$A \in \mathcal{P}_{n-1}$ with the same $A \in \mathcal{P}_n$ and each $A \in \mathcal{P}_{n-2}$ with $A \cup \{n\} \in \mathcal{P}_n$, we see that $|\mathcal{P}_n| = |\mathcal{P}_{n-1}| + |\mathcal{P}_{n-2}|$, so $|\mathcal{P}_n| = F_{n+2}$, the $(n+2)$th Fibonacci number. Now suppose that $n \geqslant 4$. This time, pair each $B \in \mathcal{P}_{n-1}$ with the same $B \in \mathcal{C}_n$ and each $B \in \mathcal{P}_{n-3}$ with $\{b+1 : b \in B\} \cup \{n\} \in \mathcal{C}_n$. It is easy to see that this gives a one-to-one correspondence between $\mathcal{C}_n$ and the disjoint union of $\mathcal{P}_{n-1}$ and $\mathcal{P}_{n-3}$. Therefore, $|\mathcal{C}_n| = |\mathcal{P}_{n-1}| + |\mathcal{P}_{n-3}|$, so $|\mathcal{C}_n| = F_{n+1} + F_{n-1}$. This way of counting $|\mathcal{C}_n|$ has been given in [1] and [2], the authors of which then use induction or Binet's formulae to prove that $F_{n+1} + F_{n-1} = L_n$ and conclude that $|\mathcal{C}_n| = L_n$ for all $n \geqslant 2$.

Our second way of counting $|\mathcal{C}_n|$ is more direct, but trickier to discover. Suppose that $n \geqslant 4$. Now pair each $B \in \mathcal{C}_{n-1}$ with the same $B \in \mathcal{C}_n$, pair each $B \in \mathcal{C}_{n-2}$ containing 1 with $B \cup \{n-1\} \in \mathcal{C}_n$, and pair each $B \in \mathcal{C}_{n-2}$ not containing 1 with $B \cup \{n\} \in \mathcal{C}_n$. This apparent one-to-one correspondence between $\mathcal{C}_n$ and the disjoint union of $\mathcal{C}_{n-1}$ and $\mathcal{C}_{n-2}$ implies that $|\mathcal{C}_n| = |\mathcal{C}_{n-1}| + |\mathcal{C}_{n-2}|$, so $|\mathcal{C}_n| = L_n$.

As a consequence of counting $|\mathcal{C}_n|$ in two different ways, we have $F_{n+1} + F_{n-1} = L_n$ for all $n \geqslant 2$, without any inductive or algebraic manipulations.

References

1. R. Honsberger, *Mathematical Gems III*, MAA (1985) pp. 120-122.
2. T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, Inc., (2001) pp. 28-29.

10.1017/mag.2025.10150 © The Authors, 2025
Published by Cambridge University Press on behalf of The Mathematical Association

LI ZHOU
Polk State College, 999 Ave. H NE, Winter Haven, FL 33881, USA
e-mail: *lzhou@polk.edu*

109.50 Functions of the square matrices with distinct non-zero eigenvalues

Introduction and main result

Finding a general formula for the nth power of a square matrix $\mathbf{A}$, and then $f(\mathbf{A})$ for any function f admitting a suitable expansion, is a classical problem with several solutions. We refer the interested reader to [1, Chap. 10], [2, Chap. 5], [3, Chap. 11], [4, Sec. 7.3, 7.9], and [5]. Also, see [6, 7, 8, 9, 10, 11] for some special matrix function computations and applications.

This paper was inspired by my attempt to obtain powers of a square matrix of order 2 in an elementary Linear Algebra course at the University of Zanjan for engineering students, where I developed a general elementary method based on the Cayley-Hamilton theorem for the case that the eigenvalues of the matrix are distinct and nonzero, hence the Vandermonde determinant of the eigenvalues is non-zero, and I could apply Cramer's rule to solve the relevant system of equations. Although, the following result coincides with the general theory of finding $\mathbf{A}^n$ in the spectral theory of matrices, the proof is elementary and only needs the above mentioned materials.

https://doi.org/10.1017/mag.2025.10151 Published online by Cambridge University Press

Theorem: Let $\mathbf{A}$ be a square matrix of order m with distinct non-zero eigenvalues $\lambda_1, \ldots \lambda_m$, and

$$\mathbf{V} = \mathbf{V}(\lambda_1, \ldots \lambda_m) = \begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{m-1} \\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{m-1} \\ 1 & \lambda_3 & \lambda_3^2 & \ldots & \lambda_3^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \lambda_m & \lambda_m^2 & \ldots & \lambda_m^{m-1} \end{bmatrix}_{m \times m}$$

be the square Vandermonde matrix of $\lambda_1, \ldots \lambda_m$. Also, for $i = 1, \ldots, m$ we define the square matrices $\mathbf{A}_i$ of order m by

$$\mathbf{A}_i = \sum_{j=1}^{m} (-1)^{i+j} M_{i,j} \mathbf{A}^{j-1}, \tag{1}$$

with $\mathbf{A}^0 = \mathbf{I}_m$ and $M_{i,j}$ denoting the (i, j)-minor of the matrix $\mathbf{V}$, i.e., the determinant of the submatrix formed by deleting the i-th row and j-th column of the Vandermonde matrix $\mathbf{V}$. Then, for any integer $n \geqslant 1$ we have

$$\mathbf{A}^n = \frac{1}{\det(\mathbf{V})} \sum_{i=1}^{m} \lambda_i^n \mathbf{A}_i. \tag{2}$$

Proof: Let $P(\lambda)$ be the characteristic polynomial of $\mathbf{A}$. Let

$$c_0 + c_1\lambda + \ldots + c_{m-1}\lambda^{m-1}$$

be the remainder polynomial when λ^n is divided by $P(\lambda)$ and define column vectors

$$\mathbf{c} = \begin{bmatrix} c_0 \\ \vdots \\ c_{m-1} \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} \lambda_1^n \\ \vdots \\ \lambda_m^n \end{bmatrix}.$$

Then $\mathbf{Vc} = \mathbf{v}$ and, by Cramer's Rule, $\det(\mathbf{V})\, c_{j-1} = \det(\mathbf{V}_j)$ where $\mathbf{V}_j$ is the matrix formed by replacing the j-th column of $\mathbf{V}$ by $\mathbf{v}$. Expanding $\det(\mathbf{V}_j)$ by its j-th column, gives

$$\det(\mathbf{V})\, c_{j-1} = \sum_{i=1}^{m} (-1)^{i+j} M_{i,j} \lambda_i^n.$$

By the Cayley-Hamilton Theorem

$$\begin{aligned} \mathbf{A}^n &= \sum_{j=1}^{m} c_{j-1} \mathbf{A}^{j-1} \\ &= \frac{1}{\det(\mathbf{V})} \sum_{j=1}^{m} \left(\sum_{i=1}^{m} (-1)^{i+j} M_{i,j} \lambda_i^n \right) \mathbf{A}^{j-1} \\ &= \frac{1}{\det(\mathbf{V})} \sum_{i=1}^{m} \lambda_i^n \left(\sum_{j=1}^{m} (-1)^{i+j} M_{i,j} \mathbf{A}^{j-1} \right) \\ &= \frac{1}{\det(\mathbf{V})} \sum_{i=1}^{m} \lambda_i^n \mathbf{A}_i. \end{aligned}$$

This completes the proof.

https://doi.org/10.1017/mag.2025.10151 Published online by Cambridge University Press

Corollary: If the function $f(z)$ is analytic on the disc $|z| < R$ with radius R larger than $\max\{|\lambda_i|,\ i = 1, \ldots, m\}$, then

$$f(\mathbf{A}) = \frac{1}{\det(\mathbf{V})} \sum_{i=1}^{m} f(\lambda_i)\,\mathbf{A}_i. \tag{3}$$

Proof: We mention that considering the expansion $f(z) = \sum_{n \geqslant 0} a_n z^n$, the matrix $f(\mathbf{A})$ is defined as $f(\mathbf{A}) = \sum_{n \geqslant 0} a_n \mathbf{A}^n$. Thus, by using (2) we obtain

$$f(\mathbf{A}) = \sum_{n \geqslant 0} a_n \left(\frac{1}{\det(\mathbf{V})} \sum_{i=1}^{m} \lambda_i^n \mathbf{A}_i \right) = \frac{1}{\det(\mathbf{V})} \sum_{i=1}^{m} \left(\sum_{n \geqslant 0} a_n \lambda_i^n \right) \mathbf{A}_i.$$

All of λ_i are assumed to be inside the disk of convergence of $f(z)$. This implies that $\sum_{n \geqslant 0} a_n \lambda_i^n = f(\lambda_i)$, and hence (3).

An example

As a simple example, shedding light on the above method and its machinery, we consider the following square matrix

$$\mathbf{A} = \begin{bmatrix} 29 & -42 & 42 \\ 20 & -29 & 30 \\ 4 & -6 & 6 \end{bmatrix}.$$

The characteristic polynomial of $\mathbf{A}$ is $P(\lambda) = \lambda^3 - 6\lambda^2 + 11\lambda - 6$, and its roots are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$. We have

$$\mathbf{V} = \mathbf{V}(1, 2, 3) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}.$$

Since $\operatorname{adj}(\mathbf{V}) = \left[(-1)^{i+j} M_{i,j}\right]^T$, we deduce that

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_3 & \mathbf{A} & \mathbf{A}^2 \end{bmatrix} \operatorname{adj}(\mathbf{V}).$$

Also, we have $\operatorname{adj}(\mathbf{V}) = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$.

Thus, $\mathbf{A}_1 = 6\mathbf{I}_3 - 5\mathbf{A} + \mathbf{A}^2$, $\mathbf{A}_2 = -6\mathbf{I}_3 + 8\mathbf{A} - 2\mathbf{A}^2$, $\mathrm{A}_3 = 2\mathbf{I}_3 - 3\mathbf{A} + \mathbf{A}^2$, and so

$$\mathbf{A}_1 = \begin{bmatrix} 30 & -42 & 0 \\ 20 & -28 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} -112 & 168 & -84 \\ -80 & 120 & -60 \\ -8 & 12 & -6 \end{bmatrix}, \mathbf{A}_3 = \begin{bmatrix} 84 & -126 & 84 \\ 60 & -90 & 60 \\ 8 & -12 & 8 \end{bmatrix}.$$

Note that $\det(\mathbf{V}) = 2$. By using (3) we deduce that if the function $f(z)$ is analytic on the disc $|z| < 3 + \varepsilon$ for some $\varepsilon > 0$, then

$$f(\mathbf{A}) = f(1)\begin{bmatrix} 15 & -21 & 0 \\ 10 & -14 & 0 \\ 0 & 0 & 0 \end{bmatrix} + f(2)\begin{bmatrix} -56 & 84 & -42 \\ -40 & 60 & -30 \\ -4 & 6 & -3 \end{bmatrix} + f(3)\begin{bmatrix} 42 & -63 & 42 \\ 30 & -45 & 30 \\ 4 & -6 & 4 \end{bmatrix}.$$

https://doi.org/10.1017/mag.2025.10151 Published online by Cambridge University Press

More precisely, taking $f(z) = z^n$ we obtain

$$\mathbf{A}^n = \begin{bmatrix} 15 - 56(2^n) + 42(3^n) & -21 + 84(2^n) - 63(3^n) & -42(2^n) + 42(3^n) \\ 10 - 40(2^n) + 30(3^n) & -14 + 60(2^n) - 45(3^n) & -30(2^n) + 30(3^n) \\ -4(2^n) + 4(3^n) & 6(2^n) - 6(3^n) & -3(2^n) + 4(3^n) \end{bmatrix}.$$

Further remarks

Remark 1: Let $\hat{\mathbf{A}}$ to be the row vector consisting of the matrices $\mathbf{A}^0, \ldots, \mathbf{A}^{m-1}$. Then (1) asserts that $\hat{\mathbf{A}}\mathbf{V}^{-1}$ is the row vector consisting of the matrices $\mathbf{A}_1, \ldots, \mathbf{A}_m$. Hence, if $\Lambda_f = \left(f(\lambda_1), \ldots, f(\lambda_m)\right)$, then we may rewrite (3) as the following dot product form

$$f(\mathbf{A}) = \left(\hat{\mathbf{A}}\mathbf{V}^{-1}\right) \cdot \Lambda_f.$$

In order to compute $\mathbf{V}^{-1}$, we let

$$L_j(x) = \prod_{\substack{0 \leqslant i \leqslant m-1 \\ i \neq j}} \frac{x - \lambda_{i+1}}{\lambda_{j+1} - \lambda_{i+1}} = \sum_{i=0}^{m-1} v_{i,j} x^i$$

be the Lagrange interpolation polynomials in $\lambda_1, \ldots, \lambda_m$. Thus, Lagrange interpolation (for example see [12, Chap. 4]) implies $L_{j+1}(\lambda_{i+1}) = 0$ for $i \neq j$ and $L_{j+1}(\lambda_{j+1}) = 1$. Hence, if $\mathbf{B} = [v_{i,j}]_{0 \leqslant i,j \leqslant m-1}$, then

$$\mathbf{VB} = \left[L_{j+1}(\lambda_{i+1})\right]_{0 \leqslant i,j \leqslant m-1} = \mathbf{I}_m.$$

Therefore, $\mathbf{V}^{-1} = \mathbf{B}$, and this asserts that the columns of $\mathbf{V}^{-1}$ are the coefficients of the Lagrange polynomials $L_j(x)$.

Remark 2: In the above example, we note that $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = 2\mathbf{I}_3$. This is not an accidental event. In the spectral theory of matrices (for example see [4, p. 603]), the matrices $\mathbf{A}_i$ (up to a normalizing coefficient, which here is $\frac{1}{\det(\mathbf{V})}$), are known as the spectral projectors, having several properties including

$$\mathbf{A}_1 + \ldots + \mathbf{A}_m = \det(\mathbf{V})\,\mathbf{I}_m, \tag{4}$$

and

$$\mathbf{A}_i \cdot \mathbf{A}_j = \mathbf{0}_{m \times m} \qquad (i \neq j), \tag{5}$$

where $\mathbf{0}_{m \times m}$ is the square zero matrix of order m. As an elementary counterpart to spectral theory of matrices, we prove these properties. As we showed in the above remark, the columns of $\mathbf{V}^{-1}$ are the coefficients of the Lagrange polynomials $L_j(x)$. Since the sum of the coefficients of the Lagrange interpolation polynomials is equal to 1, we get (4). Also, one may conclude (5) by using the fact that the inner product of one row (or column) in any given square matrix with the cofactors of a different row (or column) always is zero [4, p. 481].

Remark 3: Another observation is that by substituting (1) in (3) we deduce that $f(\mathbf{A})$ eventually is a linear combination of the matrices $\mathrm{A}^0, \ldots, \mathbf{A}^{m-1}$.

https://doi.org/10.1017/mag.2025.10151 Published online by Cambridge University Press

This property also holds in general, asserting that for a function $f(z)$ defined at $\mathbf{A}$ there exists a polynomial $p(z)$ such that $f(\mathbf{A}) = p(\mathbf{A})$ (see [4, p. 606]).

Acknowledgments

The author is greatly indebted to the referee for a thorough reading of the manuscript and very helpful comments improving the presentation of this Note.

References

1. D. S. Bernstein, *Matrix mathematics (theory, facts, and formulas)*, (2nd edn.), Princeton University Press, Princeton, NJ, 2009.
2. F. R. Gantmacher, *The theory of matrices*, Volume 1, Amer. Math. Soc., Chelsea Publication, Providence, Rhode Island, 2022.
3. G. H. Golub and C. F. Van Loan, *Matrix computations*, (4th edn.), Johns Hopkins Studies in the Mathematical Sciences, Johns Hopkins University Press, Baltimore, MD, 2013.
4. C. D. Meyer, *Matrix analysis and applied linear algebra*, (2nd edn.), Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2023.
5. I. L. Kantor and I. M. Trishin, Formulas for powers and functions of matrices, *Linear Algebra Appl.*, **186** (1993), pp. 1-13.
6. H. Belbachir and F. Bencherif, Linear recurrent sequences and powers of a square matrix, *Integers*, **6** (2006), Paper A12.
7. N. N. Chanta and P. Rochanakul, A combinatorial formula for powers of 3×3 matrices and some combinatorial identities, *Thai J. Math.*, **18** (2020) pp. 372-383.
8. L. Jódar and J. C. Cortés, Some properties of gamma and beta matrix functions, *Appl. Math. Lett.*, 11 (1998) pp. 89-93.
9. J. Kirillov, Power of a 3×3 matrix, subgroups and tiling, *Integers*, **22** (2022), Paper A39.
10. J. McLaughlin and B. Sury, Powers of a matrix and combinatorial identities, *Integers*, **5** (2005), Paper A13.
11. K. Williams, The nth power of a 2×2 matrix, *Math. Mag.*, *65* (1992) p. 336.
12. D. E. Stewart, *Numerical analysis: a graduate course*, CMS/CAIMS Books in Mathematics, 4, Springer, Cham, 2022.

10.1017/mag.2025.10151 © The Authors, 2025
Published by Cambridge University Press
on behalf of The Mathematical Association

MEHDI HASSANI
Department of Mathematics
and Computer Science
University of Zanjan
University Blvd., 45371-38791,
Zanjan, Iran
e-mail: *mehdi.hassani@znu.ac.ir*

https://doi.org/10.1017/mag.2025.10151 Published online by Cambridge University Press