)$ is called a subshift. We write $\sigma|_\Lambda$ as σ for simplicity. We sometimes write subshift (Λ, σ) as Λ. We denote by X_Λ the set of all right infinite sequences that appear in Λ. The dynamical system (X_Λ, σ) is called the (right) one-sided subshift for Λ.

A finite sequence $\mu = (\mu_1, \ldots, \mu_k)$ of elements $\mu_j \in \Sigma$ is called a block or a word. We denote by $|\mu|$ the length k of μ. A block $\mu = (\mu_1, \ldots, \mu_k)$ is said to occur or to

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appear in $x = (x_i) \in \Sigma^{\mathbb{Z}}$ if $x_m = \mu_1, \ldots, x_{m+k-1} = \mu_k$ for some $m \in \mathbb{Z}$. We write it as $x_{[m,m+k-1]}$. For $x \in \Sigma^{\mathbb{Z}}$ and $m \in \mathbb{Z}$, we denote by $x_{[m,\infty)}$ and $x_{(-\infty,m]}$ the right infinite sequence $(x_m, x_{m+1}, \ldots,)$ and the left infinite sequence $(\ldots, x_{m-1}, x_m)$ respectively.

For a subshift (Λ, σ) and $k \in \mathbb{N}$, let Λ^k be the set of all words with length k in $\Sigma^{\mathbb{Z}}$ occurring in some $x \in \Lambda$. We set $\Lambda_l = \cup_{k=0}^{l} \Lambda^k$ and $\Lambda^* = \cup_{k=0}^{\infty} \Lambda^k$ where Λ^0 denotes the empty word $\emptyset$. Put

$$\Lambda_l(x) = \{\mu \in \Lambda_l \mid \mu x \in X_\Lambda\}, \quad \text{for } x \in X_\Lambda,\ l \in \mathbb{N}.$$

We define equivalence relations in X_Λ. For $l \in \mathbb{N}$, two points $x, y \in X_\Lambda$ are said to be l-past equivalent if $\Lambda_l(x) = \Lambda_l(y)$ [**Ma3**]. We write this equivalence as $x \sim_l y$. Let F_i^l, $i = 1, \ldots, m(l)$, be the set of all l-past equivalence classes of X_Λ. Hence we have $X_\Lambda = \cup_{i=1}^{m(l)} F_i^l$. We denote by $\Omega_l = X_\Lambda/\sim_l$ the quotient space of the l-past equivalence classes of X_Λ. We endow Ω_l with discrete topology for each $l \in \mathbb{N}$. As there exists a sequence of natural surjections :

$$\Omega_1 \leftarrow \Omega_2 \leftarrow \cdots \leftarrow \Omega_{l+1} \leftarrow \Omega_{l+2} \leftarrow \cdots,$$

we may set

$$\Omega_\Lambda = \varprojlim \Omega_l$$

the projective limit as a topological space along the above sequence of surjections.

Now let us recall the construction of the C^*-algebra associated with subshift (Λ, σ) introduced in [**Ma**].

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of an n-dimensional Hilbert space $\mathbb{C}^n$. We put:

- $\mathfrak{H}_\Lambda^0 = \mathbb{C}e_0$ where e_0 is the vacuum vector,
- $\mathfrak{H}_\Lambda^k$ = the Hilbert space spanned by the vectors $e_\mu = e_{\mu_1} \otimes \cdots \otimes e_{\mu_k}$, for $\mu = (\mu_1, \ldots, \mu_k) \in \Lambda^k$,
- $\mathfrak{H}_\Lambda = \oplus_{k=0}^{\infty} \mathfrak{H}_\Lambda^k$ (Hilbert space direct sum).

We denote by T_ν, $(\nu \in \Lambda^*)$ the creation operator on $\mathfrak{H}_\Lambda$ of e_ν, $\nu \in \Lambda^* (\nu \neq \emptyset)$ defined by

$$T_\nu e_0 = e_\nu \quad \text{and} \quad T_\nu e_\mu = \begin{cases} e_\nu \otimes e_\mu, & (\nu\mu \in \Lambda^*) \\ 0, & \text{else,} \end{cases}$$

which is a partial isometry. We put $T_\nu = 1$ for $\nu = \emptyset$. Let $\mathbb{P}_0$ the rank one projection onto the vacuum vector e_0. It immediately follows that $\sum_{i=1}^n T_i T_i^* + \mathbb{P}_0 = 1$. We then easily see that for $\mu, \nu \in \Lambda^*$, the operator $T_\mu \mathbb{P}_0 T_\nu^*$ is the rank one partial isometry from the vector e_ν to e_μ. Hence, the C^*-algebra generated by elements of the form $T_\mu \mathbb{P}_0 T_\nu^*$, $\mu, \nu \in \Lambda^*$ is nothing but the C^*-algebra $\mathcal{K}(\mathfrak{H}_\Lambda)$ of all compact operators on $\mathfrak{H}_\Lambda$. Let $\mathcal{T}_\Lambda$ be the C^*-algebra on $\mathfrak{H}_\Lambda$ generated by the elements T_ν, $\nu \in \Lambda^*$.

Definition [**Ma**]. The C^*-algebra $\mathcal{O}_\Lambda$ associated with subshift (Λ, σ) is defined as the quotient C^*-algebra $\mathcal{T}_\Lambda/\mathcal{K}(\mathfrak{H}_\Lambda)$ of $\mathcal{T}_\Lambda$ by $\mathcal{K}(\mathfrak{H}_\Lambda)$.

We denote by S_i and S_μ the quotient image of the operator T_i, $i \in \Sigma$ and T_μ, $\mu \in \Lambda^*$ respectively. Hence $\mathcal{O}_\Lambda$ is generated by n partial isometries $S_1, \ldots, S_n$ with relation $\sum_{i=1}^n S_i S_i^* = 1$.

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If a subshift is a topological Markov shift (Λ_A, σ), the associated C^*-algebra is nothing but the Cuntz–Krieger algebra $\mathcal{O}_A$ (cf. [**CK, EFW, Ev**]).

We review the notation and basic facts for studying the C^*-algebra $\mathcal{O}_\Lambda$ [**Ma**].

Put $a_\mu = S_\mu^* S_\mu$, $\mu \in \Lambda^*$. Since $T_\nu T_\nu^*$ commutes with $T_\mu^* T_\mu$, $\mu, \nu \in \Lambda^*$, the following identities hold

$$a_\mu S_\nu = S_\nu a_{\mu\nu}, \quad \mu, \nu \in \Lambda^*. \tag{$*$}$$

We notice that for $\mu, \nu \in \Lambda^*$ with $|\mu| = |\nu|$,

$$S_\mu^* S_\nu \neq 0 \quad \text{if and only if } \mu = \nu.$$

We will use the following notation. Let k, l be natural numbers with $k \leq l$.

- $\mathcal{A}_l$ is the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by a_μ, $\mu \in \Lambda_l$.
- $\mathcal{A}_\Lambda$ is the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by a_μ, $\mu \in \Lambda^*$.
- $\mathcal{F}_k^l$ is the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^k$, $a \in \mathcal{A}_l$.
- $\mathcal{F}_k^\infty$ is the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^k$, $a \in \mathcal{A}_\Lambda$.
- $\mathcal{F}_\Lambda$ is the C^*-subalgebra of $\mathcal{O}_\Lambda$ generated by $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^*$, $|\mu| = |\nu|$, $a \in \mathcal{A}_\Lambda$.

The projections $\{T_\mu^* T_\mu; \mu \in \Lambda^*\}$ are mutually commutative so that the C^*-algebras $\mathcal{A}_l$, $l \in \mathbb{N}$ are commutative. We can regard a minimal projection of $\mathcal{A}_l$ as the characteristic function of a subset of the subshift X_Λ. Then the support of the function is one of the set F_i^l, $i = 1, 2, \ldots, m(l)$ of all l-past equivalence classes of X_Λ. Hence the set of all minimal projections E_i^l, $i = 1, 2, \ldots, m(l)$ of $\mathcal{A}_l$ exactly corresponds to the set F_i^l, $i = 1, 2, \ldots, m(l)$ of Ω_l. Thus we see the following lemma (cf. [**Ma**, §3; **Ma3**, §4]).

LEMMA 2.1.

(i) *$\mathcal{A}_l$ is isomorphic to the commutative C^*-algebra $C(\Omega_l)$ of all continuous functions on Ω_l.*

(ii) *$\mathcal{A}_l$ is naturally embedded into $\mathcal{A}_{l+1}$ so that $\mathcal{A}_\Lambda = \varinjlim \mathcal{A}_l$ is isomorphic to the C^*-algebra $C(\Omega_\Lambda)$.*

(iii) *Each element of $\mathcal{F}_k^l$ is a finite linear combination of elements of the form $S_\mu a S_\nu^*$, $\mu, \nu \in \Lambda^k$, $a \in \mathcal{A}_l$. Hence $\mathcal{F}_k^l$ is finite-dimensional.*

(iv) *There are two embeddings in $\{\mathcal{F}_k^l\}_{k \leq l}$:*
(iv-a) *$\iota_l : \mathcal{F}_k^l \subset \mathcal{F}_k^{l+1}$ through the embedding $\mathcal{A}_l \subset \mathcal{A}_{l+1}$ and*
(iv-b) *$\lambda_k : \mathcal{F}_k^l \subset \mathcal{F}_{k+1}^{l+1}$ through the identity*

$$S_\mu a S_\nu^* = \sum_{j=1}^n S_{\mu j} S_j^* a S_j S_{\nu j}^*, \quad \mu, \nu \in \Lambda^k, a \in \mathcal{A}_l.$$

(v) *Both $\mathcal{F}_k^\infty = \lim_{l \to \infty} \mathcal{F}_k^l$ and $\mathcal{F}_\Lambda = \lim_{k \to \infty} \mathcal{F}_k^\infty$ are* AF*-algebras.*

The correspondence $S_i \to e^{2\pi\sqrt{-1}t} S_i$ for $t \in \mathbb{R}/\mathbb{Z}$, $i = 1, \ldots, n$ yields an action on $\mathcal{O}_\Lambda$ of the one-dimensional torus group $\mathbb{T}(= \mathbb{R}/\mathbb{Z})$. We denote the action by α and call it the gauge action. The AF-algebra $\mathcal{F}_\Lambda$ is realized as the fixed point algebra ${\mathcal{O}_\Lambda}^\alpha$ under gauge action α [**Ma**, Proposition 3.11].

We will describe a structure theorem for the C^*-algebra $\mathcal{O}_\Lambda$.

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LEMMA 2.2. [**Ma**, Theorems 4.9 and 5.2; **Ma3**]. *Let $\mathfrak{A}$ be a unital C^*-algebra. Suppose that there is a unital *-homomorphism π from $\mathcal{A}_\Lambda$ to $\mathfrak{A}$ and there are n partial isometries $s_1, \dots, s_n \in \mathfrak{A}$ satisfying the following relations*

(a) $\displaystyle\sum_{j=1}^{n} s_j s_j^* = 1,\ s_\mu^* s_\mu s_\nu = s_\nu s_{\mu\nu}^* s_{\mu\nu},\ \mu, \nu \in \Lambda^*,$

(b) $s_\mu^* s_\mu = \pi(S_\mu^* S_\mu),\ \mu \in \Lambda^*$

*where $s_\mu = s_{\mu_1} \cdots s_{\mu_k}$, $\mu = (\mu_1, \dots, \mu_k)$. Then there exists a unital *-homomorphism $\tilde{\pi}$ from $\mathcal{O}_\Lambda$ to $\mathfrak{A}$ such that $\tilde{\pi}(S_i) = s_i$, $i = 1, \dots, n$ and its restriction to $\mathcal{A}_\Lambda$ coincides with π. In addition, if the corresponding one-sided subshift (X_Λ, σ) satisfies the condition (I) below, this extended homomorphism $\tilde{\pi}$ becomes injective whenever π is injective.*

Condition (I). for any $l \in \mathbb{N}$ and $x \in X_\Lambda$, there exists $y \in X_\Lambda$ such that $y \neq x$ and $y \sim_l x$.

As in [**Ma3**], the condition (I) for subshift (X_Λ, σ) is equivalent to the condition (I_Λ) for the algebra $\mathcal{O}_\Lambda$ defined in [**Ma**].

Suppose that (X_Λ, σ) satisfies condition (I). Then $\mathcal{O}_\Lambda$ is simple if and only if (X_Λ, σ) is irreducible in past equivalence. If (X_Λ, σ) is aperiodic in past equivalence, $\mathcal{O}_\Lambda$ is purely infinite and $\mathcal{F}_\Lambda$ is simple [**Ma3**], where:

(i) (X_Λ, σ) is irreducible in past equivalence if for any $l \in \mathbb{N}$, $y \in X_\Lambda$ and a sequence $(x^k)_{k\in\mathbb{N}}$ of X_Λ with $x^k \sim_k x^{k+1}$, there exists a number N and a word $\mu \in \Lambda^N$ such that $y \sim_l \mu x^{l+N}$;

(ii) (X_Λ, σ) is aperiodic in past equivalence if for any $l \in \mathbb{N}$ there exists a number $N \in \mathbb{N}$ such that for any pair $x, y \in X_\Lambda$, there exists a word $\mu \in \Lambda^N$ such that $y \sim_l \mu x$.

If a subshift (X_Λ, σ) is aperiodic or irreducible in past equivalence with an aperiodic point, then it satisfies condition (I).

We notice the following (cf. [**CK**, Proposition 2.17]):

LEMMA 2.3. [**Ma**, Proposition 5.8; **Ma3**, Lemma 4.5]. *Let (Λ_1, σ) and (Λ_2, σ) be subshifts satisfying condition (I). If the associated one-sided subshifts (X_{Λ_1}, σ) and (X_{Λ_2}, σ) are topologically conjugate, then there exists an isomorphism Φ from $\mathcal{O}_{\Lambda_1}$ onto $\mathcal{O}_{\Lambda_2}$ such that $\Phi \circ \alpha_z^1 = \alpha_z^2 \circ \Phi$, $z \in \mathbb{T}$, where α^i is the gauge action on $\mathcal{O}_{\Lambda_i}$, $i = 1, 2$ respectively.*

3. *AF-algebras constructed from one-sided subshifts*

In this section, we represent the AF-algebra $\mathcal{F}_\Lambda$ by a crossed product of a C^*-algebra of functions on X_Λ by a group action arising from exchanging words of X_Λ.

Fix an arbitrary subshift (Λ, σ) over $\Sigma = \{1, \dots, n\}$. Let us recall that the subsets F_i^l of X_Λ, $i = 1, 2, \dots, m(l)$ denote the set of all l-past equivalence classes of X_Λ so that we have a sequence $\{F_i^l \mid i = 1, 2, \dots, m(l)\}_{l\in\mathbb{N}}$ of refinements of the space X_Λ. For fixed k, $l = 0, 1, \dots$, with $k \leq l$, we denote by $\mathfrak{B}_{\Lambda_l^k}$ the Boolean algebra of sets generated by all subsets of X_Λ of the form

$$\mu F_i^l = \{(x_1, \dots, x_k, x_{k+1}, \dots) \in X_\Lambda \mid x_j = \mu_j,\ j = 1, 2, \dots, k, (x_{k+1}, \dots) \in F_i^l\}$$

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for $\mu = (\mu_1, \dots, \mu_k) \in \Lambda^k$ and $i = 1, 2, \dots, m(l)$. Let $\mathfrak{B}_{\Lambda_\infty^k}$ be the Boolean algebra of sets generated by $\cup_{l=0}^\infty \mathfrak{B}_{\Lambda_l^k}$ and $\mathfrak{B}_\Lambda$ the Boolean algebra of sets generated by $\cup_{k=0}^\infty \mathfrak{B}_{\Lambda_\infty^k}$. Put

$$A_i^l = \{j = 1, \dots, m(l+1) \mid F_j^{l+1} \subset F_i^l\}$$

so that we have refinements

$$F_i^l = \cup_{j \in A_i^l} F_j^{l+1} \quad \text{and hence} \quad \mu F_i^l = \cup_{j \in A_i^l} \mu F_j^{l+1} \tag{3.1}$$

for $\mu \in \Lambda^k$ with $\mu F_i^l \subset X_\Lambda$.

LEMMA 3.1. *For any $r \in \Sigma = \{1, 2, \dots, n\}$ and $j = 1, 2, \dots, m(l+1)$, we have $rF_j^{l+1} \subset F_{\lambda(r,j)}^l$ for some $\lambda(r, j) = 1, 2, \dots, m(l)$ or $rF_j^{l+1} = \emptyset$.*

Put

$$B_i^l = \{(r, j) \in \Sigma \times \{1, 2, \dots, m(l+1)\} \mid rF_j^{l+1} \subset F_i^l\}$$

so that we have refinements

$$F_i^l = \cup_{(r,j) \in B_i^l} rF_j^{l+1} \quad \text{and hence} \quad \mu F_i^l = \cup_{(r,j) \in B_i^l} \mu r F_j^{l+1} \tag{3.2}$$

for $\mu \in \Lambda^k$ with $\mu F_i^l \subset X_\Lambda$.

LEMMA 3.2. *The above refinements (3.1) and (3.2) yield inclusion maps of the Boolean algebras*

$$\iota_k : \mathfrak{B}_{\Lambda_l^k} \to \mathfrak{B}_{\Lambda_{l+1}^k} \quad \textit{and} \quad \lambda_k : \mathfrak{B}_{\Lambda_l^k} \to \mathfrak{B}_{\Lambda_{l+1}^{k+1}}$$

respectively.

We set:

- $\mathcal{D}_k^l = \operatorname{Diag} \mathcal{F}_k^l =$ the C^*-algebra generated by $S_\mu a S_\mu^*$ for $\mu \in \Lambda^k$, $a \in \mathcal{A}_l$;
- $\mathcal{D}_k^\infty = \operatorname{Diag} \mathcal{F}_k^\infty =$ the C^*-algebra generated by $S_\mu a S_\mu^*$ for $\mu \in \Lambda^k$, $a \in \mathcal{A}_\Lambda$;
- $\mathcal{D}_\Lambda = \operatorname{Diag} \mathcal{F}_\Lambda =$ the C^*-algebra generated by $S_\mu a S_\mu^*$ for $\mu \in \Lambda^*$, $a \in \mathcal{A}_\Lambda$.

For a subset $U \subset X_\Lambda$, we denote by χ_U the characteristic function of U on X_Λ. Let $B(X_\Lambda)$ be the C^*-algebra of all bounded functions on X_Λ. We consider the following C^*-subalgebras of $B(X_\Lambda)$:

- $C^*(\mathfrak{B}_{\Lambda_l^k}) =$ the C^*-algebra on X_Λ generated by χ_U for $U \in \mathfrak{B}_{\Lambda_l^k}$;
- $C^*(\mathfrak{B}_{\Lambda_\infty^k}) =$ the C^*-algebra on X_Λ generated by χ_U for $U \in \mathfrak{B}_{\Lambda_\infty^k}$;
- $C^*(\mathfrak{B}_\Lambda) =$ the C^*-algebra on X_Λ generated by χ_U for $U \in \mathfrak{B}_\Lambda$.

LEMMA 3.3. *The correspondence*

$$\chi_{\mu F_i^l} \to S_\mu E_i^l S_\mu^*, \quad \textit{for } \mu \in \Lambda^k,\ i = 1, 2, \dots, m(l)$$

yields the following isomorphisms of C^*-algebras:

(i) $C^*(\mathfrak{B}_{\Lambda_l^0}) \cong \mathcal{A}_l$ *and* $C^*(\mathfrak{B}_{\Lambda_\infty^0}) \cong \mathcal{A}_\Lambda$;

(ii) $C^*(\mathfrak{B}_{\Lambda_l^k}) \cong \mathcal{D}_k^l$, $C^*(\mathfrak{B}_{\Lambda_\infty^k}) \cong \mathcal{D}_k^\infty$ *and* $C^*(\mathfrak{B}_\Lambda) \cong \mathcal{D}_\Lambda$.

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Now let Γ_k^l be the set of all bijections u on X_Λ such that $u(\mathfrak{B}_{\Lambda_l^k}) = \mathfrak{B}_{\Lambda_l^k}$ and

$$u((x_i))_j = x_j, \quad \text{for all } j > k,\ (x_i) \in X_\Lambda.$$

Then Γ_k^l becomes a group naturally and we see

$$\Gamma_k^l = \oplus_{i=1}^{m(l)} S(|\Lambda_l^k(i)|)$$

where $S(N)$ denotes the symmetric group on N-elements and $|\Lambda_l^k(i)|$ denotes the cardinality of the set

$$\Lambda_l^k(i) = \{\mu \in \Lambda^k \mid \mu x \in X_\Lambda \text{ for } x \in F_i^l\}.$$

Since each element u of Γ_k^l gives rise to an automorphism on the Boolean algebra $\mathfrak{B}_{\Lambda_l^k}$, the group acts on the algebra $C^*(\mathfrak{B}_{\Lambda_l^k})$. Let $u_t, t \in \Gamma_k^l$ be the unitary representation of Γ_k^l in the multiplier algebra of the crossed product $C^*(\mathfrak{B}_{\Lambda_l^k}) \times \Gamma_k^l$. Let J_k^l be the ideal in the algebra $C^*(\mathfrak{B}_{\Lambda_l^k}) \times \Gamma_k^l$ generated by elements of the form $u_t \chi_W - u_s \chi_W$ for $t, s \in \Gamma_k^l$ and $W \in \mathfrak{B}_{\Lambda_l^k}$ such that $t|_W = s|_W$.

LEMMA 3.4. $C^*(\mathfrak{B}_{\Lambda_l^k}) \times \Gamma_k^l / J_k^l \cong \mathcal{F}_k^l$.

Let Γ_k be the set of all bijections u on X_Λ such that $u(\mathfrak{B}_{\Lambda_\infty^k}) = \mathfrak{B}_{\Lambda_\infty^k}$ and

$$u((x_i))_j = x_j, \quad \text{for all } j > k,\ (x_i) \in X_\Lambda.$$

Let J_k be the ideal in the algebra $C^*(\mathfrak{B}_{\Lambda_\infty^k}) \times \Gamma_k$ generated by elements of the form $u_t \chi_W - u_s \chi_W$ for $t, s \in \Gamma_k$ and $W \in \mathfrak{B}_{\Lambda_\infty^k}$ such that $t|_W = s|_W$. We directly see the following lemma.

LEMMA 3.5. *There are natural inclusion relations that make the following sequences commutative.*

(i)

$$\begin{array}{ccc} \mathcal{F}_k^l & \xrightarrow{\iota_k} & \mathcal{F}_k^{l+1} \\ \| & & \| \\ C^*(\mathfrak{B}_{\Lambda_l^k}) \times \Gamma_k^l / J_k^l & \longrightarrow & C^*(\mathfrak{B}_{\Lambda_{l+1}^k}) \times \Gamma_k^{l+1} / J_k^{l+1}. \end{array}$$

(ii)

$$\begin{array}{ccc} \mathcal{F}_k^\infty & \xrightarrow{\iota_k} & \mathcal{F}_{k+1}^\infty \\ \| & & \| \\ C^*(\mathfrak{B}_{\Lambda_\infty^k}) \times \Gamma_k / J_k & \longrightarrow & C^*(\mathfrak{B}_{\Lambda_\infty^{k+1}}) \times \Gamma_{k+1} / J_{k+1}. \end{array}$$

Let Γ_Λ be the set of all bijections u on X_Λ such that $u(\mathfrak{B}_\Lambda) = \mathfrak{B}_\Lambda$ and for some $k \in \mathbb{N}$

$$u((x_i))_j = x_j, \quad \text{for all } j > k,\ (x_i) \in X_\Lambda.$$

Let J be the ideal in the algebra $C^*(\mathfrak{B}_\Lambda) \times \Gamma$ generated by elements of the form $u_t \chi_W - u_s \chi_W$ for $t, s \in \Gamma$ and $W \in \mathfrak{B}_\Lambda$ such that $t|_W = s|_W$. By the above lemma with Lemma 2.1, we may represent the AF-algebra $\mathcal{F}_\Lambda$ as a crossed product of an action of homeomorphisms on the space X_Λ in the following way.

PROPOSITION 3.6. $\mathcal{F}_\Lambda \cong C^*(\mathfrak{B}_\Lambda) \times \Gamma_\Lambda / J$.

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4. *Unstable manifolds and dimensionally covering sets*

In this section, we introduce the notion of a dimensionally block covering set for a finite or countably infinite σ-invariant set $H \subset \Lambda$. We then prove that the notion is invariant for topological conjugacy. We also study Boolean algebras associated with the unstable manifold $W^u(H)$ of H. We denote by X_Λ^- the set of all left infinite sequences appearing in Λ. That is,

$$X_\Lambda^- = \left\{ (\ldots, z_2, z_1, z_0) \in \prod_{i=-\infty}^{0} \Sigma_i \;\middle|\; (z_i) \in \Lambda \right\}.$$

We first provide some notation following [**CK, C2, Kr3**]. For an element $h \in \Lambda$ and a subset $H \subset \Lambda$, we set

$$\mathcal{L}(h) = \{y \in X_\Lambda^- \mid y_k = h_k \text{ for all } k \leq -m, \text{ for some } m \in \mathbb{N}\},$$
$$\mathcal{L}(H) = \cup_{h \in H} \mathcal{L}(h).$$

For $a \in \mathcal{L}(h)$, put

$$Z(a) = \{y \in \Lambda \mid (y_k)_{k \leq 0} = a\}.$$

Let d be a metric defining the topology on $\Sigma^{\mathbb{Z}}$. Since Λ is a subset of $\Sigma^{\mathbb{Z}}$, it has the induced metric. Let $H \subset \Lambda$ be a finite or countably infinite σ-invariant set. Put for $j \in \mathbb{Z}$ and $h \in H$,

$$W_j(h) = \{x \in \Lambda \mid x_m = h_m \text{ for all } m \leq j\},$$
$$W_j(H) = \{x \in \Lambda \mid x_m = h_m \text{ for all } m \leq j \text{ for some } h \in H\}$$
$$= \cup_{h \in H} W_j(h).$$

Consider the unstable manifold $W^u(H)$ of H,

$$W^u(H) = \cup_{j \in \mathbb{Z}} W_j(H).$$

Hence, we have

$$W^u(H) = \{x \in \Lambda \mid d(\sigma^n(x), \sigma^n(h)) \to 0 \text{ as } n \to \infty \text{ for some } h \in H\}.$$

Each set $W_j(H)$ is endowed with the topology from the space Λ. As there exists a sequence of inclusion relations:

$$W^u(H) \supset \cdots \supset W_{j-1}(H) \supset W_j(H) \supset \cdots \tag{4.1}$$

and the inclusions $W_{j-1}(H) \hookleftarrow W_j(H)$ are continuous, we may put $W^u(H)$ the inductive limit topology of the sequence $W_j(H)$, $j \in \mathbb{Z}$. Since each $W_j(h)$ is compact and satisfies $\sigma(W_j(H)) = W_{j-1}(H)$, we know that $W^u(H)$ is second countable, Hausdorff and σ-compact and invariant under the shift. For $j \in \mathbb{Z}$, we consider the following subsets of the form

$$U_j(h; \mu, \nu) = \{x = (x_m) \in W_j(h) \mid x_{[j+1, j+k]} = \mu, \nu x_{[j+k+1, \infty)} \in X_\Lambda\}$$

for $h \in H$, $\mu \in \Lambda^k$ and $\nu \in \Lambda^l$ with $k \leq l$. Let $\mathfrak{B}_j(H)$ be the Boolean algebra of sets in $W_j(H)$ generated by all sets of the above form. The elements of $\mathfrak{B}_j(H)$ consist of the set

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of all finite unions and finite intersections and complements in the sets of the above form. Corresponding to the sequence (4.1), we have a natural sequence of inclusion relations of the Boolean algebras:

$$\cdots \supset \mathfrak{B}_{j-1}(H) \supset \mathfrak{B}_j(H) \supset \cdots .$$

We denote by $\mathfrak{B}(H)$ the inductive limit $\varinjlim \mathfrak{B}_j(H) = \cup_{j\in\mathbb{Z}}\mathfrak{B}_j(H)$. For subshifts Λ, Λ' and subsets $H \subset \Lambda$, $H' \subset \Lambda'$, a bijection from $\mathfrak{B}(H)$ onto $\mathfrak{B}(H')$ is said to be an isomorphism if for each $j \in \mathbb{Z}$ the restriction to the Boolean algebra $\mathfrak{B}_j(H)$ is an isomorphism as Boolean algebra.

A shift commuting homeomorphism between subshifts is called a topological conjugacy or simply a conjugacy. It is well known that a conjugacy between subshifts is given by a block map (cf. [**He, LM, Ki**]).

LEMMA 4.1. *Let (Λ, σ), (Λ', σ) be subshifts. If φ is a conjugacy from Λ to Λ', its restriction to $W^u(H)$ yields a homeomorphism between $W^u(H)$ and $W^u(\varphi(H))$ that gives rise to an isomorphism from $\mathfrak{B}(H)$ onto $\mathfrak{B}(\varphi(H))$.*

Proof. It is obvious that the conjugacy φ induces a homeomorphism between $W^u(H)$ and $W^u(\varphi(H))$. We will prove that it gives rise to an isomorphism from $\mathfrak{B}(H)$ onto $\mathfrak{B}(\varphi(H))$. Put for $h \in H$, $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \in \Lambda^k$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_l) \in \Lambda^l$ with $k \le l$,

$$W_{j+k}(h_{(-\infty,j]}\mu) = \{x \in \Lambda \mid h_{(-\infty,j]}\mu = x_{(-\infty,j+k]}\},$$
$$F_m(\nu) = \{x \in \Lambda \mid \nu x_{[m+1,\infty)} \in X_\Lambda\}$$

so that we have

$$U_j(h; \mu, \nu) = W_{j+k}(h_{(-\infty,j]}\mu) \cap F_{j+k}(\nu).$$

Hence the Boolean algebra $\mathfrak{B}_j(H)$ is generated by the set of the forms:

$$W_{j+k}(h_{(-\infty,j]}\mu), \quad F_{j+k}(\nu) \cap W_j(h)$$

in $W^u(H)$. As φ is induced by a block map, we see that

$$\varphi(W_{j+k}(h_{(-\infty,j]}\mu)) = W_{j'+k'}(\varphi(h)_{(-\infty,j']}\mu')$$

for some $j' \in \mathbb{Z}$ and $\mu' \in {\Lambda'}^{k'}$. For any word ν of Λ' and $i \in \mathbb{Z}$, fix an arbitrary point $y \in \varphi^{-1}(F_i(\nu))$. Put $x = \varphi(y)$ and hence $\nu x_{[i+1,\infty)} \in X_{\Lambda'}$. Take $z \in X_\Lambda^-$ and put $x' = z\nu x_{[i+1,\infty)} \in \Lambda'$ and $y' = \varphi^{-1}(x')$. Since φ^{-1} is a sliding block code, there exists a block map Φ that induces φ^{-1}. There exists $j'' \in \mathbb{Z}$ that depends only on i such that $y'_{[j''+1,\infty)} = y_{[j''+1,\infty)}$. Take $m \in \mathbb{N}$ and a block $\xi = y'_{[j-m,j]}$ in y' with $\Phi(\xi) = \nu$. We may choose j large enough so that $j'' < j$ and $\nu(= x'_{[i-l,i]})$ is a sub-block of $\Phi(\xi)$. We write $\Phi(\xi) = x'_{[i-l',i+l'']}$. Take $r \in \mathbb{Z}$ that satisfies $\varphi(b)_{[r,\infty)} = \varphi(b')_{[r,\infty)}$ for all $b, b' \in \Lambda$ with $b_{[j+1,\infty)} = b'_{[j+1,\infty)}$. For $M > r$, we may find $p > j$ such that $\Phi(\xi y_{[j,p]}) = x'_{[i-l',M]}$. We then see $\Phi(y_{[q,p]}) = x_{[i-l',M]}$ where $q = j - m$. Put $\mu = y_{[q,p]}$. We will show that

$$y \in F_j(\xi) \cap [\mu]_q^p \subset \varphi^{-1}(F_i(\nu)) \tag{4.2}$$

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where we denote $[\mu]_q^p = \{x \in \Lambda | x_{[q,p]} = \mu\}$. It is obvious that $y \in F_j(\xi)$ because $y_{[j+1,\infty)} = y'_{[j+1,\infty)}$. Hence we see $y \in F_j(\xi) \cap [\mu]_q^p$. For any $a \in F_j(\xi) \cap [\mu]_q^p$, find $z' \in X_\Lambda^-$ with $z'\xi a_{[j+1,\infty)}$ that is denoted by $a' \in \Lambda$. Hence we have

$$\varphi(a')_{[r,\infty)} = \varphi(a)_{[r,\infty)}. \tag{4.3}$$

It follows that $a'_{[q,p]} = \xi a_{[j+1,p]} = \xi y_{[j+1,p]}$. We see $\Phi(a'_{[q,p]}) = x'_{[i-l',M]}$ so that

$$\varphi(a')_{[i-l',M]} = x'_{[i-l',M]}. \tag{4.4}$$

As $a_{[q,p]} = y_{[q,p]}$, we have $\Phi(a_{[q,p]}) = x_{[i-l',M]}$ so that

$$\varphi(a)_{[i-l',M]} = x_{[i-l',M]}. \tag{4.5}$$

By the above equalities (4.4) and (4.5) and the equality $x'_{[i,r]} = x_{[i,r]}$ with $r \leq M$, we get

$$\varphi(a')_{[i,r]} = \varphi(a)_{[i,r]}. \tag{4.6}$$

Thus we have, by (4.3) and (4.6)

$$\varphi(a')_{[i,\infty)} = \varphi(a)_{[i,\infty)}.$$

By $\varphi(a')_{[i-l,i]} = \nu$, we obtain $\varphi(a)$ belongs to $F_i(\nu)$ so that the inclusion relation (4.2) holds. In the above discussion, we may take the numbers j, p, q and the length $|\xi|$ of ξ independently of the choice of y. They depend only on φ and $F_i(\nu)$. Thus we may find finitely many words $\xi(1), \ldots, \xi(L)$ and $\mu(1), \ldots, \mu(N)$ such that

$$\cup_{r=1}^{L} \cup_{s=1}^{N} (F_j(\xi(r)) \cap [\mu(s)]_q^p) = \varphi^{-1}(F_i(\nu)).$$

Hence the set of the form $\varphi^{-1}(F_i(\nu) \cap W_{j'}(\varphi(h)))$ for $j' < i$ and $h \in H$ is an element of $\mathfrak{B}(H)$ so that we have $\varphi^{-1}(\mathfrak{B}(\varphi(H)) \subset \mathfrak{B}(H)$. We similarly have $\varphi(\mathfrak{B}(H)) \subset \mathfrak{B}(\varphi(H))$. Thus we conclude $\varphi(\mathfrak{B}(H)) = \mathfrak{B}(\varphi(H))$ and φ gives rise to an isomorphism. □

For $z \in X_\Lambda^-$ and $\mu \in \Lambda^*$, we put

$$F(z) = \{b \in X_\Lambda | zb \in \Lambda\}, \quad F(\mu) = \{b \in X_\Lambda | \mu b \in X_\Lambda\}$$

which we call the future of z and the future of μ, respectively.

We easily have the following lemma.

LEMMA 4.2. *Let $\mu \in \Lambda^k$ and $y \in X_\Lambda^-$ satisfy $y\mu \in X_\Lambda^-$ and $F(y\mu) = F(\mu)$. Then for any word ν with $y = y'\nu$ for some $y' \in X_\Lambda^-$, we have $F(\nu\mu) = F(\mu)$.*

For $z_1, z_2 \in X_\Lambda^-$, we write as $z_1 \sim_F z_2$ if $F(z_1) = F(z_2)$. We call this equivalence future equivalence. Let $P_\gamma, \gamma \in \Gamma_\Lambda$ be the set of all future equivalence classes of X_Λ^-. Hence we have

$$X_\Lambda^- = \cup_{\gamma \in \Gamma_\Lambda} P_\gamma \text{ : disjoint union.}$$

Now we introduce the following condition called (E) for Λ.

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Condition (E). For any $\mu \in \Lambda^*$, there exists $y \in X_\Lambda^-$ such that

$$y\mu \in X_\Lambda^- \quad \text{and} \quad F(y\mu) = F(\mu).$$

That is, any word μ has an infinite past extension with the same future.

We say that a finite or countably infinite σ-invariant subset H of Λ is *dimensionally block covering for* Λ if it has the following two properties.

(i) For any $\mu \in \Lambda^*$, there exists $y \in \mathcal{L}(H)$ such that $y\mu \in \mathcal{L}(H)$ and $F(y\mu) = F(\mu)$.

(ii) For any $z \in \mathcal{L}(H)$, there exist $y \in \mathcal{L}(H)$ and $\mu \in \Lambda^*$ such that $z = y\mu$ and $F(z) = F(\mu)$.

Examples of dimensionally block covering sets are as follows.

Example. (a) For an irreducible topological Markov shift Λ, take an arbitrary point $h \in \Lambda$ and set $H = \{\sigma^n(h) \mid n \in \mathbb{Z}\}$. Then the H is dimensionally block covering.

(b) For a subshift Λ that contains a symbol $a \in \Sigma$ such that $a\mu$ is admissible for any block $\mu \in \Lambda^*$, put $h = (\ldots, a, a, a, \ldots) \in \Lambda$ and $H = \{h\}$. Then the H is dimensionally block covering (cf. [**Ma6**]).

This notation is an analogy to Krieger's notation of dimensionally covering for sofic subshifts defined in [**Kr3**].

We will prove that dimensionally block covering sets are preserved under topological conjugacy. For the proof, we will use a factorization of topological conjugacy between subshifts that has been proved by Nasu in [**N**]. Nasu introduced the notion of bipartite conjugacy and proved that any conjugacy can be factorized into compositions of bipartite conjugacies and symbolic conjugacies. A subshift Λ over the symbols Σ is said to be bipartite if there are disjoint subset C and D of Σ such that for any $(x_i)_{i\in\mathbb{Z}} \in \Lambda$ either $x_i \in C$, $x_{i+1} \in D$ for all $i \in \mathbb{Z}$ or $x_i \in D$, $x_{i+1} \in C$ for all $i \in \mathbb{Z}$. Then, the second power shift

$$\Lambda^{(2)} = \{(x_{2i}x_{2i+1})_{i\in\mathbb{Z}} \mid (x_i)_{i\in\mathbb{Z}} \in \Lambda\}$$

is divided into the two disjoint subshifts Λ_{CD} and Λ_{DC} where Λ_{CD} and Λ_{DC} are defined to be the set of all sequences $(c_i d_i)_{i\in\mathbb{Z}} \in \Lambda^{(2)}$ and $(d_i c_i)_{i\in\mathbb{Z}} \in \Lambda^{(2)}$ for $c_i \in C$, $d_i \in D$ and $i \in \mathbb{Z}$ respectively. The conjugacy $\zeta^+ : \Lambda_{CD} \to \Lambda_{DC}$ defined by $\zeta^+((c_i d_i)_{i\in\mathbb{Z}}) = (d_i c_{i+1})_{i\in\mathbb{Z}}$ is said to be forward bipartite conjugacy. The backward bipartite conjugacy $\zeta^- : \Lambda_{CD} \to \Lambda_{DC}$ is defined by $\zeta^-((c_i d_i)_{i\in\mathbb{Z}}) = (d_{i-1} c_i)_{i\in\mathbb{Z}}$. The symbolic conjugacy is defined as a conjugacy obtained by changing the names of the symbols.

LEMMA 4.3. *Suppose that both subshifts Λ and Λ' satisfy (E). Let φ be a conjugacy from a subshift Λ onto Λ'. If a finite or countably infinite σ-invariant set $H \subset \Lambda$ has the previous property (i) in Λ, then the subset $\varphi(H) \subset \Lambda'$ also has the property (i).*

Proof. We will show that the following property (i′) for a finite or countably infinite σ-invariant subset H is preserved by a conjugacy.

(i′) There exists $K \in \mathbb{N}$ such that for any $\mu \in \Lambda^*$ with $F(y\mu) = F(\mu)$ for some $y \in X_\Lambda^-$ and any $\nu \in \Lambda^K$ with $\mu\nu \in \Lambda^*$, there exists $w \in \mathcal{L}(H)$ such that $w\mu\nu \in \mathcal{L}(H)$ and $F(w\mu\nu) = F(\mu\nu)$.

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The shift σ clearly preserves the property (i′). Since the composition $\varphi\circ\sigma^{-l}$ is a forward conjugacy for some $l \in \mathbb{N}$ in the sense of Nasu in [**N, N2**], it suffices to prove the assertion for a forward bipartite conjugacy $\zeta^+ : \Lambda_{CD} \to \Lambda_{DC}$.

Now assume that $H \subset \Lambda_{CD}$ satisfies the above (i′). Put $K' = K + 1$. For $\mu = (\mu_{D_1}, \mu_{C_1})(\mu_{D_2}, \mu_{C_2}) \cdots (\mu_{D_l}, \mu_{C_l}) \in \Lambda^*_{DC}$ with $F(y\mu) = F(\mu)$ for some $y \in X^-_{\Lambda_{DC}}$ and $\nu = (\nu_{D_1}, \nu_{C_1})(\nu_{D_2}, \nu_{C_2}) \cdots (\nu_{D_{K'}}, \nu_{C_{K'}}) \in \Lambda^{K'}_{DC}$ with $\mu\nu \in \Lambda^*_{DC}$, set

$$\mu' = (y_{C_0}, \mu_{D_1})(\mu_{C_1}, \mu_{D_2}) \cdots (\mu_{C_{l-1}}, \mu_{D_l})(\mu_{C_l}, \nu_{D_1}) \in \Lambda^*_{CD},$$
$$\nu' = (\nu_{C_1}, \nu_{D_2})(\nu_{C_2}, \nu_{D_3}) \cdots (\nu_{C_{K'-1}}, \nu_{D_{K'}}) \in \Lambda^K_{CD},$$
$$y' = \cdots (y_{C_{-2}}, y_{D_{-1}})(y_{C_1}, y_{D_0}) \in X^-_{\Lambda_{CD}}$$

where $y = \cdots (y_{D_{-1}}, y_{C_{-1}})(y_{D_0}, y_{C_0}) \in X^-_{\Lambda_{DC}}$. As $y\mu\nu$ is admissible in Λ_{DC}, we have $\mu'\nu' \in \Lambda^*_{CD}$. Since $F(y'\mu') = F(\mu')$ in Λ_{CD} and $H \subset \Lambda_{CD}$ satisfies the condition (i′), we can find $w' \in \mathcal{L}(H) \subset X^-_{\Lambda_{CD}}$ such that $F(w'\mu'\nu') = F(\mu'\nu')$. Put

$$w = \cdots (w_{D_{-2}}, w_{C_{-1}})(w_{D_{-1}}, w_{C_0})(w_{D_0}, y_{C_0})$$

where $w' = \cdots (w_{C_{-2}}, w_{D_{-2}})(w_{C_{-1}}, w_{D_{-1}})(w_{C_0}, w_{D_0}) \in X^-_{\Lambda_{CD}}$. As $w'\mu'$ is admissible, we see that the sequence w belongs to $X^-_{\Lambda_{DC}}$ and to $\mathcal{L}(\zeta^+(H))$. It is also obvious that the sequence $w\mu$ is admissible. Then we obtain that $F(w\mu\nu) = F(\mu\nu)$. Thus we can see that the property (i′) is preserved under a conjugacy.

We may easily see that if a subshift Λ satisfies (E), the property (i′) for $H \subset \Lambda$ is equivalent to the following property (i″).

(i″) There exists $K \in \mathbb{N}$ such that for any $\mu \in \Lambda^*$ with $|\mu| > K$, there exists $w \in \mathcal{L}(H)$ such that $w\mu \in \mathcal{L}(H)$ and $F(w\mu) = F(\mu)$.

For a word $\mu \in \Lambda^*$ with $F(y\mu) = F(\mu)$ for some $y \in X^-_\Lambda$, put $\eta = y_{[-K,0]} \in \Lambda^{K+1}$ so that we have $F(y\mu) = F(\eta\mu) = F(\mu)$. We set $\xi = \eta\mu$ and hence $|\xi| > K$, $F(\xi) = F(\mu)$. Hence it follows that property (i″) implies property (i) under the condition that Λ satisfies (E). Therefore we conclude that if Λ satisfies (E), property (i) is equivalent to property (i″) and hence to property (i′). Thus property (i) for H is invariant under topological conjugacy. □

LEMMA 4.4. *Let φ be a conjugacy from Λ onto Λ'. If a finite or countably infinite σ-invariant set $H \subset \Lambda$ has the previous property (ii) in Λ, then the subset $\varphi(H) \subset \Lambda'$ also has property (ii).*

Proof. We may consider a conjugacy φ as a forward bipartite conjugacy $\zeta^+ : \Lambda_{CD} \to \Lambda_{DC}$ for a subshift Λ.

Now assume that $H \subset \Lambda_{CD}$ has property (ii). We will prove that the subset $\zeta^+(H) \subset \Lambda_{DC}$ also has property (ii). For

$$z = \cdots (d_{-2}, c_{-2})(d_{-1}, c_{-1})(d_0, c_0) \in \mathcal{L}(\zeta^+(H)),$$

put

$$w = \cdots (c_{-3}, d_{-2})(c_{-2}, d_{-1})(c_{-1}, d_0) \in X^-_{\Lambda_{CD}}.$$

We have $w \in \mathcal{L}(H)$. As H has property (ii) in Λ_{CD}, there exists $K \in \mathbb{N}$ such that

$$F(w) = F((c_{-K}, d_{-K+1})(c_{-K+1}, d_{-K+2}) \cdots (c_{-1}, d_0)).$$

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We set

$$\mu = (d_{-K}, c_{-K})(d_{-K+1}, c_{-K+1}) \cdots (d_0, c_0) \in \Lambda_{DC}^*.$$

We directly see that $F(z) = F(\mu)$ in Λ_{DC}. Thus $\zeta^+(H)$ has property (ii). □

Thus we see the following.

COROLLARY 4.5. *Suppose that both subshifts Λ and Λ' satisfy (E). Let φ be a conjugacy from a subshift Λ onto Λ'. If $H \subset \Lambda$ is dimensionally block covering, so is $\varphi(H) \subset \Lambda'$.*

We next see that the unstable manifold $W^u(H)$ for a dimensionally block covering set H does not depend on the choice of H as long as H is dimensionally block covering for Λ.

Set

$$E_\Lambda^- = \{z \in X_\Lambda^- \mid F(z) = F(z_{[-N,0]}) \text{ for some } N \in \mathbb{N}\}.$$

For a set S, we denote by $|S|$ its cardinality.

LEMMA 4.6. *For a word $\mu \in \Lambda^*$, if the cardinality of the set*

$$|\{y \in E_\Lambda^- \mid F(y) = F(\mu)\}|$$

is finite, then for any dimensionally block covering set $H \subset \Lambda$, we have

$$|\{y \in E_\Lambda^- \mid F(y) = F(\mu)\}| = |\{y \in \mathcal{L}(H) \mid F(y) = F(\mu)\}|.$$

Proof. Put

$$E_\Lambda^-(\mu) = \{y \in E_\Lambda^- \mid F(y) = F(\mu)\},$$
$$\mathcal{L}(H; \mu) = \{y \in \mathcal{L}(H) \mid F(y) = F(\mu)\}.$$

Let $E_\Lambda^-(\mu) = \{y(1), y(2), \ldots, y(m)\}$. As $\mathcal{L}(H; \mu) \subset E_\Lambda^-(\mu)$, we suppose that there exists $y(i)$ for some $i = 1, 2, \ldots, m$ such that $y(i)$ does not belong to $\mathcal{L}(H)$. Since $y(i) \neq y(j)$ for each $j = 1, 2, \ldots, m$ with $j \neq i$, we find $N \in \mathbb{N}$ so that $y(i)_{[-N,0]} \neq y(j)_{[-N,0]}$ for $j = 1, 2, \ldots, m$ with $j \neq i$ and $F(y(k)_{[-N,0]}) = F(y(k))$ for all $k = 1, \ldots, m$. Put $\xi = y(i)_{[-N,0]}$. Since H is dimensionally block covering, we can take $z \in \mathcal{L}(H)$ such that $F(z\xi) = F(\xi)$. Set $y(0) = zy(i)_{[-N,0]}$ so that we have $F(y(0)) = F(\mu)$ by Lemma 4.2. As $y(0) \in \mathcal{L}(H)$, we have $y(0) \neq y(i)$ and hence $y(0) \neq y(j)$, $j = 1, 2, \ldots, m$, a contradiction. □

We also have the following.

LEMMA 4.7. *For a word $\mu \in \Lambda^*$, if the cardinality of the set*

$$|\{y \in E_\Lambda^- \mid F(y) = F(\mu)\}|$$

is infinite, then for any dimensionally block covering set $H \subset \Lambda$, the cardinality of the set

$$|\{y \in \mathcal{L}(H) \mid F(y) = F(\mu)\}|$$

is infinite.

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Proof. Suppose that the cardinality of the set

$$|\{y \in E_\Lambda^- \mid F(y) = F(\mu)\}|$$

is infinite. Take a countable infinite family $y(1), y(2), \dots$ in the above set with $y(i) \neq y(j)$ for $i \neq j$. We may assume that any of $y(i), i \in \mathbb{N}$ does not belong to $\mathcal{L}(H)$. As $y(i) \in E_\Lambda^-$, there exists a sequence of numbers $N_i \in \mathbb{N}$ such that $F(y(i)) = F(y(i)_{[-N_i,0]})$. Take $M_1 \in \mathbb{N}$ with $M_1 \geq N_1$. Since H is dimensionally block covering, we may find $z_1 \in \mathcal{L}(H)$ such that $F(z_1 y(1)_{[-M_1,0]}) = F(y(1)_{[-M_1,0]})$. Put $x(1) = z_1 y(1)_{[-M_1,0]}$ so that we have $x(1) \in \mathcal{L}(H)$ and $F(x(1)) = F(\mu)$. Assume that we may take $x(1), x(2), \dots, x(k)$ such that $x(i) \in \mathcal{L}(H)$, $F(x(i)) = F(\mu)$ for all $i = 1, \dots, k$ and $x(i) \neq x(j)$ for $i \neq j$. Since $y(k+1)$ does not belong to $\mathcal{L}(H)$, we may take $M_{k+1} \in \mathbb{N}$ with $M_{k+1} \geq N_{k+1}$ and $y(k+1)_{[-M_{k+1},0]}$ does not coincide with any of $x(i)_{[-M_{k+1},0]}$ for $i = 1, \dots, k$. Since H is dimensionally block covering, we may find $z_{k+1} \in \mathcal{L}(H)$ such that $F(z_{k+1} y(k+1)_{[-M_{k+1},0]}) = F(y(k+1)_{[-M_{k+1},0]})$. Put $x(k+1) = z_{k+1} y(k+1)_{[-M_{k+1},0]}$ so that we have $x(k+1) \in \mathcal{L}(H)$, $F(x(k+1)) = F(\mu)$ and $x(k+1) \neq x(i)$ for $i = 1, \dots, k$. Thus we may find a countably infinite sequence in the set

$$\{y \in \mathcal{L}(H) \mid F(y) = F(\mu)\}. \qquad \square$$

COROLLARY 4.8. *If $H_1, H_2 \subset \Lambda$ are dimensionally block covering, then for any word $\mu \in \Lambda^*$, we have*

$$|\{z \in \mathcal{L}(H_1) \mid F(z) = F(\mu)\}| = |\{z \in \mathcal{L}(H_2) \mid F(z) = F(\mu)\}|.$$

PROPOSITION 4.9. *If both sets $H_1, H_2 \subset \Lambda$ are dimensionally block covering, there exists a homeomorphism φ from $W^u(H_1)$ onto $W^u(H_2)$ such that $(\varphi(x))_k = x_k$ for all $k \in \mathbb{N}$ and $\varphi(\mathfrak{B}(H_1)) = \mathfrak{B}(H_2)$.*

Proof. Let $\tilde{E}_\Lambda^-$ be the set of all future equivalence classes of E_Λ^-. For any $\gamma \in \tilde{E}_\Lambda^-$, the cardinality of the sets $\mathcal{L}(H_i) \cap \gamma, i = 1, 2$ are the same and are finite or countably infinite. Hence we can take a bijection φ_γ^- from $\mathcal{L}(H_1) \cap \gamma$ to $\mathcal{L}(H_2) \cap \gamma$. For $y = (\dots, y_{-1}, y_0, y_1, y_2, \dots) \in W^u(H_1)$, we denote $y^- = (\dots, y_{-1}, y_0)$, $y^+ = (y_1, y_2, \dots)$ so that we have $y = (y^-, y^+)$, $y^- \in \mathcal{L}(H_1)$. Since $y^- \sim_F \varphi^-(y^-)$, the map φ defined by $\varphi(y^-, y^+) = (\varphi^-(y^-), y^+)$ is a bijective map from $W^u(H_1)$ to $W^u(H_2)$. It is easy to see that it gives rise to a homeomorphism between them. We will prove that $\varphi(\mathfrak{B}(H_1)) = \mathfrak{B}(H_2)$. For $h \in H_1$, $\mu \in \Lambda^k$, $\nu \in \Lambda^l$ and $j \in \mathbb{Z}$, we consider $U_j(h; \mu, \nu)$ for $j + k < 0$ by taking negatively enough j. Let $\eta(1), \dots, \eta(m)$ be the set of all words $x_{[j+k+1,0]}$ for $x \in U_j(h; \mu, \nu)$. We then have

$$U_j(h; \mu, \nu) = \cup_{i=1}^m U_j(h; \mu\eta(i), \nu\eta(i))$$

and $h_{(-\infty,j]}\mu\eta(i) \in \mathcal{L}(H_1)$. Put $h(i)^- = \varphi^-(h_{(-\infty,j]}\mu\eta(i))$ that belongs to $\mathcal{L}(H_2)$. Take $h(i) \in \Lambda$ with $h(i)_{(-\infty,0]} = h(i)^-$. There exists a negative integer $n(i)$ such that $h(i)_{(-\infty,n(i)]} = h'(i)_{(-\infty,n(i)]}$ for some $h'(i) \in H_2$. Put $\xi(i) = h(i)_{[n(i)+1,0]}$. It then follows that

$$\varphi(U_j(h; \mu\eta(i), \nu\eta(i))) = U_{n(i)}(h'(i); \xi(i), \nu\eta(i)).$$

This implies that $\varphi(\mathfrak{B}(H_1) \subset \mathfrak{B}(H_2)$. We similarly have $\varphi^{-1}(\mathfrak{B}(H_2) \subset \mathfrak{B}(H_1)$ and φ maps $\mathfrak{B}(H_1)$ onto $\mathfrak{B}(H_2)$. $\square$

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By Lemma 4.1, we have the following.

COROLLARY 4.10. *Suppose that both subshifts Λ and Λ' satisfy (E). Let $H \subset \Lambda$, $H' \subset \Lambda'$ be dimensionally block covering. If ψ is a conjugacy from a subshift Λ onto Λ', there exists a homeomorphism $\tilde{\psi}$ from $W^u(H)$ onto $W^u(H')$ such that $(\tilde{\psi}(x))_k = \psi(x)_k$ for all $k \in \mathbb{N}$ and $\tilde{\psi}(\mathfrak{B}(H)) = \mathfrak{B}(H')$. Hence $\mathfrak{B}(H)$ and $\mathfrak{B}(H')$ are isomorphic with each other.*

5. *Crossed products by uniformly finite-dimensional homeomorphisms*

For a set $B \in \mathfrak{B}_j(H)$, let $\mathfrak{B}_B(H)$ be the Boolean subalgebra of $\mathfrak{B}_j(H)$ consisting of all subsets of B in the algebra $\mathfrak{B}_j(H)$. A bijection u from a set $B \in \mathfrak{B}(H)$ onto another set $C \in \mathfrak{B}(H)$ satisfying $u(\mathfrak{B}_B(H)) = \mathfrak{B}_C(H)$ is said to be a uniformly finite-dimensional homeomorphism if

$$\sup_{x \in B} d(\sigma^n u(x), \sigma^n(x)) \to 0, \quad \text{for } n \to \infty.$$

This condition is equivalent to the condition that there exists $l \in \mathbb{Z}$ such that $u(y)_k = y_k$ for all $k \geq l$. We denote by $\mathcal{G}_\Lambda(H)$ the set of all uniformly finite-dimensional homeomorphisms from $W^u(H)$ onto $W^u(H)$. It becomes a group in a natural way. We remark that each element of $\mathcal{G}_\Lambda(H)$ is not necessarily a homeomorphism on $W^u(H)$ unless the subshift Λ is a shift of finite type. If Λ is a shift of finite type, $\mathfrak{B}(H)$ consists of all finite unions of cylinder sets in $W^u(H)$ and hence the condition $u(\mathfrak{B}(H)) = \mathfrak{B}(H)$ is equivalent to the condition that u is a homeomorphism on $W^u(H)$ for a bijection u. The notion of uniformly finite-dimensional homeomorphism in shifts of finite type and sofic subshifts has appeared in [**CK, Kr3**] (cf. [**Kr1, Kr2**]).

The shift σ on Λ naturally induces a homeomorphism on $W^u(H)$. We still denote it by σ. The following lemma is easily seen in a similar manner to [**CK**, Section 3].

LEMMA 5.1. *Let H_1, H_2 be dimensionally block covering for Λ. Let $\varphi : W^u(H_1) \to W^u(H_2)$ be a homeomorphism that satisfies $\varphi(\mathfrak{B}(H_1)) = \mathfrak{B}(H_2)$ and $\varphi(x)_k = x_k, k \in \mathbb{N}$ for $x \in W^u(H_1)$. Then we have*

$$\varphi \mathcal{G}_\Lambda(H_1) \varphi^{-1} = \mathcal{G}_\Lambda(H_2) \quad \text{and} \quad \sigma^{-1} \varphi \sigma \varphi^{-1} \in \mathcal{G}_\Lambda(H_2).$$

Thus the space $W^u(H)$ with group action $\mathcal{G}_\Lambda(H)$ is independent of the choice of a dimensionally block covering set H. We denote $\mathcal{G}_\Lambda(H)$ by $\mathcal{G}_\Lambda$.

We denote by $\bar{\mathcal{D}}_\Lambda$ the C^*-algebra $C^*(\mathfrak{B}(H))$ generated by all characteristic functions on $W^u(H)$ of the sets in $\mathfrak{B}(H)$. Then the group $\mathcal{G}_\Lambda$ of all uniformly finite-dimensional homeomorphisms on $W^u(H)$ naturally acts on $\bar{\mathcal{D}}_\Lambda$ as automorphism groups. Let $\mathcal{C}_\Lambda$ be the crossed product of $\bar{\mathcal{D}}_\Lambda$ by the group $\mathcal{G}_\Lambda$. For every $u \in \mathcal{G}_\Lambda$, we write as $\tilde{u}$ the corresponding unitary of the multiplier algebra of $\mathcal{C}_\Lambda$. Let $\mathcal{I}$ be the closed ideal of $\mathcal{C}_\Lambda$ generated by all elements of the form $\tilde{u} P_B - \tilde{v} P_B$ where $u, v \in \mathcal{G}_\Lambda$ such that the restrictions of u, v to the set $B \in \mathfrak{B}(H)$ coincide, and where P_B denotes the characteristic function of B. We define the AF-algebra $\bar{\mathcal{F}}_\Lambda$ as the quotient C^*-algebra of $\mathcal{C}_\Lambda$ by $\mathcal{I}$. The algebra $\bar{\mathcal{D}}_\Lambda$ is regarded as a subalgebra of $\bar{\mathcal{F}}_\Lambda$. Let $\bar{\mathfrak{D}}_\Lambda$ be the C^*-algebra of all complex valued continuous functions on $W^u(H)$ vanishing at infinity that is a subalgebra $\bar{\mathcal{D}}_\Lambda$. Every uniformly finite-dimensional homeomorphism u of a set B in $\mathfrak{B}(H)$ onto a set C in $\mathfrak{B}(H)$ has an image $\tilde{u}$

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in $\bar{\mathcal{F}}_\Lambda$ as a partial isometry with range projection P_C and support projection P_B. We will know that the algebra $\bar{\mathcal{F}}_\Lambda$ is isomorphic to the stabilized C^*-algebra $\mathcal{F}_\Lambda \otimes \mathcal{K}$ by using the discussion of §3. Since the shift σ satisfies $\sigma \mathcal{G}_\Lambda \sigma^{-1} = \mathcal{G}_\Lambda$, it induces an automorphism of $\bar{\mathcal{F}}_\Lambda$. We define the C^*-algebra $\bar{\mathcal{O}}_\Lambda$ by the crossed product of $\bar{\mathcal{F}}_\Lambda$ by the automorphism. This automorphism yields a unitary $\tilde{\sigma}$ in the multiplier algebra of $\bar{\mathcal{O}}_\Lambda$. Let $\bar{\alpha}_t^\Lambda, t \in \mathbb{T}$ be the dual action of the crossed product $\bar{\mathcal{O}}_\Lambda = \bar{\mathcal{F}}_\Lambda \times_\sigma \mathbb{Z}$ that satisfies the condition $\bar{\alpha}_t^\Lambda(\tilde{\sigma}) = t\tilde{\sigma}, t \in \mathbb{T}$. By Proposition 4.9, we have the following.

LEMMA 5.2. *The C^*-dynamical system $(\bar{\mathcal{O}}_\Lambda, \bar{\alpha}^\Lambda, \mathbb{T})$ and the C^*-algebras $\bar{\mathcal{D}}_\Lambda$ and $\bar{\mathfrak{D}}_\Lambda$ do not depend on the choice of the $H \subset \Lambda$ as long as H is dimensionally block covering.*

By Corollary 4.10 and Lemma 5.2, we have the following.

COROLLARY 5.3. *The C^*-dynamical system $(\bar{\mathcal{O}}_\Lambda, \bar{\alpha}^\Lambda, \mathbb{T})$ and the C^*-algebras $\bar{\mathcal{D}}_\Lambda$ and $\bar{\mathfrak{D}}_\Lambda$ are invariant under topological conjugacy of subshift (Λ, σ) with (E).*

We will prove that the C^*-dynamical system is isomorphic to the stabilized C^*-dynamical system $(\mathcal{O}_\Lambda \otimes \mathcal{K}, \alpha \otimes \mathrm{id})$.

Now assume that (Λ, σ) satisfies (E) and a subset $H \subset \Lambda$ is dimensionally block covering. Given $a \in \mathcal{L}(H)$ and a subset $Y \subset X_\Lambda$ set

$$Z(a, Y) = \{y \in W^u(H) \mid y_j = a_j \text{ for } j \leq 0 \text{ and } (y_i)_{i \in \mathbb{N}} \in Y\}.$$

For a set $D \in \mathfrak{B}(H)$, the characteristic function P_D of D on $W^u(H)$ is an element of the multiplier algebras $M(\bar{\mathcal{D}}_\Lambda)$ of $\bar{\mathcal{D}}_\Lambda$ and $M(\bar{\mathcal{O}}_\Lambda)$ of $\bar{\mathcal{O}}_\Lambda$.

The following series of lemmas are generalizations of the series of lemmas in [**CK**]. The proofs are completely similar to [**CK**]. We will give their proofs for the sake of completeness.

LEMMA 5.4. *For sets B, B' and C in $\mathfrak{B}(H)$, put $C' = \sigma(B')$. Let $u : B \to B'$ and $v : C' \to C$ be uniformly finite-dimensional homeomorphisms. Then the C^*-algebra $\bar{\mathcal{O}}_\Lambda$ is isomorphic to the C^*-algebra $C^*(\tilde{v}\tilde{\sigma}\tilde{u}, \bar{\mathcal{F}}_\Lambda)$ generated by $\tilde{v}\tilde{\sigma}\tilde{u}$ and $\bar{\mathcal{F}}_\Lambda$.*

Proof. It suffices to show that $\tilde{\sigma} P_D \in C^*(\tilde{v}\tilde{\sigma}\tilde{u}, \bar{\mathcal{F}}_\Lambda)$ for a set $D \in \mathfrak{B}(H)$. As the set D we may take a set of the form $Z(a, vF_i^l)$ for some $a \in \mathcal{L}(H)$ and $v \in \Lambda^*$ with $vF_i^l \subset X_\Lambda$ because the set of this form finitely generates the Boolean algebras $\mathfrak{B}_j(H)$. Hence we may assume that the sets B and C are also of the above form. By refining D we may assume that there exist uniformly finite-dimensional homeomorphisms

$$u_1 : D \to D_1 \subset B \quad \text{and} \quad v_1 : v\sigma u u_1(D) \to \sigma(D).$$

By putting $w = \sigma^{-1} v_1 v \sigma u u_1$ a uniformly finite-dimensional homeomorphism, we have $\tilde{\sigma} P_D = \tilde{v_1}\tilde{v}\tilde{\sigma}\tilde{u}\tilde{u_1}\tilde{w}^{-1} P_D$ so that it belongs to the algebra $C^*(\tilde{v}\tilde{\sigma}\tilde{u}, \bar{\mathcal{F}}_\Lambda)$. □

As H is dimensionally block covering, for each $i = 1, \ldots, m(l)$, we can choose $a(i) \in \mathcal{L}(H)$ such that $a(i)x$ is admissible for all $x \in F_i^1$. Put

$$Z_\Lambda = \cup_{i=1}^{m(1)} Z(a(i), F_i^1)$$

that is an element of $\mathfrak{B}(H)$. Hence the characteristic function $P_\Lambda = \chi_{Z_\Lambda}$ belongs to the multiplier algebra $M(\bar{\mathcal{D}}_\Lambda)$.

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LEMMA 5.5. $\bar{\mathcal{O}}_\Lambda = C^*(P_\Lambda \bar{\mathcal{O}}_\Lambda P_\Lambda, \bar{\mathcal{F}}_\Lambda)$.

Proof. Put $B_0 = Z_\Lambda$. Find $r \in \Sigma(= \{1, \dots, n\})$ and $j = 1, \dots, m(1)$ such that $rF_j^2 \subset F_1^1$ and put $B_0' = Z(a(1), rF_j^2) \subset B_0$. Take i with $F_j^2 \subset F_i^1$. Put $C = Z(a(i), F_j^2) \subset B_0$. Hence there exists a uniformly finite-dimensional homeomorphism $u : \sigma(B_0') \to C$. As $\tilde{u}\tilde{\sigma} P_{B_0'}$ belongs to $P_\Lambda \bar{\mathcal{O}}_\Lambda P_\Lambda$, by the previous lemma, we see $\tilde{u}\tilde{\sigma} P_{B_0'}$ and $\bar{\mathcal{F}}_\Lambda$ generate $\bar{\mathcal{O}}_\Lambda$. Thus we have $\bar{\mathcal{O}}_\Lambda = C^*(P_\Lambda \bar{\mathcal{O}}_\Lambda P_\Lambda, \bar{\mathcal{F}}_\Lambda)$. □

LEMMA 5.6. *Put $B_0 = Z_\Lambda$. Then there exist a partition $\{B_j\}_{j=0,1,\dots}$ of $W^u(H)$ consisting of elements of $\mathfrak{B}(H)$ and uniformly finite-dimensional homeomorphisms $u_j : B_0 \to B_j$.*

Proof. Set

$$W^u(H; F_i^1) = \{y \in W^u(H) \mid (y_k)_{k \in \mathbb{N}} \in F_i^1\}.$$

We may take a countable sequence $a(i,0)(= a(i)), a(i,1), a(i,2), \dots$, in $\mathcal{L}(H)$ such that $a(i,j)x \in \Lambda$ for all $x \in F_i^1$ and $W^u(H; F_i^1) = \cup_{j=0}^\infty Z(a(i,j), F_i^1)$. Set $B_j = \cup_{i=1}^{m(1)} Z(a(i,j), F_i^1)$ so that we have $W^u(H) = \cup_{j=0}^\infty B_j$. Exchanging $a(i,0)$ by $a(i,j)$ yields an isomorphism $u(i,j) : Z(a(i;0), F_i^1) \to Z(a(i;j), F_i^1)$ between the Boolean algebras. The isomorphisms $\{u(i,j)\}_{i=1}^{m(1)}$ gives rise to a uniformly finite-dimensional homeomorphism $u_j : B_0 \to B_j$. □

Thus we obtain the following.

PROPOSITION 5.7. $(\bar{\mathcal{O}}_\Lambda, \bar{\mathcal{D}}_\Lambda) \cong (\mathcal{K} \otimes P_\Lambda \bar{\mathcal{O}}_\Lambda P_\Lambda, \mathcal{C} \otimes P_\Lambda \bar{\mathcal{D}}_\Lambda)$ *where $\mathcal{C}$ is a maximal commutative C^*-algebra of $\mathcal{K}$.*

Proof. Let a partition B_j and uniformly finite-dimensional homeomorphisms u_j be as in the previous lemma. We have a system $e_{j,k}, k, j \in \mathbb{N}$ of matrix units in $\bar{\mathcal{O}}_\Lambda$ such that

$$e_{j,j} = P_{B_j} \quad \text{and} \quad e_{j,0} = \tilde{u}_j.$$

They give rise to a desired isomorphism of C^*-algebras. □

6. *Main theorem and corollaries*

We state the main theorem.

THEOREM 6.1. *Suppose that a subshift Λ satisfies conditions (I) and (E). There exists an isomorphism*

$$\Phi(\bar{\mathcal{O}}_\Lambda, \bar{\mathcal{D}}_\Lambda) \to (\mathcal{K} \otimes \mathcal{O}_\Lambda, \mathcal{C} \otimes \mathcal{D}_\Lambda)$$

such that $\Phi \circ \bar{\alpha}_t^\Lambda \circ \Phi^{-1} = \mathrm{id} \otimes \alpha_t, t \in \mathbb{T}$.

Proof. In order to prove that $P_\Lambda \bar{\mathcal{O}}_\Lambda P_\Lambda$ is isomorphic to the C^*-algebra $\mathcal{O}_\Lambda$, we will find corresponding generating partial isometries $\bar{S}_1, \dots, \bar{S}_n$ in $P_\Lambda \bar{\mathcal{O}}_\Lambda P_\Lambda$. For $i = 1, \dots, n$ and $j = 1, \dots, m(2)$ with ix is admissible for all $x \in F_j^2$, there exist $2(j), \lambda(i,j) = 1, \dots, m(1)$ such that $F_j^2 \subset F_{2(j)}^1$ and $iF_j^2 \subset F_{\lambda(i,j)}^1$. Take a uniformly finite-dimensional homeomorphism

$$u_i(j) : Z(a(2(j)), F_j^2) \to Z(a(\lambda(i,j)), iF_j^2).$$

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We set

$$\bar{S}_i = \sum_{j,\Lambda_2(i;j)=1} \tilde{\sigma}^{-1}\tilde{u}_i(j)$$

where

$$\Lambda_2(i;j) = \begin{cases} 1 & \text{if } iF_j^2 \text{ is admissible} \\ 0 & \text{otherwise.} \end{cases}$$

Then it follows that the correspondences $\bar{S}_\mu^*\bar{S}_\mu \leftrightarrow \chi_{\sigma^{|\mu|}(U_\mu)}$ yield an isomorphism between the C^*-algebra generated by the projections $\bar{S}_\mu^*\bar{S}_\mu, \mu \in \Lambda^*$ and the subalgebra $\mathcal{A}_\Lambda$ of $\mathcal{O}_\Lambda$. Therefore, by the universality of the algebra $\mathcal{O}_\Lambda$ as in Lemma 2.2 or [**Ma4**], the map $\bar{S}_i \to S_i$ gives rise to an isomorphism from $P_\Lambda\bar{\mathcal{O}}_\Lambda P_\Lambda$ to $\mathcal{O}_\Lambda$. That induces a desired isomorphism. □

COROLLARY 6.2. *Let two subshifts (Λ_1,σ) and (Λ_2,σ) both satisfy condition (I) and (E). If they are topologically conjugate, then we have an isomorphism*

$$(\mathcal{K}\otimes\mathcal{O}_{\Lambda_1}, \mathcal{C}\otimes\mathcal{D}_{\Lambda_1}, \mathrm{id}\otimes\alpha^1) \cong (\mathcal{K}\otimes\mathcal{O}_{\Lambda_2}, \mathcal{C}\otimes\mathcal{D}_{\Lambda_2}, \mathrm{id}\otimes\alpha^2)$$

where α^i denotes the gauge action on $\mathcal{O}_{\Lambda_i}, i = 1,2$. Namely, the above triplet is a topological conjugacy invariant for two-sided subshifts. We in particular have $\mathcal{K}\otimes\mathcal{F}_{\Lambda_1} \cong \mathcal{K}\otimes\mathcal{F}_{\Lambda_2}$.

For a subshift Λ, let $\mathbb{Z}^{m(l)}$ be the free abelian group generated by the l-past equivalence classes $F_i^l, i = 1,2,\dots,m(l)$. Let $I_{l,l+1}$ be the homomorphism from $\mathbb{Z}^{m(l+1)}(=\oplus_{j=1}^{m(l+1)}\mathbb{Z}[F_j^{l+1}])$ to $\mathbb{Z}^{m(l)}(=\oplus_{i=1}^{m(l)}\mathbb{Z}[F_i^l])$ induced by the inclusions of the refinements of X_Λ. That is, the (i,j)-component $I_{l,l+1}(i,j)$ of the matrix $I_{l,l+1}$ is one if $F_j^{l+1} \subset F_i^l$ otherwise zero. We denote by $A_{l,l+1}(i,j)$ the cardinality of the set $\{r \in \Sigma \mid rx \in F_i^l \text{ for some } x \in F_j^{l+1}\}$. Hence we have two $m(l)\times m(l+1)$ matrices $I_{l,l+1}$ with entries in $\{0,1\}$ and $A_{l,l+1}$ with entries in non-negative integers for each $l \in \mathbb{N}$. As in [**Ma2, Ma3**], K-groups for the C^*-algebras have been computed as

(i) $K_0(\mathcal{O}_\Lambda) \cong \varinjlim \mathbb{Z}^{m(l+1)}/(I_{l,l+1}^t - A_{l,l+1}^t)\mathbb{Z}^{m(l)}$,

(ii) $K_1(\mathcal{O}_\Lambda) \cong \varinjlim \mathrm{Ker}(I_{l,l+1}^t - A_{l,l+1}^t)$ in $\mathbb{Z}^{m(l)}$,

where the sequences of homomorphisms in $\varinjlim$ of (i) and (ii) are taken along the inclusions $I_{l,l+1}^t : \mathbb{Z}^{m(l)} \hookrightarrow \mathbb{Z}^{m(l+1)}$. We set the inductive limit of the ordered groups

$$\mathbb{Z}_\Lambda = \varinjlim\{I_{l,l+1}^t : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+1)}\}$$

where the positive cone of $\mathbb{Z}^{m(l)}$ is $(\mathbb{Z}_+)^{m(l)}$ for the set of all non-negative integers $\mathbb{Z}_+$. As the sequence of matrices $\{A_{l,l+1}\}_{l\in\mathbb{N}}$ induces an endomorphism on the ordered group $\mathbb{Z}_\Lambda$ that we denote by λ_Λ. Then we see by [**Ma2**],

$$\begin{aligned} K_0(\mathcal{F}_\Lambda) &\cong \varinjlim\{\lambda_\Lambda : \mathbb{Z}_\Lambda \to \mathbb{Z}_\Lambda\} \\ &\cong \varinjlim\{I_{l+1,l+2}^t A_{l,l+1}^t : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+2)}\}. \end{aligned}$$

Since K-groups for C^*-algebras are invariant for stable isomorphism class, we have the following.

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COROLLARY 6.3. (cf. [**Ma2, Ma3, Ma4**]). *If two subshifts (Λ_1, σ) and (Λ_2, σ) are topologically conjugate, then we have*

$$K_0(\mathcal{O}_{\Lambda_1}) \cong K_0(\mathcal{O}_{\Lambda_2}), \quad K_1(\mathcal{O}_{\Lambda_1}) \cong K_1(\mathcal{O}_{\Lambda_2}) \quad \textit{and} \quad K_0(\mathcal{F}_{\Lambda_1}) \cong K_0(\mathcal{F}_{\Lambda_2}).$$

We see that λ_Λ naturally acts as an automorphism on the ordered group $K_0(\mathcal{F}_\Lambda)$. We also know that, by Corollary 6.2, the crossed product with the dual action

$$(\mathcal{K} \otimes (\mathcal{O}_\Lambda \times_\alpha \mathbb{T}), \mathrm{id} \otimes \hat{\alpha}, \mathbb{Z})$$

is a topological conjugacy invariant of subshift (Λ, σ). By the discussion of [**Ma2**], we know the algebra $\mathcal{K} \otimes (\mathcal{O}_\Lambda \times_\alpha \mathbb{T})$ is isomorphic to the algebra $\mathcal{K} \otimes \mathcal{F}_\Lambda$. The automorphism $\mathrm{id} \otimes \hat{\alpha}$ on $\mathcal{K} \otimes (\mathcal{O}_\Lambda \times_\alpha \mathbb{T})$ induces an automorphism on $K_0(\mathcal{F}_\Lambda)$ that is realized as the shift automorphism σ_*. Therefore we have an isomorphic pair

$$(K_0(\mathcal{F}_\Lambda), \sigma_*) \cong (K_0(\mathcal{F}_\Lambda), \lambda_\Lambda)$$

that is a topological conjugacy invariant of subshift (Λ, σ).

The ordered group $K_0(\mathcal{F}_\Lambda)$ with the automorphism σ_* is defined to be the dimension group with dimension automorphism for subshift Λ in [**Ma2, Ma3**] as in the case of topological Markov shifts (cf. [**Kr1, Kr2, Le**]).

In [**Ma5**], the author has defined the Bowen–Franks group $BF(\Lambda)$ for subshift Λ as the Ext-group $Ext(\mathcal{O}_\Lambda)$ for the algebra $\mathcal{O}_\Lambda$ that is a generalization of the Bowen–Franks group for topological Markov shifts [**BF**]. By the universal coefficient theorem of K-theory for C^*-algebras, the Ext-group for C^*-algebras is determined by their K-groups (cf. [**RS, Bl**]). Hence, by the previous corollary, we see that the Bowen–Franks group $BF(\Lambda)$ for subshifts are also invariant under topological conjugacy.

Concerning on automorphisms on two-sided subshifts, we directly have the following.

COROLLARY 6.4. *Let τ be an automorphism on the two-sided subshift Λ. Then it yields an automorphism on the C^*-algebra $\mathcal{O}_\Lambda \otimes \mathcal{K}$ that commutes with the action $\alpha_t \otimes \mathrm{id}$, $t \in \mathbb{T}$. Hence τ gives rise to an automorphism of the following abelian groups*

$$\varinjlim \mathbb{Z}^{m(l+1)}/(I^t_{l,l+1} - A^t_{l,l+1})\mathbb{Z}^{m(l)}, \quad \varinjlim \mathrm{Ker}(I^t_{l,l+1} - A^t_{l,l+1}) \textit{ in } \mathbb{Z}^{m(l)}$$

and

$$\varinjlim \{I^t_{l+1,l+2} A^t_{l,l+1} : \mathbb{Z}^{m(l)} \to \mathbb{Z}^{m(l+2)}\}.$$

We note that, by a discussion of [**Ma7**] (cf. [**Ma5**]), it is possible to directly prove the above K-theoretic groups

$$K_0(\mathcal{O}_\Lambda), \quad K_1(\mathcal{O}_\Lambda), \quad Ext(\mathcal{O}_\Lambda), \quad K_0(\mathcal{F}_\Lambda)$$

are invariant for topological conjugacy of two-sided subshifts by essentially using Nasu's factorization theorem of conjugacy without using C^*-algebra theory.

7. *Remark*

We remark on another construction of the algebra $\bar{\mathcal{F}}_\Lambda$ from a groupoid point of view that was suggested by the referee. The C^*-algebra $\bar{\mathcal{F}}_\Lambda$ constructed in §5 is an AF-algebra. It

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has been defined to be a crossed product algebra of a commutative C^*-algebra of Boolean algebras of sets in an unstable manifold of the subshift by uniformly finite-dimensional homeomorphisms. It is possible to realize it as a C^*-algebra of a groupoid consisting of transversals that are unstable sets of the points in a dimensionally block covering set H. Each element of the groupoid exactly corresponds to a uniformly finite-dimensional homeomorphism on $W^u(H)$. The unit space of the groupoid is regarded as the Boolean algebras $\mathfrak{B}_j(H)$. Ruelle in [**Ru**], Putnam in [**P**] and Putnam and Spielberg in [**PS**] used these techniques of construction of C^*-algebras by groupoids for large classes of dynamics, including subshifts of finite type (cf. [**Re**]). We finally remark that the algebra $\mathcal{O}_\Lambda$ itself is the C^*-algebra of an r-discrete groupoid (see [**AnD, Re**]).

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