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The stratum of random mapping classes

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Abstract. We consider random walks on the mapping class group that have finite first moment with respect to the word metric, whose support generates a non-elementary subgroup and contains a pseudo-Anosov map whose invariant Teichmüller geodesic is in the principal stratum. For such random walks, we show that mapping classes along almost every infinite sample path are eventually pseudo-Anosov, with invariant Teichmüller geodesics in the principal stratum. This provides an answer to a question of Kapovich and Pfaff [*Internat. J. Algebra Comput.* **25**, 2015 (5) 745–798].

1. *Introduction*

Let S be an orientable surface of finite type, and let $\mathrm{Mod}(S)$ denote the mapping class group of S, consisting of orientation-preserving diffeomorphisms on S modulo isotopy. Let $\mathcal{T}(S)$ be the Teichmüller space of marked conformal structures on S, and let $\mathcal{Q}(S)$ be the space of quadratic differentials, which may be identified with the cotangent bundle of $\mathcal{T}(S)$. For punctured surfaces, the quadratic differentials in $\mathcal{Q}(S)$ are assumed to be meromorphic with poles only at punctures and with every puncture a simple pole. The space $\mathcal{Q}(S)$ is stratified by the order of the zeros of the quadratic differential; the principal stratum consists of those quadratic differentials whose zeros are all simple. By the Thurston classification, mapping classes are periodic, reducible or pseudo-Anosov. A pseudo-Anosov map has a unique invariant Teichmüller geodesic. If the invariant Teichmüller geodesic is given by a quadratic differential that lies in the principal stratum, then we say that the pseudo-Anosov map is in the principal stratum.

We consider random walks on the mapping class group $\mathrm{Mod}(S)$ that have finite first moment with respect to word metric and whose support generates a non-elementary

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subgroup of $\mathrm{Mod}(S)$: that is, the subgroup generated by the support of the initial distribution that contains a pair of pseudo-Anosov maps with distinct stable and unstable measured foliations. In independent work, Maher [**Mah11**] and Rivin [**Riv08**] showed that the probability that a random walk gives a pseudo-Anosov map tends to one in the length of the sample path. As a refinement, Kapovich and Pfaff raise the following question. What is the stratum of quadratic differentials for the invariant Teichmüller geodesic of a random pseudo-Anosov element? See [**KP15**, Question 1.5], [**DHM15**, Question 6.1].

As a step towards answering the question, we prove the following result. We shall write d_{Mod} for the word metric on $\mathrm{Mod}(S)$ with respect to a choice of finite generating set.

THEOREM 1.1. *Let S be a connected orientable surface of finite type whose Teichmüller space $\mathcal{T}(S)$ has complex dimension at least two. Let μ be a probability distribution on $Mod(S)$ such that:*

(1) *μ has finite first moment with respect to d_{Mod};*
(2) *$Supp(\mu)$ generates a non-elementary subgroup H of $Mod(S)$; and*
(3) *the semigroup generated by $Supp(\mu)$ contains a pseudo-Anosov g such that the invariant Teichmüller geodesic γ_g for g lies in the principal stratum of quadratic differentials.*

Then, for almost every bi-infinite sample path $\omega = (w_n)_{n\in\mathbb{Z}}$, there is positive integer N such that, for all $n \geqslant N$, the mapping class w_n is a pseudo-Anosov map in the principal stratum: that is, its invariant Teichmüller geodesic is given by a quadratic differential with simple zeros and poles. Furthermore, almost every bi-infinite sample path determines a unique Teichmüller geodesic γ_ω with the same limit points as the bi-infinite sample path, and this geodesic also lies in the principal stratum.

In the statement of Theorem 1.1, hypothesis (3) for the distribution μ is necessary. There are random walks for which μ satisfies hypothesis (1) and (2) but the conclusion of Theorem 1.1 is false. One such example is a random walk supported on a non-principal Teichmüller curve, that is, on a Veech lattice whose Teichmüller disc is not contained in the principal stratum. See [**Wri15**] for the precise definition of a Teichmüller curve.

We only consider surfaces whose Teichmüller space has complex dimension at least two. This excludes the sphere with four or fewer punctures and the torus with one or fewer punctures. In the case of the four-punctured sphere, and the torus with at most one puncture, $\mathcal{T}(S)$ has complex dimension one, and the principal stratum consists of all of $\mathcal{T}(S)$. So the result holds trivially; the remaining surfaces have finite $\mathrm{Mod}(S)$ and the result does not apply.

We have described the principal stratum in terms of the zeros of quadratic differentials, but it may also be described in terms of flat structures, foliations or laminations, as we now discuss. A quadratic differential equips the surface with a flat structure with a pair of measured foliations (or measured laminations), corresponding to the vertical and horizontal directions. The condition of having simple poles at punctures is equivalent to the flat metric having cone angle π around every puncture. The foliations thus have one-pronged singularities at the punctures, and the laminations have a complementary region which is a punctured monogon for each singularity. In the principal stratum, all zeros are simple. This corresponds to the flat metric having all other cone points of

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angle 3π; the corresponding measured foliations will have all other singularities trivalent, and the measured laminations will have all of the other complementary regions consisting of triangles.

If we consider PMF(S) as Thurston's boundary for $\mathcal{T}(S)$, then there is a natural Lebesgue measure class on PMF(S), defined in terms of train track coordinate charts. Also, Hubbard and Masur in [**HM79**] showed that, at each point X in $\mathcal{T}(S)$, the unit cotangent space may also be identified with PMF(S), and then the Masur–Veech measure [**Mas82**, **Vee82**] on the space of unit area quadratic differentials gives a conditional measure s_X on PMF(S), which Athreya *et al*, in [**ABEM12**], showed is absolutely continuous with respect to the Lebesgue measure class. As we now explain, the final statement of Theorem 1.1 is known for bi-infinite geodesics chosen according to this Lebesgue measure class. Kerckhoff, Masur and Smillie [**KMS86**] showed that the uniquely ergodic foliations have full measure with respect to the Lebesgue measure class. So any two foliations chosen according to the Lebesgue measures almost surely determine a unique bi-infinite Teichmüller geodesic. Finally, the principal stratum foliations have full measure in PMF(S) with respect to these measures. This is most easily seen using train track coordinates: the collection of non-principal strata foliations is contained in a countable union of positive co-dimension sets, and so has Lebesgue measure zero.

Kaimanovich and Masur [**KM96**] showed that, for any basepoint X in $\mathcal{T}(S)$ and for almost every infinite sample path $\omega = (w_n)$, the sequence $w_n X$ converges to the Thurston boundary PMF(S). This defines a harmonic measure on PMF(S) giving the distribution of the limit points of infinite sample paths. Furthermore, Kaimanovich and Masur showed that the measure is non-atomic and the set of uniquely ergodic foliations have full measure. However, such harmonic measures are singular with respect to the Lebesgue measure class on PMF(S). See [**Gad14**] for singularity for finitely supported random walks and [**GMT**] for singularity for random walks with finite first moment with respect to the word metric. Therefore the final statement of Theorem 1.1 does not follow immediately from the Lebesgue measure case.

There are countably many pseudo-Anosov elements, so the union of their endpoints has measure zero with respect to any of the measures discussed above. Hence, the results for pseudo-Anosov elements do not follow immediately from the results for bi-infinite geodesics. Apart from random walks, there are other ways to study generic foliations of mapping class group elements. For example, every conjugacy class of a pseudo-Anosov element corresponds to a closed geodesic in moduli space. Eskin and Mirzakhani [**EM11**] showed that the number of closed geodesics in moduli space of Teichmüller length at most R grows asymptotically as e^{hR}/hR, and Eskin, Mirzakhani and Rafi [**EMR12**] showed that the proportion of the closed geodesics of length at most R which do not lie in the principal stratum tends to zero exponentially quickly in R. They also show that the number of lattice points in the ball of radius R in Teichmüller space grows asymptotically as e^{hR}, and the proportion of these lattice points which lie close to a geodesic contained in the non-principal stratum also tends to zero exponentially fast.

The proof of Theorem 1.1 follows the following strategy. Let g be a pseudo-Anosov element whose invariant Teichmüller geodesic γ_g lies in the principal stratum. We show that any Teichmüller geodesic that fellow travels γ_g for a sufficiently large distance D,

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depending on g, also lies in the principal stratum. Next, we show that if g lies in the semigroup generated by the support of μ, there is a positive probability that the geodesic γ_ω tracked by a sample path ω fellow travels the invariant geodesic γ_g for distance at least D. Ergodicity of the shift map on $\mathrm{Mod}(S)^{\mathbb{Z}}$ then implies that a positive proportion of subsegments of γ_ω of length D fellow travel some translate of γ_g. We then use work of Dahmani and Horbez [**DH15**] which shows that, for almost all sample paths ω, for sufficiently large n, all elements w_n are pseudo-Anosov, with invariant geodesics γ_{w_n} which fellow travel γ_ω for a distance which grows linearly in n. In particular, this implies that γ_{w_n} fellow travels a sufficiently long subsegment of a translate of γ_g, and so lies in the principal stratum.

2. *Teichmüller preliminaries*

Let S be an orientable surface of finite type, whose Teichmüller space has complex dimension at least two, that is, it is not a sphere with at most four punctures or a torus with at most two punctures. The Teichmüller metric is given by

$$d_{\mathcal{T}}(X, Y) = \frac{1}{2} \inf_f \log K(f),$$

where the infimum is taken over all quasiconformal maps $f\colon X \to Y$ in the given homotopy class and $K(f)$ is the quasiconformal constant of f. As there is a unique Teichmüller geodesic connecting any pair of points in Teichmüller space, we may sometimes write $[X, Y]$ to denote the Teichmüller geodesic segment from X to Y. For detailed background information about the Teichmüller metric and the geometry of quadratic differentials, see, for example, [**Wri15**].

The complex of curves $\mathcal{C}(S)$ is an infinite graph with vertices that are isotopy classes of simple closed curves on S. Two vertices $[\alpha]$, $[\beta]$ are separated by an edge if the curves α and β can be isotoped to be disjoint. The graph $\mathcal{C}(S)$ is locally infinite and has infinite diameter, and Masur and Minsky showed that $\mathcal{C}(S)$ is δ-hyperbolic in the sense of Gromov [**MM99**].

By the uniformization theorem, a conformal class X determines a unique hyperbolic metric on S, which we shall also denote by X. For a hyperbolic surface X, a systole of X is a simple closed curve that has the shortest length in the hyperbolic metric. The set of systoles of X is a finite set whose diameter in $\mathcal{C}(S)$ is bounded above by a constant that depends only on the topology of S. Thus, the systole defines a coarse projection map $\pi : \mathcal{T}(S) \to \mathcal{C}(S)$. For notational simplicity, we will use upper case letters for points X in $\mathcal{T}(S)$ and the corresponding lower case letters $x = \pi(X)$ for their projections to the curve complex. Masur and Minsky, in [**MM99**, 6.1], showed that π is coarsely Lipschitz: that is, there are constants $M_1 > 0$, $A_1 > 0$ that depend only on S such that, for any pair of points $X, Y \in \mathcal{T}(S)$,

$$d_{\mathcal{C}}(x, y) < M_1 d_{\mathcal{T}}(X, Y) + A_1. \tag{2.1}$$

Moreover, Masur and Minsky also show that Teichmüller geodesics γ project to uniformly unparametrized quasigeodesics in $\mathcal{C}(S)$. Let (M_2, A_2)-be the quasigeodesicity constants for the projection of a Teichmüller geodesic, and let these constants depend only on S.

The set of hyperbolic surfaces $X \in \mathcal{T}(S)$ for which the length of the systole is less than ϵ form the ϵ-thin part $\mathcal{T}(S)_\epsilon$ of Teichmüller space. The complement $K_\epsilon = \mathcal{T}(S) \setminus \mathcal{T}(S)_\epsilon$ is

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called the thick part. By Mumford compactness, $\mathrm{Mod}(S)\backslash K_\epsilon$ is compact and, furthermore, a metric regular neighbourhood of the thick part is contained in a larger thick part. More precisely, for any $\epsilon > 0$ and any $D \geqslant 0$, there is a constant ϵ', depending on ϵ, D and the surface S, such that a metric D-neighbourhood of K_ϵ, in the Teichmüller metric, is contained in $K_{\epsilon'}$.

Let γ and γ' be two geodesics in a metric space (M, d). If there are are choices of (not necessarily unit speed) parametrizations $\gamma(t)$ and $\gamma'(t)$ such that there is a constant E with $d(\gamma(t), \gamma'(t)) \leqslant E$ for all t, then we say that γ and γ' are fellow travellers with fellow travelling constant E or that they E-fellow travel. If $d(\gamma(t), \gamma'(t)) \leqslant E$ for all t, for the unit speed parametrizations of γ and γ', we then say that γ and γ' are parametrized E-fellow travellers.

Let γ and γ' be two Teichmüller geodesics whose projections to the curve complex $\pi(\gamma)$ and $\pi(\gamma')$ fellow travel. In general, this does not imply that the original Teichmüller geodesics fellow travel in Teichmüller space. However, we now show, in the following lemma, that if γ is contained in a thick part K_ϵ and $\pi(\gamma')$ fellow travels $\pi(\gamma)$ for a sufficiently long distance in $\mathcal{C}(S)$, then γ' contains a point that is close to γ in Teichmüller space.

LEMMA 2.2. *For any constants $\epsilon > 0$ and $E \geqslant 0$, there are constants $L > 0$ and $F > 0$, depending on ϵ, E and the surface S such that, if $\gamma = [X, Y]$ is a Teichmüller geodesic segment contained in the thick part K_ϵ, of length at least L, and $\gamma' = [X', Y']$ is a Teichmüller geodesic segment whose endpoints x', y' in $\mathcal{C}(S)$ are at a distance of at most E from the endpoints x, y of $\pi(\gamma)$, that is, $d_{\mathcal{C}(S)}(x, x') \leqslant E$ and $d_{\mathcal{C}(S)}(y, y') \leqslant E$, then there is a point Z on γ' such that $d_{\mathcal{T}}(Z, \gamma) \leqslant F$.*

This result may also be deduced from work of Horbez [**Hor15**, Proposition 3.10] and Dowdall, Duchin and Masur [**DDM14**, Theorem A], extending that of Rafi in [**Raf14**], but, for the convenience of the reader, we provide a direct proof of this result in §5, that relies only on Rafi [**Raf14**]. In particular, we will make extensive use of the following fellow travelling result for Teichmüller geodesics whose endpoints are close together in the thick part.

THEOREM 2.3. [**Raf14**, Theorem 7.1] *For any constants $\epsilon > 0$ and $A \geqslant 0$, there is a constant B, depending only on ϵ, A and the surface S, such that if $[X, Y]$ and $[X', Y']$ are two Teichmüller geodesics, with X and Y in the ϵ-thick part, and*

$$d_{\mathcal{T}}(X, X') \leqslant A \text{ and } d_{\mathcal{T}}(Y, Y') \leqslant A,$$

then $[X, Y]$ and $[X', Y']$ are parametrized $B(\epsilon, A)$-fellow travellers.

We now continue with the proof of Theorem 1.1, assuming Lemma 2.2. Recall that the Gromov product based at a point $u \in \mathcal{C}(S)$ is defined to be

$$(x, y)_u = \tfrac{1}{2}(d_{\mathcal{C}}(u, x) + d_{\mathcal{C}}(u, y) - d_{\mathcal{C}}(x, y)).$$

Given points $x, y \in \mathcal{C}(S)$ and a constant $R > 0$, the R-shadow of y is defined to be

$$S_x(y, R) = \{z \in \mathcal{C}(S) \mid (y, z)_x \geqslant d_{\mathcal{C}}(x, y) - R\}.$$

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The definition we use here for shadows follows [**MT14**], and may differ slightly from other sources. The following lemma follows from the thin triangles property of Gromov hyperbolic spaces, and we give a proof for the convenience of the reader.

LEMMA 2.4. *There is a constant D, which only depends on δ, and a constant E, which only depends on R and δ, such that if $d_{\mathcal{C}}(x, y) \geqslant 2R + D$, then, for any $x' \in S_y(x, R)$ and any $y' \in S_x(y, R)$, any geodesic segment $[x', y']$ contains a subsegment which E-fellow travels $[x, y]$.*

Proof. We shall write $O(\delta)$ to denote a constant which only depends (not necessarily linearly) on δ.

Let p be the nearest point projection of x' to $[x, y]$, and let q be the nearest point projection of y' to $[x, y]$. The nearest point projection of the shadow $S_x(y, R)$ is contained in an $(R + O(\delta))$-neighbourhood of y (see, for example, [**MT14**, Proposition 2.4]), so $d_{\mathcal{C}}(x, p) \leqslant R + O(\delta)$ and $d_{\mathcal{C}}(y, q) \leqslant R + O(\delta)$. Recall that if $d_{\mathcal{C}}(p, q) \geqslant O(\delta)$, then any geodesic from x' to y' passes within an $O(\delta)$-neighbourhood of both p and q (see, for example, [**MT14**, Proposition 2.3]). Therefore, if $d(x, y) \geqslant 2R + O(\delta)$, then this implies that if p' is the closest point on $[x', y']$ to p and q' is the closest point on $[x', y']$ to q, then $[p', q']$ E-fellow travels $[x, y]$, where E is a constant which only depends on R and δ, as required. □

Remark 2.5. One can replace the geodesic segments $[x, y]$ and $[x', y']$ by (M_2, A_2)-quasigeodesic segments. The constants D and E now change and, in addition to R and δ, they now depend on the quasigeodesicity constants.

We shall write $\mathrm{PMF}(S)$ for the set of projective measured foliations on the surface S, which is Thurston's boundary for Teichmüller space. A projective measured foliation is uniquely ergodic if the foliation supports a unique projective measure class. Let $\mathrm{UE}(S)$ be the subset of $\mathrm{PMF}(S)$ consisting of uniquely ergodic foliations. We shall give $\mathrm{UE}(S)$ the corresponding subspace topology. A uniquely ergodic foliation determines a class of mutually asymptotic geodesic rays in $\mathcal{T}(S)$, as shown by Masur [**Mas80**]. These rays project to a class of mutually asymptotic quasigeodesic rays in $\mathcal{C}(S)$, and so determine a point in the Gromov boundary of the curve complex. This boundary map is injective on uniquely ergodic foliations (see, for example, Hubbard and Masur [**HM79**]). Thus $\mathrm{UE}(S)$ is also a subset of $\partial\mathcal{C}(S)$. Klarriech [**Kla**] showed that $\partial\mathcal{C}(S)$ is homeomorphic to the quotient of the set of minimal foliations in $\mathrm{PMF}(S)$ by the equivalence relation which forgets the measure. In particular, this implies that the two subspace topologies on $\mathrm{UE}(S)$, induced from inclusions in $\mathrm{PMF}(S)$ and $\partial\mathcal{C}(S)$, are the same.

Let γ be a Teichmüller geodesic in a thick part K_ϵ. Let λ^+ and λ^- be the projective classes of vertical and horizontal measured foliations of γ. By the work of Kerckhoff, Masur and Smillie, in [**KMS86**, Theorem 3], vertical foliations of Teichmüller rays that are recurrent to a thick part are uniquely ergodic, so the foliations λ^+ and λ^- are uniquely ergodic, and, by Hubbard and Masur [**HM79**], such a pair (λ^-, λ^+) determines a unique bi-infinite Teichmüller geodesic. Given two points X and Y in Teichmüller space and a constant $r \geqslant 0$, we define $\Gamma_r(X, Y)$ to be the set of all oriented geodesics with uniquely ergodic vertical and horizontal foliations, which intersect both $B_r(X)$ and

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$B_r(Y)$ and, furthermore, whose first point of intersection with either $B_r(X)$ or $B_r(Y)$ lies in $B_r(X)$. A Teichmüller geodesic with uniquely ergodic vertical foliation λ^+ and uniquely ergodic horizontal foliation λ^- determines a point (λ^-, λ^+) in $\mathrm{UE}(S) \times \mathrm{UE}(S)$. Therefore $\Gamma_r(X, Y)$ determines a subset of $\mathrm{UE}(S) \times \mathrm{UE}(S)$, which, by abuse of notation, we shall also denote by $\Gamma_r(X, Y)$.

PROPOSITION 2.6. *For any Teichmüller geodesic γ contained in a thick part K_ϵ, with vertical foliation λ^+ and horizontal foliation λ^-, there is a constant $r > 0$, depending on ϵ, such that for any pair of points X and Y on γ, the set $\Gamma_r(X, Y)$ contains an open neighbourhood of (λ^-, λ^+) in $UE(S) \times UE(S)$.*

Proof. As $\mathrm{Mod}(S)$ acts coarsely transitively on the curve complex $\mathcal{C}(S)$, there is a constant $R > 0$, depending only on S, such that, for all x and y in $\mathcal{C}(S)$, the limit set of the shadow $\overline{S_x(y, R)}$ contains a non-empty open set in $\partial\mathcal{C}(S)$ (see, for example, [**MT14**, Propositions 3.18–19]). Given such an R, let D and E be the constants in Lemma 2.4 such that if $d(x, y) \geqslant D$, then, for any $x' \in S_y(x, R)$ and $y' \in S_y(x, R)$, a geodesic $[x', y']$ has a subsegment which E-fellow travels with $[x, y]$. Given ϵ and E, let L and F be the constants in Lemma 2.2: that is, if γ and γ'' are two Teichmüller geodesics of length at least L, whose endpoints in $\mathcal{C}(S)$ are a distance of at most E apart, then the distance from γ to γ' is at most F.

As γ lies in the thick part K_ϵ, there is a constant D', depending only on ϵ, such that if $d_{\mathcal{T}}(X, Y) \geqslant D'$, then $d_{\mathcal{C}}(x, y) \geqslant D$. Let Z_1 and Z_2 be points along γ such that $[X, Y] \subset [Z_1, Z_2]$, the orientations of the segments agree, $d_{\mathcal{T}}(X, Y) \geqslant D'$, $d_{\mathcal{T}}(Z_1, X) > L$ and $d_{\mathcal{T}}(Y, Z_2) > L$. Consider the limit sets $\overline{S_{z_1}(z_2, R)}$ and $\overline{S_{z_2}(z_1, R)}$ in $\partial\mathcal{C}(S)$ and let ξ^+ and ξ^- be uniquely ergodic foliations in $\overline{S_{z_1}(z_2, R)}$ and $\overline{S_{z_2}(z_1, R)}$, respectively. Let γ' be the Teichmüller geodesic with vertical foliation ξ^+ and the horizontal foliation ξ^-. By Lemma 2.4, the projection $\pi(\gamma')$ fellow travels $\pi(\gamma)$ with constant E between z_1 and z_2. For clarity, denote by Z_1', X', Y' and Z_2' the points of γ' whose projections z_1', x', y' and z_2' are coarsely the closest points to z_1, x, y and z_2, respectively: that is, the distances $d_{\mathcal{C}}(z_1', z_1)$, $d_{\mathcal{C}}(x', x)$, $d_{\mathcal{C}}(y', y)$ and $d_{\mathcal{C}}(z_2', z_2)$ are all at most E. By Lemma 2.2 applied to the segments $[Z_1', X']$ and $[Z_1, X]$, there is a point $W_1 \in [Z_1', X']$ such that $d_{\mathcal{T}}(W_1, [Z_1, X]) \leqslant F$. Similarly, there is a point $W_2 \in [Y', Z_2']$ such that $d_{\mathcal{T}}(W_2, [Y, Z_2]) \leqslant F$.

By the fellow travelling result, Theorem 2.3, the Teichmüller geodesic segment $[W_1, W_2] \subset \gamma'$ fellow travels γ with the constant $r = B(\epsilon, F)$. In particular, γ' passes through $B_r(X)$ and $B_r(Y)$, and hence lies in $\Gamma_r(X, Y)$, and so this set contains an open neighbourhood of (λ^-, λ^+). We have shown this as long as $d_{\mathcal{T}}(X, Y) \geqslant D'$, but, for $r' = 2r + D'$, every pair of balls $B_{r'}(X')$ and $B_{r'}(Y')$ contain smaller balls $B_r(X)$ and $B_r(Y)$ with $d_{\mathcal{T}}(X, Y) \geqslant D'$, so the stated result follows. □

3. *Fellow travelling of invariant and tracked geodesics*

In this section, we establish that along almost every sample path ω, for sufficiently large n, the invariant Teichmüller geodesic for the pseudo-Anosov element w_n, has a subsegment, whose length grows linearly in n, which fellow travels the Teichmüller

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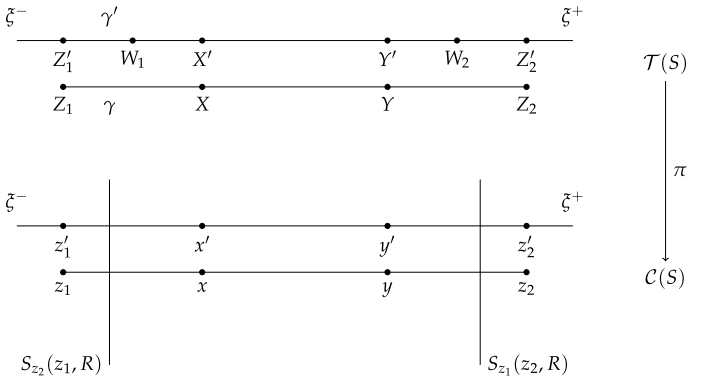


FIGURE 1. Shadows in $\mathcal{C}(S)$.

geodesic sublinearly tracked by ω. This uses a result of Dahmani and Horbez [**DH15**] and the fellow travelling result, Theorem 2.3. We fix a basepoint $X \in \mathcal{T}(S)$.

We require a slight rephrasing of a result of Dahmani and Horbez. Let ℓ be the drift of the random walk in the Teichmüller metric. Kaimanovich and Masur [**KM96**] showed that almost every bi-infinite sample path ω converges to distinct uniquely ergodic measured foliations λ_ω^+ and λ_ω^-, with $w_n X$ converging to λ_ω^+ and $w_{-n} X$ converging to λ_ω^- as $n \to \infty$. Let γ_ω be the unique bi-infinite Teichmüller geodesic determined by these foliations. We shall give γ_ω a unit speed parametrization such that $\gamma_\omega(0)$ is a closest point on γ_ω to X and, as $t \to \infty$, the geodesic $\gamma_\omega(t)$ converges to λ^+. If w_n is pseudo-Anosov, then we shall write γ_{ω_n} for its invariant Teichmüller geodesic.

To compare notation with [**DH15**, Theorem 2.6], we shall choose their group G to be $\mathrm{Mod}(S)$. The hyperbolic G-electrification (Y, X) they use is, in our case, the electrification with $Y = \mathcal{T}(S)$ and $X = \mathcal{C}(S)$. We do not have to worry about metric symmetrization as the Teichmüller metric is symmetric. The space $\mathrm{PMF}(S)$ of projective measured foliations on S gives the bordification of $\mathcal{T}(S)$. Then steps 1 and 3 in the proof of [**DH15**, Theorem 2.6], stated in the context of Teichmüller space, can be rephrased as the following proposition.

PROPOSITION 3.1. *Given $\epsilon > 0$, there are constants $F > 0$ and $0 < e < \frac{1}{2}$ such that, for almost every ω, there exists N such that, for all $n \geqslant N$, there are points Y_0 and Y_1 of γ_{w_n} and points $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$ of γ_ω, such that:*

(1) $d_{\mathcal{T}}(\gamma_\omega(T_0), Y_0) \leqslant F$;
(2) $d_{\mathcal{T}}(\gamma_\omega(T_1), Y_1) \leqslant F$;
(3) $0 \leqslant T_0 \leqslant e\ell n \leqslant (1-e)\ell n \leqslant T_1 \leqslant \ell n$; *and*
(4) $\gamma_\omega(T_0)$ *and* $\gamma_\omega(T_1)$ *are in the thick part* K_ϵ.

For completeness, we summarize specifically in the Teichmüller context the key ideas behind [**DH15**, Theorem 2.6]. The summary differs slightly from the proof given in [**DH15**] as we describe the special case which applies to Proposition 3.1. First, the Teichmüller translation length for a pseudo-Anosov map f is the infimum (which is realized in $\mathcal{T}(S)$) of $d_{\mathcal{T}}(Y, fY)$ as Y varies over $\mathcal{T}(S)$. The definition directly implies

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that, for almost every sample path ω, the quantity ℓn is an upper bound for the Teichmüller translation length of w_n as $n \to \infty$. Next, we summarize the argument for the analogous lower bound.

Almost every sample path ω makes linear progress in $\mathcal{T}(S)$. By the work of Tiozzo [**Tio15**], almost every ω sublinearly tracks a Teichmüller geodesic γ_ω. Thus, given $0 < e < 1/2$, for all n large enough depending on ω, the Teichmüller distance of w_n from $\gamma_\omega(\ell n)$ is at most $e\ell n$. By the ergodicity of the shift map on $\mathrm{Mod}(S)^{\mathbb{Z}}$, almost every ω recurs infinitely often to a fixed neighbourhood of γ_ω. Furthermore, the recurrence has an asymptotic frequency. The point of γ_ω closest to such a recurrent point is also in the thick part because of its proximity to a point of the sample path. In particular, similar arguments imply that, given $0 < e < 1/2$, there is n large enough depending on ω such that inside the segment $[\gamma_\omega(0), \gamma_\omega(e\ell n)]$ one can find a subsegment of a definite size contained entirely in the ϵ-thick part. A thick Teichmüller segment of a definite size makes definite progress under projection to the curve complex. The proximity of w_n to $\gamma_\omega(\ell n)$ and the hyperbolicity of $\mathcal{C}(S)$ implies that the vertical foliation for γ_{w_n} as a point in $\partial\mathcal{C}(S)$ is contained in the limit set of some shadow set of the point $\pi(\gamma_\omega(\ell n))$. In particular, this implies that the projection $\pi(\gamma_{w_n})$ of the invariant geodesic for w_n fellow travels $\pi(\gamma_\omega)$ over a segment that contains this segment of definite progress. Hence, by [**DH15**, Proposition 3.7] attributed to Dowdall, Duchin and Masur [**DDM14**], the fellow travelling in the curve complex lifts to a fellow travelling of Teichmüller geodesics γ_{w_n} and γ_ω over the thick subsegment. In particular, there is a point Y_0 of γ_{w_n} that is a bounded Teichmüller distance away from some point $\gamma_\omega(T_0)$ with $0 \leqslant T_0 \leqslant e\ell n$. A symmetric argument shows that there is a point Y_1 of γ_{w_n} that is a bounded Teichmüller distance away from some point $\gamma_\omega(T_1)$ with $(1-e)\ell n \leqslant T_1 \leqslant \ell n$. Proposition 3.1 gives a precise version of these statements. By a simple triangle inequality, it then follows that ℓn is asymptotically a lower bound for the Teichmüller translation length of w_n.

Dahmani and Horbez state condition (4) in terms of a 'contraction' property that they define as follows: $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$ are 'contraction' points on γ_ω for the projection map to the curve complex. In effect, the property being used by them is that, under the projection to the curve complex, γ_ω makes definite progress at $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$. See the discussion related to [**DH15**, Propositions 3.6 and 3.7]. We will now recall their precise definition [**DH15**, Definition 3.5] of definite progress.

Definition 3.2. Given constants $B, C > 0$, a Teichmüller geodesic γ makes (B, C)-progress at a point $Y = \gamma(T)$ if the image under π of the subsegment of γ of length B starting at Y has diameter at least C in the curve complex.

For completeness, we prove that definite progress implies thickness.

LEMMA 3.3. *If γ makes (B, C)-progress at Y, then there is a constant $\epsilon > 0$, which depends on B and C, such that Y lies in the thick part K_ϵ.*

Proof. Let α be the systole for the hyperbolic surface Y. For any point Y' on the subsegment, Wolpert's lemma implies that

$$\ell_{Y'}(\alpha) \leqslant e^B \ell_Y(\alpha).$$

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We will use the following version of the collar lemma, due to Matelski [**Mat76**], which states that a simple closed geodesic of length ℓ is contained in an embedded annular collar neighbourhood of width at least w_ℓ, where a lower bound for w_ℓ is given by

$$\sinh^{-1}\left(\frac{1}{\sinh(\ell/2)}\right),$$

and, furthermore, this lower bounds holds for all $\ell > 0$. Thus the width of the collar neighbourhood for α in the hyperbolic metric corresponding to Y' is bounded below by

$$\sinh^{-1}\left(\frac{1}{\sinh(e^B \ell_Y(\alpha)/2)}\right),$$

and the bound tends to infinity monotonically as $\ell_Y(\alpha)$ tends to zero. Suppose that β is the systole at Y' and $d_{\mathcal{C}}(\alpha, \beta) \geqslant C$. This implies that the intersection number satisfies

$$i(\alpha, \beta) \geqslant \frac{C-1}{2}.$$

From the lower bound on the width of the collar, the length of β has to satisfy

$$\ell_{Y'}(\beta) \geqslant \frac{C-1}{2} \sinh^{-1}\left(\frac{1}{\sinh(e^B \ell_Y(\alpha)/2)}\right).$$

Since β is the systole at Y', the length of β at Y' is at most the length of α at Y', so

$$e^B \ell_Y(\alpha) \geqslant \frac{C-1}{2} \sinh^{-1}\left(\frac{1}{\sinh(e^B \ell_Y(\alpha)/2)}\right).$$

Note that sinh is monotonically increasing, zero at zero, and unbounded, so this gives a lower bound ϵ on how small $\ell_Y(\alpha)$ can be, which depends on B and C. □

Remark 3.4. Lemma 3.3 implies that the points $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$ in Proposition 3.1 are in a thick part K_ϵ. By the fellow travelling result, Theorem 2.3, the geodesics γ_ω and γ_{w_n} fellow travel between $\gamma_\omega(T_0)$ and $\gamma_\omega(T_1)$. Let $s = B(\epsilon, F)$ be the constant for fellow travelling of γ_ω and γ_{w_n}.

4. *Ubiquity of segments contained in the principal stratum*

We now show that, for a pseudo-Anosov element g in the support of μ, there is a positive probability that the geodesic γ_ω fellow travels the invariant geodesic γ_g. We shall write ν for the harmonic measure on $\mathrm{UE}(S)$ and $\check{\nu}$ for the reflected harmonic measure, that is, the harmonic measure arising from the random walk generated by the probability distribution $\check{\mu}(g) = \mu(g^{-1})$.

LEMMA 4.1. *Let g be a pseudo-Anosov element contained in the support of μ with invariant Teichmüller geodesic γ_g. Then there is a constant $r > 0$ such that $\check{\nu} \times \nu(\Gamma_r(X, Y)) > 0$ for all X and Y on γ_g.*

Furthermore, there is a constant $\rho > 0$, depending on g, such that, for all constants $D \geqslant 0$, there is a positive probability (that depends on D) for the subsegment of γ_ω of length D, centred at a closest point on γ_ω to the basepoint, to ρ-fellow travel with γ_g.

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Proof. Let λ^+ and $\lambda^- \in \mathrm{PMF}(S)$ be the vertical and horizontal foliations of γ_g. Fix an $\epsilon > 0$ such that the thick part K_ϵ contains the geodesic γ_g. Let r be the constant in Proposition 2.6: that is, for any points X and Y on γ_g, the set $\Gamma_r(X, Y)$ contains an open neighbourhood of (λ^-, λ^+). We recall the following proposition.

PROPOSITION 4.2. [**MT14**, Proposition 5.4] *Let G be a non-elementary, countable group acting by isometries on a separable Gromov hyperbolic space X, and let μ be a non-elementary probability distribution on G. Then there is a number R_0 such that, for any group element g in the semigroup generated by the support of μ, the closure of the shadow $S_{x_0}(gx_0, R_0)$ has positive hitting measure for the random walk determined by μ.*

Let $x_0 = \pi(X_0)$ be the projection of the basepoint X_0 into the curve complex. We may assume that $\Gamma_r(X, Y)$ contains an open neighbourhood of (λ^-, λ^+) of the form $U^- \times U^+$, where U^- is an open neighbourhood of λ^- in $\mathrm{UE}(S)$ and U^+ is an open neighbourhood of λ^+ in $\mathrm{UE}(S)$. As

$$\bigcap_{i\in\mathbb{N}} \overline{S_{x_0}(g^{-i}x_0, R_0)} = \lambda^- \quad \text{and} \quad \bigcap_{i\in\mathbb{N}} \overline{S_{x_0}(g^{i}x_0, R_0)} = \lambda^+,$$

there is an integer i, such that the limit sets of the shadows are contained in the open neighbourhoods of λ^+ and λ^-: that is,

$$\overline{S_{x_0}(g^{-i}x_0, R_0)} \cap \mathrm{UE}(S) \subset U^- \quad \text{and} \quad \overline{S_{x_0}(g^{i}x_0, R_0)} \cap \mathrm{UE}(S) \subset U^+.$$

The element g^{-1} is in the semigroup generated by the inverses of $\mathrm{Supp}(\mu)$: that is, $g^{-1} \in \mathrm{Supp}(\check{\mu})$. Hence, by Proposition 4.2,

$$\check{\nu} \times \nu\big(\overline{S_{x_0}(g^{-i}x_0, R_0)} \times \overline{S_{x_0}(g^{i}x_0, R_0)}\big) > 0,$$

and so $\check{\nu} \times \nu(\Gamma_r(X, Y)) > 0$, as required.

The final statement then follows from Theorem 2.3, which implies that there is a $\rho > 0$ such that any geodesic in $\Gamma_r(X, Y)$ must ρ-fellow travel $[X, Y]$, as required. Here we may choose X and Y on γ_g such that the geodesic $[X, Y]$ contains a subsegment of length D centred at any closest point on γ_g to the basepoint X_0; as γ_g is contained in a thick part K_ϵ, the set of closest points on γ_g to X_0 has bounded diameter, depending only on ϵ and the surface S. □

We now make use of the principal stratum assumption, that is, that the semigroup generated by $\mathrm{Supp}(\mu)$ contains a pseudo-Anosov g whose invariant Teichmüller geodesic γ_g lies in the principal stratum. We first prove the following proposition.

PROPOSITION 4.3. *Let g be a pseudo-Anosov element of $\mathrm{Mod}(S)$ whose invariant Teichmüller geodesic is contained in the principal stratum. For any $\rho > 0$, there is a constant $D > 0$, depending on ρ and g, such that, for any pair of points X, Y on γ_g with $d_{\mathcal{T}}(X, Y) \geqslant D$, any Teichmüller geodesic in $\Gamma_\rho(X, Y)$ lies in the principal stratum.*

Proof. The invariant geodesic γ_g projects to a closed geodesic in moduli space, and so lies in the thick part K_ϵ for some ϵ depending on g. If a geodesic γ passes through $B_\rho(X)$ and $B_\rho(Y)$ for $X, Y \in \gamma_g$, then, by the fellow travelling result, Theorem 2.3, it $B(\epsilon, \rho)$-fellow travels $[X, Y]$.

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To derive a contradiction, suppose that there is a sequence ϕ_n of geodesic segments in non-principal strata such that the ϕ_n fellow travel γ_g for distances d_n with $d_n \to \infty$ as $n \to \infty$. As the cyclic group generated by g acts coarsely transitively on γ_g, we may assume that the midpoints of the ϕ_n are all a bounded distance from the basepoint X in Teichmüller space. By convergence on compact sets, we can pass to a limiting geodesic ϕ which lies in a non-principal strata, as the principal stratum is open. The geodesics ϕ and γ_g fellow travel in the forward direction for all times. By [**Mas80**, Theorem 2], this implies that ϕ and γ_g have the same vertical foliation. This is a contradiction since ϕ is in a non-principal stratum. □

We now complete the proof of Theorem 1.1.

Proof of Theorem 1.1. We fix a pseudo-Anosov element g in the support of μ for which the invariant Teichmüller geodesic γ_g is contained in the principal stratum. Without loss of generality, we fix the basepoint X to be on γ_g.

Let $\epsilon > 0$ be sufficiently small such that γ_g is contained in the thick part K_ϵ. Given this ϵ, let $F_0 > 0$ and $0 < e_0 < \frac{1}{2}$ be the constants from Proposition 3.1. Let $\rho > 0$ be the constant in Lemma 4.1 that ensures that ρ-fellow travels for any length $D > 0$ between γ_w and γ_g with a positive probability, depending on D. By Proposition 4.3, there is a D_0 such that any Teichmüller geodesic for which $(\rho + F_0)$-fellow travels with γ_g a distance of at least D_0 is contained in the principal stratum. We shall set $D = D_0 + 2F_0$.

Let $k > 0$ be the smallest positive integer such that $d_{\mathcal{T}}(g^{-k}X, g^k X) \geqslant D$. By Theorem 2.3, any geodesic in $\Gamma_r(g^{-k}X, g^k X)$ ρ-fellow travels the subsegment $[g^{-k}X, g^k X]$ of γ_g. Let $\Omega \subset \mathrm{Mod}(S)^{\mathbb{Z}}$ consist of those sample paths ω such that the sequences $w_{-n}X$ and $w_n X$ converge to distinct uniquely ergodic foliations $(\lambda^-, \lambda^+) \in \Gamma_r(g^{-k}X, g^k X)$. Lemma 4.1 implies that the subset Ω has positive probability $p > 0$.

Let $\sigma : \mathrm{Mod}(S)^{\mathbb{Z}} \to \mathrm{Mod}(S)^{\mathbb{Z}}$ be the shift map. Ergodicity of σ implies that, for almost every ω, there is some $n \geqslant 0$ such that $\sigma^n(\omega) \in \Omega$. For such n, the subsegment of γ_ω of length D, centred at the closest point on γ_ω to the point $w_n X$, ρ-fellow travels with a translate of $w_n \gamma_g$. In particular, this implies that γ_ω lies in the principal stratum, giving the final claim in Theorem 1.1.

For almost every ω, the proportion of times $1 \leqslant n \leqslant N$ such that $\sigma^n(\omega) \in \Omega$ tends to p as $N \to \infty$. Choose numbers e_1 and e_2 such that $e_0 < e_1 < e_2 < \frac{1}{2}$. Then this also holds for N replaced with either $e_1 N$ or $(1 - e_1)N$. So this implies that the proportion of times $e_1 N \leqslant n \leqslant (1 - e_1)N$ with this property also tends to p as $N \to \infty$. This implies that, given ω, there is an N_0 such that, for all $N \geqslant N_0$, there is an n with $e_1 N \leqslant n \leqslant (1 - e_1)N$ and $\sigma^n(\omega) \in \Omega$.

Recall that, by sublinear tracking in Teichmüller space, due to Tiozzo [**Tio15**], there is a constant $\ell > 0$ such that, for almost all ω,

$$\lim_{n \to \infty} \frac{1}{n} d_{\mathcal{T}}(w_n X, \gamma_\omega(\ell n)) = 0,$$

where γ_ω is parametrized such that $\gamma_\omega(0)$ is a closest point on γ_ω to the basepoint. Therefore, possibly replacing N_0 with a larger number, we may also assume that $d_{\mathcal{T}}(x_N X, \gamma_\omega(\ell N)) \leqslant (e_2 - e_1)N$ for all $N \geqslant N_0$.

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Choose numbers ℓ_1 and ℓ_2, with $\ell_1 < \ell < \ell_2$, and choose them sufficiently close to ℓ so that $e_0\ell < e_1\ell_1$ and $(1-e_1)\ell_2 < (1-e_0)\ell$. Therefore the geodesic $[\gamma_\omega(e_2\ell_1 N - \rho), \gamma_\omega((1-e_2)\ell_2 N + \rho)]$ contains a subsegment of length at least D which ρ-fellow travels with a translate of γ_g. By our choice of ℓ_1 and ℓ_2, the geodesic $[\gamma_\omega(e_0\ell_1 N - \rho), \gamma_\omega((1-e_0)\ell_2 N + \rho)]$ is contained in $[\gamma_\omega(e_2\ell N), \gamma_\omega((1-e_2)\ell N)]$ for N sufficiently large. Now, using Proposition 3.1, this implies that the invariant geodesic γ_{w_n} $(\rho + F_0)$-fellow travels with γ_g for a distance of at least $D - 2F_0 \geqslant D_0$. Then, by Proposition 4.3, γ_{w_n} is contained in the principal stratum, as required.

5. *Fellow travelling in Teichmüller space*

We now provide a direct proof of Lemma 2.2, relying only on results from Rafi [**Raf14**]. The first result we shall use is the fellow travelling result for Teichmüller geodesics with endpoints in the thick part (Theorem 2.3). The second result is a thin triangles theorem for triangles in Teichmüller space, where one side has a large segment contained in the thick part.

THEOREM 5.1. [**Raf14**, Theorem 8.1] *For every $\epsilon > 0$, there are constants C and L, depending only on ϵ and S, such that the following holds. Let X, Y and Z be three points in $\mathcal{T}(S)$, and let $[X', Y']$ be a segment of $[X, Y]$ with $d_{\mathcal{T}}(X', Y') > L$ such that $[X', Y']$ is contained in the ϵ-thick part of $\mathcal{T}(S)$. Then there is a point $W \in [X', Y']$, such that*

$$\min\{d_{\mathcal{T}}(W, [X, Z]), d_{\mathcal{T}}(W, [Y, Z])\} \leqslant C.$$

We now prove Lemma 2.2.

Proof. The projection of an ϵ_i-thick Teichmüller geodesic makes definite progress in the curve complex: that is, there exist constants P_i and Q_i, depending on ϵ_i and the surface S, such that, for any points X, Y on γ, we have the estimate

$$d_{\mathcal{C}}(x, y) \geqslant P_i d_{\mathcal{T}}(X, Y) - Q_i. \tag{5.2}$$

Set $\epsilon_1 = \epsilon$. Let L_1 and C_1 be the corresponding constants from the thin triangle result, Theorem 5.1. Let $B_1 = B(\epsilon_1, C_1 + L_1/2)$ be the constant in the fellow travelling theorem, Theorem 2.3. Set $\epsilon_2 = \epsilon(\epsilon_1, B_1)$: that is, the B_1-neighbourhood of K_{ϵ_1} is contained in K_{ϵ_2}. Given this ϵ_2, let L_2 and C_2 be the corresponding constants from the thin triangle result, Theorem 5.1. Now that all the constants we need are defined, we shall choose L to be the maximum of the three terms

$$\begin{gathered}\frac{3}{P_1}(M_1C_1 + Q_1 + M_2E + A_2 + A_1) + \frac{3}{2}L_1,\\ 3L_2 + 3L_1 + 6C_1,\\ \frac{3}{P_2}(M_1C_2 + Q_2 + M_2E + A_2 + A_1) + \frac{3}{2}L_1 + 3B_1.\end{gathered} \tag{5.3}$$

Let Z_1 be the point that is $1/3$ of the way along $[X, Y]$. Let γ_1 be the geodesic segment of γ centred at Z_1 with length L_1. Similarly, let Z_2 be the point that is $2/3$ of the way along $[X, Y]$. Let γ_2 be the geodesic segment of γ centred at Z_2 with length L_1. The second term of (5.3) implies that $L > 3L_1$. Figure 2 illustrates this set-up.

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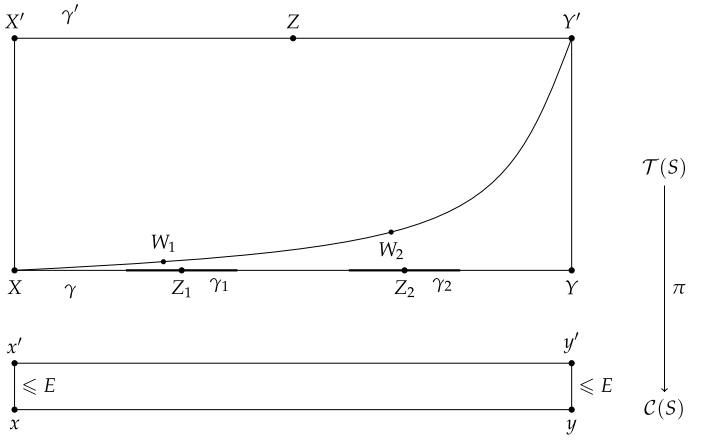


FIGURE 2. Fellow travelling geodesics in $\mathcal{T}(S)$.

Applying the thin triangles result, Theorem 5.1, to X, Y and Y', there is a point W_1 on $[X, Y'] \cup [Y, Y']$ within distance C_1 of γ_1. Similarly, there is a point W_2 on $[X, Y'] \cup [Y, Y']$ within distance C_1 of γ_2.

We now show that there is a lower bound on the distance of γ_2 from $[Y, Y']$. In particular, the same is true for the distance of γ_1 from $[Y, Y']$.

CLAIM 5.4. *The Teichmüller distance from γ_2 to $[Y, Y']$ is at least C_1.*

Proof. The Teichmüller distance of Y from γ_2 is at least $\frac{1}{3}L - \frac{1}{2}L_1$: that is,

$$d_{\mathcal{T}}(\gamma_2, Y) \geqslant \tfrac{1}{3}L - \tfrac{1}{2}L_1.$$

As ϵ_1-thick geodesics make definite progress in $\mathcal{C}(S)$, (5.2), this implies that

$$d_{\mathcal{C}}(\pi(\gamma_2), y) \geqslant P_1(\tfrac{1}{3}L - \tfrac{1}{2}L_1) - Q_1.$$

Teichmüller geodesic segments project to (M_2, A_2)-quasigeodesics in $\mathcal{C}(S)$. Since the endpoints of γ and γ' are distance at most E apart in $\mathcal{C}(S)$, this implies that

$$d_{\mathcal{C}}(\pi(\gamma_2), \pi([Y, Y'])) \geqslant P_1(\tfrac{1}{3}L - \tfrac{1}{2}L_1) - Q_1 - M_2E - A_2.$$

As the curve complex distance is a coarse lower bound on the Teichmüller distance, (2.1), this implies that

$$d_{\mathcal{T}}(\gamma_2, [Y, Y']) \geqslant \frac{1}{M_1}\left(P_1\left(\frac{1}{3}L - \frac{1}{2}L_1\right) - Q_1 - M_2E - A_2 - A_1\right).$$

Finally, a comparison with the first term of (5.3) shows that

$$d_{\mathcal{T}}(\gamma_2, [Y, Y']) > C_1,$$

as required. □

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This implies that W_2 lies on $[X, Y']$ and not on $[Y, Y']$. As γ_1 is further away from $[Y, Y']$ along γ than γ_2, the same argument implies that W_1 lies on $[X, Y']$. Furthermore, $d_{\mathcal{T}}(W_1, Z_1) \leqslant C_1 + L_1/2$. Similarly, $d_{\mathcal{T}}(W_2, Z_2) \leqslant C_1 + L_1/2$.

The segment $[X, Z_2]$ is in the ϵ_1-thick part. The endpoints of $[X, W_2]$ are within a distance of $C_1 + L_1/2$ of the endpoints of $[X, Z_2]$. So, by the fellow travelling result, that is, Theorem 2.3, $[X, W_2]$ and $[X, Z_2]$ are B_1-fellow travellers, where $B_1 = B(\epsilon_1, C_1 + L_1/2)$. Recall that B_1 depends on ϵ_1, $C_1 + L_1/2$ and the surface S.

Recall that $\epsilon_2 = \epsilon'(\epsilon_1, B_1)$, that is, the B_1-neighbourhood of K_{ϵ_1} is contained in K_{ϵ_2}. Note that ϵ_2 depends only on the constants $\epsilon = \epsilon_1$, B_1 and the surface S. In particular, the geodesic $[X, W_2]$ is contained in the ϵ_2-thick part. Given ϵ_2, recall that L_2 and C_2 are the corresponding constants from the thin triangle result, Theorem 5.1.

By the triangle inequality,

$$d_{\mathcal{T}}(Z_1, W_1) + d_{\mathcal{T}}(W_1, W_2) + d_{\mathcal{T}}(W_2, Z_2) \geqslant d_{\mathcal{T}}(Z_1, Z_2).$$

Thus, the Teichmüller distance between W_1 and W_2 is at least

$$d_{\mathcal{T}}(W_1, W_2) \geqslant \tfrac{1}{3}L - 2C_1 - L_1.$$

The second term of (5.3) implies that the right-hand side above is at least L_2. So we may apply the thin triangles result, Theorem 5.1, to X, X' and Y' to conclude that there is a point Z on $[X, X'] \cup [X', Y']$ within a distance C_2 of $[W_1, W_2]$.

We now show a lower bound for the distance between $[W_1, W_2]$ and $[X, X']$.

CLAIM 5.5. *The distance between* $[W_1, W_2]$ *and* $[X, X']$ *is at least* C_2.

Proof. Let W be a point of $[W_1, W_2]$ that is closest to X. Let V be the point of γ that is closest to W. Then

$$B_1 \geqslant d_{\mathcal{T}}(W, V) \quad \text{and} \quad d_{\mathcal{T}}(X, V) \geqslant \tfrac{1}{3}L - \tfrac{1}{2}L_1.$$

Thus, by the triangle inequality,

$$d_{\mathcal{T}}(X, W) \geqslant d_{\mathcal{T}}(X, V) - d_{\mathcal{T}}(W, V) > \tfrac{1}{3}L - \tfrac{1}{2}L_1 - B_1,$$

or, equivalently,

$$d_{\mathcal{T}}([W_1, W_2], X) \geqslant \tfrac{1}{3}L - \tfrac{1}{2}L_1 - B_1.$$

As ϵ_2-thick geodesics make definite progress in $\mathcal{C}(S)$, (5.2) implies that

$$d_{\mathcal{C}}(\pi([W_1, W_2]), x) \geqslant P_2(\tfrac{1}{3}L - \tfrac{1}{2}L_1 - B_1) - Q_2.$$

As the distance between x and x' in $\mathcal{C}(S)$ is at most E, this implies that

$$d_{\mathcal{C}}(\pi([W_1, W_2]), \pi([X, X'])) \geqslant P_2(\tfrac{1}{3}L - \tfrac{3}{2}L_1 - C_1) - Q_2 - M_2E - A_2.$$

As the curve complex distance is a coarse lower bound on the Teichmüller metric (2.1), this implies that

$$d_{\mathcal{T}}([W_1, W_2], [X, X']) \geqslant \frac{1}{M_1}\left(P_2\left(\frac{1}{3}L - \frac{1}{2}L_1 - B_1\right) - Q_2 - M_2E - A_2 - A_1\right).$$

A comparison with the third term in (5.3) then shows that

$$d_{\mathcal{T}}([W_1, W_2], [Y, Y']) > C_2,$$

as required. □

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Claim 5.5 implies that Z lies on $[X', Y']$ and not on $[X, X']$. The segments $[W_1, W_2]$ and $[Z_1, Z_2]$ are B_1-fellow travellers. As Z lies within distance C_2 of $[W_1, W_2]$, the distance of Z from γ is at most $C_2 + B_1$. To conclude the proof of Lemma 2.2, we may set $F = C_2 + B_1$, which depends only on ϵ, A and the surface S, as required. □

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