INTERLAYER COUPLING EFFECT ON BUCKLING MODES OF SPHERICAL BILAYERS

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ABSTRACT

This study examined the critical buckling characteristics of hydrostatically pressurized double-walled complete spherical shells. An analytical model based on small deflection thin shell theory is presented; the equations are solved in conjunction with variational principles. Axisymmetric and inextensional assumptions are not initially used in the exact formulation. This approach therefore avoids any discussion about the validity of the solution and allows the model to be extended to cover more generic nonaxisymmetric cases with relative ease. The analytical results are presented using illustrative buckling modes. Based on the developed formulation, only axisymmetric eigenmodes were found to occur despite the inclusion of the effect of interactions between outer and inner shells. Critical modes that are symmetric or antisymmetric about the equator may be determined depending on the combination of the stiffness connecting the outer and inner shells and the radius-to-wall thickness ratios.

Keywords: Complete spherical shell, Buckling, Double-walled structure.

1. INTRODUCTION

Theoretical analysis of thin shell structures has been a challenge in applied mechanics and in many engineering branches. Of many choices of geometry, spherical shells show non-trivial mechanical responses to external compressive pressure. Seminal works done by Thompson [1], Hutchinson [2], and Koiter [3] demonstrated that a pressure is increased beyond a certain critical value, the spherical shell suddenly buckled into a non-spherical equilibrium state. The obtained buckled patterns as well as the critical pressure are strongly dependent on the geometric conditions of the shells, which is the reason why a number of investigations have been performed on the buckling problem during the last five decades. To date, the buckling nature of spherical shells was found to be relevant to a wide variety of fields including nano-science and biomechanics: Pressure vessels, spherical honeycombs (Tarnai [4]), carbon onions (Baowan *et al*. [5]), and spherical viruses (Ru [6]) are only a few to mention. Nowadays it is broadly accepted that continuum shell approximation is effective to a degree for describing mechanical stability of those spherical-shaped materials.

When a rigid core is enclosed by a spherical shell, the interaction between the inner core and outer shell causes novel buckling patterns of the core/shell structure under hydrostatic pressure. One interesting example was presented by Yin *et al*. [7], in which a series of stress-driven buckling patterns in spheroidal core/shell structures was evaluated using the finite element approach. They proposed an intriguing idea that the shapes of many natural fruits and vegetables can be reproduced by anisotropic stress-driven buckling on the spheroidal core/shell system. The role of internal core on the spherical shell buckling was also considered by Sato *et al*. [8]; they derived analytic solutions for the

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buckling pressures and patterns of a complete spherical shell filled with an elastic medium (Winkler foundations) using exact formulations. Their formulation clearly showed that the interaction between shells and Winkler foundations causes only axisymmetric buckling modes and that no non-axisymmetric modes can occur.

The results above-mentioned pose a question as to what occurs if the internal core is replaced by a hollow spherical shell. It results in a by-layered spherical shell, and layer-to-layer coupling is expected to play a primary role in determining its buckling mode under pressure. Such multiply-layered spherical shells can be regarded as approximate models of carbon onions (Los *et al*. [9], Pudlak and Pincak [10]), core/shell semiconductor nanoparticles (Trallero-Giner *et al*. [11]), and the ‘‘nano-matryoshka’’ (Prodan *et al*. [12]), all of which consist of a stack of concentric spherical layers. In particular, the last system,comprising metallic and dielectric materials, shows great tenability in the optical response. Therefore,it is of great importance to examine how pressure-induced deformation affects the electromagnetic properties of nanoshells and their resonant frequencies is of interest. Other than the case of nano-matryoshka, furthermore, deformation-based control of physical properties peculiar to multiply spherical shells may open an avenue of next-generation material design. Toward the aim, it is crucial to obtain a thorough understanding of the bucking nature of multiply spherical shells under pressure.

In this contribution, we present the critical buckling characteristics of hydrostatically pressurized double-walled sandwich-type complete spherical shells. In this model, the outer and inner shells are linked by a Winkler foundation. This modeling of the Winkler foundation corresponds to a van del Waars interaction for carbon onions or a core material for sandwich spherical shells. An analytical model based on small deflection thin shell theory is presented where the equations are solved in conjunction with variational principles. Axisymmetric and inextensional assumptions are not initially used in the exact formulation. This approach therefore avoids any discussion about the validity of the solution and allows the model to be extended to cover more generic nonaxisymmetric cases with relative ease. The analytical results are presented using illustrative buckling modes. Based on the developed formulation, only axisymmetric eigenmodes were found to occur despite the inclusion of the effect of interactions between outer and inner shells. Critical modes that are symmetric or antisymmetric about the equator may be determined depending on the combination of the stiffness connecting outer and inner shells and the radius-to-wall thickness ratios.

2. ANALYTICAL MODEL

The critical pitchfork bifurcation phenomenon of the hydrostatically pressurized complete spherical shell was investigated, as shown in Fig. 1. The spherical shell was constructed from a homogeneous and isotropic

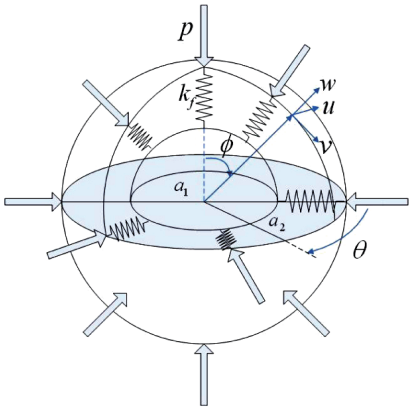


Fig. 1 Analytical model fordouble-walled complete spherical shell.

linear elastic material with Young’s modulus E and Poisson’s ratio ν; it was filled with an elastic material that was modeled as a Winkler foundation, *i.e.*, with uncoupled springs in the radial direction and a constant foundation modulus k_f. For a complete spherical shell with radius and thickness h, spherical angular coordinates in the latitude and meridian directions θ and ϕ were used. Displacement functions u, v, and w were in the θ, ϕ, and (outward) radial directions, respectively.

3. ENERGY FORMULATION

3.1 Energy Formulation

The total potential energy V is expressed by the sum of the strain energy and work done by the external load:

$$V = \sum_{i=1}^{2}\left\{U_{M(i)} + U_{M(i)}\right\} + U_F + \Omega \tag{1}$$

where i = 1, 2 correspond to the quantities of the inner and outer spherical shells, respectively, and

$$U_{M(i)} = \frac{C_i}{2}\int_A \left(\varepsilon_{\varphi(i)}^2 + \varepsilon_{\theta(i)}^2 + 2\nu_{(i)}\,\varepsilon_{\varphi(i)}\,\varepsilon_{\theta(i)} + \frac{1-\nu_{(i)}}{2}\gamma_{\varphi\theta(i)}^2\right) dA_i \tag{2}$$

$$U_{B(i)} = \frac{D_i}{2}\int_A \left[\chi_{\varphi(i)}^2 + \chi_{\theta(i)}^2 + 2\nu_{(i)}\,\chi_{\varphi(i)}\,\chi_{\theta(i)} + 2(1-\nu_1)\chi_{\varphi\theta(i)}^2\right] dA_i \tag{3}$$

$$U_F = \frac{1}{2}k_f\int_A (w_1 - w_2)^2\, dA \tag{4}$$

$$\Omega = -\int_{A_2} p w_2\, dA_2 \tag{5}$$

$$C_i = \frac{E_i\, h_i}{1-\nu_i^2} \tag{6}$$

$$D_i = \frac{E_i\, h_i^3}{12(1-\nu_1^2)}. \tag{7}$$

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$$dA_i = a_i^2 \sin\varphi \, d\varphi \, d\theta \tag{8}$$

$$dA = \left(\frac{a_1 + a_2}{2} \right)^2 \sin\varphi \, d\varphi \, d\theta \tag{9}$$

Based on the assumptions of thin shell theory (Brush and Almroth [13]), the strain-displacement relations can be expressed as

$$\varepsilon_\varphi = e_{\varphi\varphi} + \frac{1}{2}\beta_\varphi{}^2 \tag{10}$$

$$\varepsilon_\theta = e_{\theta\theta} + \frac{1}{2}\beta_\theta^2 \tag{11}$$

$$\gamma_{\varphi\theta} = e_{\varphi\theta} + \beta_\varphi \beta_\theta \tag{12}$$

$$\chi_\varphi = \frac{1}{a^2}(-w_{,\varphi\varphi} + v_{,\varphi}) \tag{13}$$

$$\chi_\theta = \frac{1}{a^2 \sin\varphi}\left(-w_{,\theta\theta}\frac{1}{\sin\varphi} + u_{,\theta} - w_{,\varphi}\cos\varphi + v\cos\varphi \right) \tag{14}$$

$$\chi_{\varphi\theta} = \frac{1}{2a^2 \sin\varphi}(u_{,\varphi}\sin\varphi - u\cos\varphi - 2w_{,\varphi\theta} + 2w_{,\theta}\cot\varphi + v_{,\theta}) \tag{15}$$

where ", x" denotes differentiation with respect to x and

$$e_{\varphi\varphi} = \frac{1}{a}(v_{,\varphi} + w) \tag{16}$$

$$e_{\theta\theta} = \frac{1}{a}\left(v\cot\varphi + \frac{1}{\sin\varphi}u_{,\theta} + w \right) \tag{17}$$

$$e_{\varphi\theta} = \frac{1}{a}\left(\frac{1}{\sin\varphi}v_{,\theta} + u_{,\varphi} - u\cot\varphi \right) \tag{18}$$

$$\beta_\varphi = \frac{1}{a}(-w_{,\varphi} + v) \tag{19}$$

$$\beta_\theta = \frac{1}{a}\left(-\frac{1}{\sin\varphi}w_{,\theta} + u \right) \tag{20}$$

3.2 Fundamental State

When the uniform hydrostatic pressure p is acting on the spherical shell, only the inward radial static displacement $w_i^{(0)}$ can occur in the prebuckling state.

Hence, the stress-strain relationship of the spherical shell was assumed to be linear up to the point of instability. In the fundamental prebuckling state, the potential energies $V^{(0)}$ are given by

$$V^{(0)} = \int_\varphi \int_\theta F^{(0)} d\varphi \, d\theta \tag{21}$$

where

$$F^{(0)} = \left[\sum_{i=1}^{2} C_i (1+\nu_i) w_i^{(0)2} + \frac{1}{2}k_f \left(\frac{a_1 + a_2}{2} \right)^2 \left(w_1^{(0)} - w_2^{(0)} \right)^2 \right] \sin\varphi \tag{22}$$

For an equilibrium state,

$$\frac{\partial F}{\partial w_i^{(0)}} = 0 \tag{23}$$

These conditions give the following equations:

$$C_1(1+\nu_1)w_1^{(0)} + k_f \left(\frac{a_1 + a_2}{2} \right)^2 \left(w_1^{(0)} - w_2^{(0)} \right) = 0 \tag{24}$$

$$C_2(1+\nu_2)w_2^{(0)} + a_2^2 p + k_f \left(\frac{a_1 + a_2}{2} \right)^2 \left(w_2^{(0)} - w_1^{(0)} \right) = 0 \tag{25}$$

The static displacements $w_1^{(0)}$ and $w_2^{(0)}$ can be obtained by solving the above simultaneous equations with regard to $w_1^{(0)}$ and $w_2^{(0)}$.

3.3 Critical Buckling Analysis

To obtain an expression for the second variation of the total potential energy, the following infinitesimally small increments are defined:

$$u_i \to u_i^{(1)}, \; v_i \to v_i^{(1)}, \; w_i \to w_i^{(0)} + w_i^{(1)} \tag{26}$$

where the superscript (1) corresponds to buckling displacement modes. The expression for the potential energy due to the external force is a linear functional of the displacement components and makes no contribution to the second variation expression, which is $\delta^2\Omega = 0$ (Brush and Almroth [13]). Consequently, the second variation of the total potential energy becomes

$$\delta^2 V = \sum_{i=1}^{2}\left(\delta^2 U_{M(i)} + \delta^2 U_{B(i)} \right) + \delta^2 U_F + \delta^2 \Omega = \int_\varphi \int_\theta F d\varphi \, d\theta \tag{27}$$

where

$$\begin{aligned}\delta^2 U_{M(i)} = {} & C_i(1+\nu_i)\frac{w_i^{(0)}}{a_i}\sin\varphi\left[\left(v_i^{(1)} - w_i^{(1)}{}_{,\varphi} \right)^2 + \left(u_i^{(1)} - w_i^{(1)}{}_{,\theta}\csc\varphi \right)^2 \right] + C_i \sin\varphi\left[2(1+\nu_i)w_i^{(1)}\left(v_i^{(1)}\cot\varphi + w_i^{(1)} + u_i^{(1)}{}_{,\theta}\csc\varphi + v_i^{(1)}{}_{,\varphi} \right) \right. \\ & + \left(v_i^{(1)}\cot\varphi + u_i^{(1)}{}_{,\theta}\csc\varphi \right)^2 + v_i^{(1)}{}_{,\varphi}\left(2\nu_i \, v_i^{(1)}\cot\varphi + 2\nu_i \, u_i^{(1)}{}_{,\theta}\csc\varphi + v_i^{(1)}{}_{,\varphi} \right) \\ & \left. + \frac{1}{2}C_i(\nu_i - 1)\sin\varphi\left(-u_i^{(1)}\cot\varphi + v_i^{(1)}{}_{,\theta}\csc\varphi + u_i^{(1)}{}_{,\varphi} \right)^2 \right]\end{aligned} \tag{28}$$

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$$\begin{aligned}\delta^2 U_{B(i)} = \frac{D_i}{a_i^2}\sin\varphi\Big[&\left(v_i^{(1)}\cot\varphi + u_i^{(1)}{}_{,\theta}\csc\varphi - w_i^{(1)}{}_{,\theta\theta}\csc^2\varphi - w_i^{(1)}{}_{,\varphi}\cot\varphi\right)^2 + \left(v_i^{(1)}{}_{,\varphi} - w_i^{(1)}{}_{,\varphi\varphi}\right)\left\{\left(v_i^{(1)}{}_{,\varphi} - w_i^{(1)}{}_{,\varphi\varphi}\right)\right.\\ &\left.+ 2\nu_i\left(v_i^{(1)}\cot\varphi + u_i^{(1)}{}_{,\theta}\csc\varphi - w_i^{(1)}{}_{,\theta\theta}\csc^2\varphi - w_i^{(1)}{}_{,\varphi}\cot\varphi\right)\right\}\Big]\\ &+\frac{D_i(\nu_i - 1)}{2a_i^2}\csc^3\varphi\left(u_i^{(1)}\sin\varphi\cos\varphi - v_i^{(1)}{}_{,\theta}\sin\varphi\right.\\ &\left.-2w_i^{(1)}{}_{,\theta}\cos\varphi - u_i^{(1)}{}_{,\varphi}\sin^2\varphi + 2w_i^{(1)}{}_{,\varphi\theta}\sin\varphi\right)^2\end{aligned} \tag{29}$$

$$\delta^2 U_F = \left(\frac{a_1 + a_2}{2}\right)^2 k_f\left(w_1^{(1)} - w_2^{(1)}\right)^2\sin\varphi \tag{30}$$

$$\delta^2\Omega = 0 \tag{31}$$

According to the Trefftz criterion, the buckling equations can be obtained by introducing F in Eq. (27) into the Euler-Lagrange equations with the calculus of variations. The Euler-Lagrange equations in this case are as follows (Brush and Almroth [13]):

$$\begin{aligned}&\frac{\partial F}{\partial u_i^{(1)}} - \frac{\partial}{\partial\varphi}\frac{\partial F}{\partial u_i^{(1)}{}_{,\varphi}} - \frac{\partial}{\partial\theta}\frac{\partial F}{\partial u_i^{(1)}{}_{,\theta}} = 0\\ &\frac{\partial F}{\partial v_i^{(1)}} - \frac{\partial}{\partial\varphi}\frac{\partial F}{\partial v_i^{(1)}{}_{,\varphi}} - \frac{\partial}{\partial\theta}\frac{\partial F}{\partial v_i^{(1)}{}_{,\theta}} = 0\\ &\frac{\partial F}{\partial w_i^{(1)}} - \frac{\partial}{\partial\varphi}\frac{\partial F}{\partial w_i^{(1)}{}_{,\varphi}} - \frac{\partial}{\partial\theta}\frac{\partial F}{\partial w_i^{(1)}{}_{,\theta}} + \frac{\partial^2}{\partial\varphi^2}\frac{\partial F}{\partial w_i^{(1)}{}_{,\varphi\varphi}} + \frac{\partial^2}{\partial\varphi\partial\theta}\frac{\partial F}{\partial w_i^{(1)}{}_{,\varphi\theta}} + \frac{\partial^2}{\partial\theta^2}\frac{\partial F}{\partial w_i^{(1)}{}_{,\theta\theta}} = 0\end{aligned} \tag{32}$$

Substitution of a solution in the form

$$\begin{aligned}u_i^{(1)}(\varphi,\theta) = \sum_{n=2}^{\infty} u_{m(i)}(\varphi)\cos n\theta,\ v_i^{(1)}(\varphi,\theta) = \sum_{n=2}^{\infty} v_{m(i)}(\varphi)\sin n\theta,\ w_i^{(1)}(\varphi,\theta)\\ = \sum_{n=2}^{\infty} w_{m(i)}(\varphi)\cos n\theta\end{aligned} \tag{33}$$

where n is a positive integer into Eq. (27) and some rearrangement gives the following differential equations for an arbitrary n:

$$\begin{aligned}&(1+k_i)\left[\frac{1+\nu_i}{2}nu'_{m(i)} - \frac{3-\nu_i}{2}nu_{m(i)}\cot\varphi + v''_{m(i)}\sin\varphi + v'_{m(i)}\cos\varphi - \frac{v_{m(i)}}{\sin\varphi}\left(\cos^2\varphi + \nu_i\sin^2\varphi + \frac{1-\nu_i}{2}n^2\right) + (1+\nu_i)w'_{m(i)}\sin\varphi\right]\\ &-k_i\left[w'''_{m(i)}\sin\varphi + w''_{m(i)}\cos\varphi + w'_{m(i)}\left(2 - \frac{1+n^2}{\sin^2\varphi}\right)\sin\varphi + 2n^2 w_{m(i)}\frac{\cos\varphi}{\sin^2\varphi}\right]\\ &-\frac{w_i^{(0)}}{a_i}(1+\nu_i)\sin\varphi(v_{m(i)} - w'_{m(i)}) = 0\end{aligned} \tag{34}$$

$$\begin{aligned}&(1+k_i)\left[\frac{1-\nu_i}{2}\left(u''_{m(i)}\sin\varphi + u'_{m(i)}\cos\varphi\right) - u_{m(i)}\left\{\frac{n^2}{\sin\varphi} - \frac{1-\nu_i}{2}(1-\cot^2\varphi)\sin\varphi\right\}\right.\\ &\left.-\frac{1+\nu_i}{2}nv'_{m(i)} - \frac{3-\nu_i}{2}nv_{m(i)}\cot\varphi - (1+\nu_i)nw_{m(i)}\right] + k_i n\left[w''_{m(i)} + w'_{m(i)}\cot\varphi + w_{m(i)}\left(2 - \frac{n^2}{\sin^2\varphi}\right)\right]\\ &-\frac{w_i^{(0)}}{a_i}(1+\nu_i)\sin\varphi\left[u_{m(i)}\sin\varphi - nw_{m(i)}\right] = 0\end{aligned} \tag{35}$$

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$$
\begin{aligned}
&(1+k_i)(1+\nu_i)\left[nu_{m(i)}+v'_{m(i)}\sin\varphi+v_{m(i)}\cos\varphi+2w_{m(i)}\sin\varphi\right]-k_i\left[nu''_{m(i)}-nu'_{m(i)}\cot\varphi+nu_{m(i)}\left(2+\frac{1-n^2}{\sin^2\varphi}\right)\right.\\
&\left.+v'''_{m(i)}\sin\varphi+2v''_{m(i)}\cos\varphi-\frac{v_{m(i)}}{\sin\varphi}(n^2+\cos^2\varphi)+v_{m(i)}\cos\varphi\left(2+\frac{1-n^2}{\sin^2\varphi}\right)\right]\\
&+k_i\left[w''''_{m(i)}\sin\varphi+2w'''_{m(i)}\cos\varphi-\frac{w_{m(i)}}{\sin\varphi}(1+2n^2+\nu_i\sin^2\varphi)\right.\\
&\left.+w_{m(i)}\left(1-\nu_i+\frac{1+2n^2}{\sin^2\varphi}\right)\cos\varphi-w_{m(i)}\left(2(1+\nu_i)\sin\varphi+n^2\frac{3-\nu_i+4\cot^2\varphi}{\sin\varphi}-\frac{n^4}{\sin^3\varphi}\right)\right]\\
&+\frac{w_i^{(0)}}{a_1}(1+\nu_i)\left[v'_{m(i)}\sin\varphi+v_{m(i)}\cos\varphi+nu_{m(i)}-\sin\varphi\left(w''_{m(i)}+w'_{m(i)}\cot\varphi-\frac{n^2}{\sin\varphi}w_{m(i)}\right)\right]\\
&+\frac{(a_1+a_2)^2k_f}{4C_i}\left(w_{m(i)}-w_{m(j)}\right)=0
\end{aligned}
\tag{36}
$$

where the primes denotes differentiation with respect to ϕ, $(i, j) = (1, 2)$ for the equations regarding the inner shell and $(i, j) = (2, 1)$ for those regarding the outer one. These governing differential equations are difficult to solve exactly by only assuming periodic functions in u, v, and w directly.

However, the definition of the following special linear differential operator H enables the solution of these equations with an exact approach:

$$
H(\ldots)=(\ldots)''+(\ldots)'\cot\phi+(\ldots)\left(m(\mathrm{m}+1)-\frac{n^2}{\sin^2\phi}\right)
\tag{37}
$$

By using the operator H, some rearrangement gives (for details see Sato *et al.* [8])

$$
\begin{aligned}
&(1+k_i)H\left(\omega_{m(i)}\right)-(1+k_i)(1+\nu_i)\omega_{m(i)}-k_iHH\left(w_{m(i)}\right)+(1+k_i)(1+\nu_i)H\left(w_{m(i)}\right)+2k_iH\left(w_{m(i)}\right)\\
&-2(1+k_i)(1+\nu_i)w_{m(i)}-\frac{w_i^{(0)}}{a_i}(1+\nu_i)\left[\omega_{m(i)}-H\left(w_{m(i)}\right)+2w_{m(i)}\right]=0
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
&(1+k_i)(1+\nu_i)\left[\omega_{m(i)}+2w_{m(i)}\right]-k_iH\left(\omega_{m(i)}\right)+k_i\left[HH\left(w_{m(i)}\right)-\nu_iH\left(w_{m(i)}\right)-3H\left(w_{m(i)}\right)\right]\\
&+(1+\nu_i)\frac{w_i^{(0)}}{a_i}\left[\omega_{m(i)}-H\left(w_{m(i)}\right)+2w_{m(i)}\right]+\frac{(a_1+a_2)^2k_f}{4C_i}w_{m(i)}=0
\end{aligned}
\tag{39}
$$

where

$$
\omega_{m(i)}==\frac{v'_{m(i)}\sin\varphi+v_{m(i)}\cos\varphi+nu_{m(i)}}{\sin\varphi}
\tag{40}
$$

Equations (38) and (39) are the governing equations to be solved. Now, the solutions of w and ω of Eqs. (38) and (39) were assumed to be in the form of

$$
w_{m(i)}=\sum_{m=2}^{\infty}W_{m(i)}\,Q_m^n(\cos\varphi)\,,\quad \omega_{m(i)}=\sum_{m=2}^{\infty}\Omega_{m(i)}\,Q_m^n(\cos\varphi)
\tag{41}
$$

where $W_{m(i)}$ and $\Omega_{m(i)}$ are the constant deformation amplitudes and the spherical harmonic Q_m^n is a series of Legendre functions that satisfies the following Legendre differential equation:

$$
Q_m^{n\prime\prime}+Q_m^{n\prime}\cot\varphi+Q_m^n\left[m(m+1)-\frac{n^2}{\sin^2\varphi}\right]=0
\tag{42}
$$

By using the operator H defined above, the following relation can be obtained:

$$
H(Q_m^n)=-\lambda_mQ_m^n\,,\quad HH(Q_m^n)=\lambda_m^2\,Q_m^n
\tag{43}
$$

where

$$
\lambda_m=m(m+1)-2
\tag{44}
$$

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Substituting the series of Eq. (41) into Eqs. (38) and (39) and using Eq. (43), the following is obtained:

$$\left[(1+k_i)\,\lambda_m+(1+k_i)(1+\nu_i)+\frac{w_i^{(0)}}{a_i}(1+\nu_i)\right]W_{m(i)}+\left[k_i\,\lambda_m^2+(1+k_i)(1+\nu_i)\,\lambda_m+2k_i\,\lambda_m+2(1+k_i)(1+\nu_i)+\frac{w_i^{(0)}}{a_i}(1+\nu_i)(1+\lambda_m)\right]\Omega_{m(i)}=0 \tag{45}$$

$$\left[(1+k_i)(1+\nu_i)+k_i\,\lambda_{m(i)}+\frac{w_i^{(0)}}{a_i}(1+\nu_i)\right]\Omega_{m(i)}+\left[2(1+k_i)(1+\nu_i)+k_i\,\lambda_m(\lambda_m+\nu_i+3)+\frac{w_i^{(0)}}{a_i}(1+\nu_i)(\lambda_m+2)+\frac{(a_1+a_2)^2k_f}{4C_i}\right]W_{m(i)}-\frac{(a_1+a_2)^2k_f}{4C_i}W_{m(j)}=0 \tag{46}$$

Hence, homogeneous linear equations are obtained for all m as follows:

$$\mathbf{C}\begin{Bmatrix}W_{m(1)}\\ \Omega_{m(1)}\\ W_{m(2)}\\ \Omega_{m(2)}\end{Bmatrix}=\mathbf{0} \tag{47}$$

This is a standard eigenvalue problem for which nontrivial values are obtained when the coefficient matrix in Eq. (47) becomes singular. The corresponding pressure p is thus calculated from the determinantal equations:

$$\det\mathbf{C}=0 \tag{48}$$

By solving Eq. (48), the pressure p with the corresponding mode m for several combinations of geometric and material constants can be obtained as an eigenvalue. For the particular values of the constants, the minimum value of the eigenvalues is the critical buckling pressure pcr. Note that the above equation is independent of the mode number n despite the effect of n in Eq. (33) being assumed; this appears in Eq. (37). Thus, only axisymmetric modes can occur in these cases.

4. ANALYTICAL RESULTS AND DISCUSSION

Figure 2 shows the variation of the nondimensional critical buckling pressure p_{cr}/p_0 where p_0 is the critical pressure for the empty complete spherical shell and is given by

$$p_0=\frac{2E_2}{\sqrt{3(1-\nu_2^2)}}\left(\frac{h_2}{a_2}\right)^2 \tag{49}$$

This figure shows that a relatively hard elastic medium necessarily leads to high buckling pressure. In the range of $a_2k_f/E>0.03\sim0.04$, the buckling pressures are almost constant. Figures 3 and 4 illustrate the buckling eigenmodes for double-walled complete spherical shells where $a_2k_f/E=0.01$ and 0.1, respectively. Characteristic wavy-shaped axisymmetric buckling deformation can be found in both cases. In Fig. 3, which shows a low a_2k_f/E value, the deformation amplitude of the inner shell is relatively small compared to that of the outer one. In contrast, Fig. 4, which shows a relatively high a_2k_f/E value, the outer and inner walls deform with almost the same amplitude. Detailed results are given in Fig. 5. Amplitude ratios converged to unity as a_2k_f/E increased. Moreover, the results in Figs. 3 and 4 suggest that the eigenmodes change corresponding to the mode number m. When m is odd, which corresponds to Fig. 3, the deformation is antisymmetric about the equator: That is, an inward

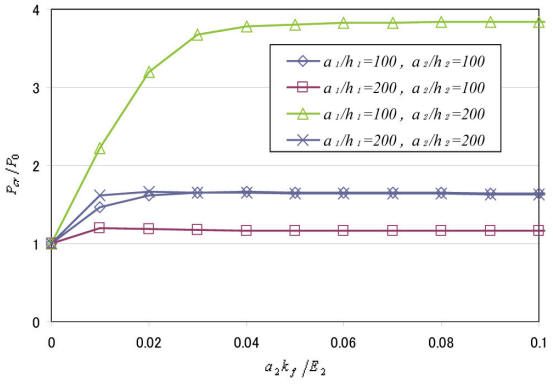


Fig. 2 Variation ofnondimensional critical buckling pressure p_{cr}/p_0.

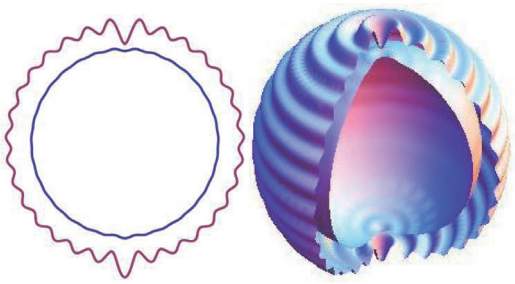


Fig. 3 Buckling modes ($a_2k_f/E=0.01$, $a_1/h_1=100$, $a_2/h_2=200$): (left) cross-sectional view (right) 3-D view. Mode number $m_{cr}=31$ can be found.

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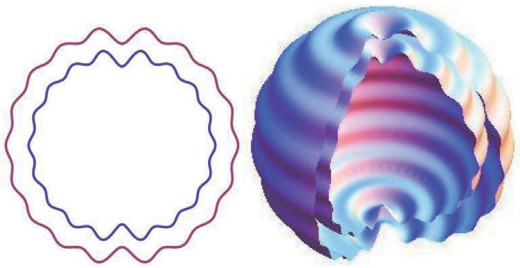


Fig. 4 Buckling modes ($a_2k_f / E = 0.01$, $a_1 / h_1 = 100$, $a_2 / h_2 = 100$): (left) cross-sectional view (right) 3-D view. Mode number $m_{cr} = 18$ can be found.

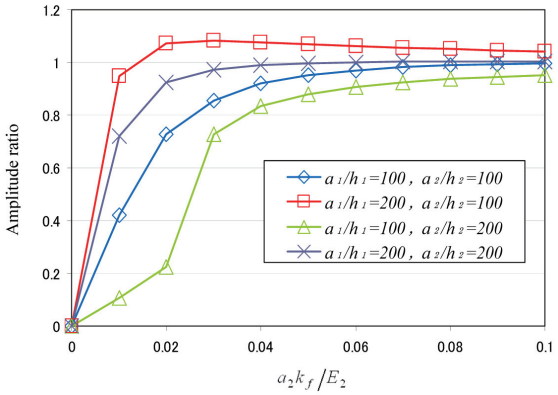


Fig. 5 Amplitude ratio (inner/outer).

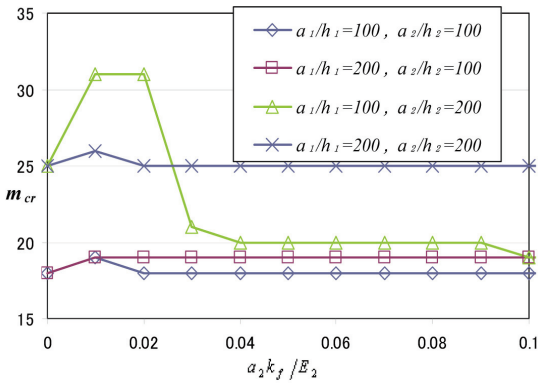


Fig. 6 Critical buckling wave number.

deformation at the north pole is accompanied by an outward deformation at the south pole. On the other hand, when m is even, the corresponding deformation of the spherical shell is symmetric about the equator, as shown in Fig. 4. This is due to the properties of the geometry of the sphere and the Legendre functions. The change in critical buckling wave number m_{cr} against a_2k_f / E is shown in Fig. 6. Over the range of $a_2k_f / E > 0.02 \sim 0.03$, m_{cr} gives constant values.

5. SUMMARY

The characteristic buckling eigenmodes in hydrostatically pressurized double-walled complete spherical shells were demonstrated to have an exact governing equation. The developed formulation showed that only axisymmetric eigenmodes can occur despite double-walled shells having an inner medium. The characteristic buckling mode can be found, and the critical modes that are symmetric or antisymmetric about the equator may be determined depending on the combination of the foundation stiffness and the radius-to-wall thickness ratios.

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