Proc. Camb. Phil. Soc. (1974), **75**, 77
PCPS 75–1
Printed in Great Britain

Simultaneous approximation of two additive forms

By MING-CHIT LIU
Department of Mathematics, University of Hong Kong, Hong Kong, B.C.C.

(*Received* 5 *April* 1973)

1. *Introduction.* In 1948, H. Heibronn(6) improved Vinogradov's result (11) and obtained that for any $\epsilon > 0$ there exists some positive constant $C(\epsilon)$ which depends on ϵ only such that, for any real number θ and any integer $N \geqslant 1$, there is an integer x satisfying

$$1 \leqslant x \leqslant N, \quad \|\theta x^2\| < C(\epsilon)\, N^{-(1/2)+\epsilon}. \tag{1·1}$$

Here $\|t\|$ means the distance from t to the nearest integer.

Recently, (1·1) has been improved and extended in (1–4, 9, 10). Among these results, in particular we have:

THEOREM L (9). *For any integer $k \geqslant 2$ put $K = 2^{k-1}$. Let δ be any arbitrary positive number. For any $\epsilon > 0$ there are some positive constants $C(\delta, \epsilon)$ and $C(k, \epsilon)$ such that for any two real numbers θ, ϕ and integer $N \geqslant 1$, there exists an integer x satisfying*

$$\left.\begin{aligned} &1 \leqslant x \leqslant N, \\ &\max(\|\theta x^k\|, \|\phi x^k\|) < \begin{cases} C(\delta, \epsilon)\, N^{-(1/(7+\delta))+\epsilon} & \text{if} \quad k = 2; \\ C(k, \epsilon)\, N^{-(1/(3K+1))+\epsilon} & \text{if} \quad k \geqslant 3. \end{cases} \end{aligned}\right\} \tag{1·2}$$

In this paper, applying a method akin to that in (9) we shall extend Theorem L to two additive forms of degree k in s variables with some improvement. We have

THEOREM. *For any integers $k \geqslant 2$ and $s \geqslant 1$, put $K = 2^{k-1}$ and $X = (x_1, x_2, \ldots, x_s)$, a vector of s real components. Let $f(X)$, $g(X)$ be any two additive forms of degree k in s variables,*

$$f(X) = \sum_{i=1}^{s} \theta_i x_i^k, \quad g(X) = \sum_{i=1}^{s} \phi_i x_i^k,$$

where $\theta_1, \ldots, \theta_s$; $\phi_1, \ldots, \phi_s$ are arbitrary real numbers. Then for any $\epsilon > 0$ and any integer $N > 1$ there are integers $x_1, \ldots, x_s$ not all zero and some positive constant $C(\epsilon, k, s)$ which depends on ϵ, k, s only such that

$$\left.\begin{aligned} &1 \leqslant \max_{1 \leqslant i \leqslant s} |x_i| \leqslant N, \\ &\max(\|f(X)\|, \|g(X)\|) < C(\epsilon, k, s)\, N^{-(1/F(k,s))+\epsilon}, \end{aligned}\right\} \tag{1·3}$$

where

$$F(k, s) = \begin{cases} 7 & \text{if} \quad s = 1, k = 2, \\ 3K + (1/k) & \text{if} \quad s = 1, k \geqslant 3, \\ 2K + 1 + (K/ks) & \text{if} \quad s \geqslant 2. \end{cases} \tag{1·4}$$

We see that for the special case $s = 1$

$$F(k, 1) = \begin{cases} 7 & \text{if} \quad k = 2, \\ 3K + (1/k) & \text{if} \quad k \geqslant 3. \end{cases}$$

https://doi.org/10.1017/S030500410004826X Published online by Cambridge University Press

So our Theorem is a generalization of Theorem L with some improvement. On the other hand, (1·3) is an analogy to Cook's result (1). He obtained that for one additive form

$$\|f(X)\| < C(\epsilon, k)\, N^{-(s/K)+\epsilon}$$

if $1 \leqslant s \leqslant K$.

The author wishes to thank the referee for his very helpful suggestions which brought improvement in the presentation of this paper.

2. *Preliminary Lemmas.* In what follows we always assume that ϵ is a small enough positive quantity which is not the same ϵ given and N is a sufficiently large positive integer, say $N > N_0(\epsilon, k, s)$, such that all the subsequent asymptotic approximations and inequalities in section 3 are satisfied. So it is difficult to define ϵ and N_0 at the beginning or at any particular point. If $y > 0$ we use $x \ll y$ to denote $|x| < Cy$, where C is some positive constant which can depend on ϵ, k, s. $[t]$ is the integral part of t. For real t we write $e(t) = \exp(2\pi t i)$.

We need several lemmas.

LEMMA 1. *Suppose Δ is a number satisfying $0 < \Delta < \frac{1}{2}$ and r is a positive integer. Then there exists a real valued function $\psi(x)$, periodic with period 1, which satisfies*

$$\left.\begin{aligned} &\psi(x) = 0 \quad \text{if} \quad \|x\| > \Delta, \\ &\psi(x) = \sum_{u=-\infty}^{\infty} c_u\, e(ux), \end{aligned}\right\} \tag{2·1}$$

where c_u are real and

$$\left.\begin{aligned} &c_0 = \Delta, \\ &|c_u| < C(r) \min(\Delta, \Delta^{-r}|u|^{-r-1}) \quad \text{if} \quad u \neq 0, \end{aligned}\right\} \tag{2·2}$$

where $C(r)$ is some positive constant depending on r only.

Proof. This is a particular case of Lemma 12 in (12, p. 32) with $\beta = -\alpha = \frac{1}{2}\Delta$.

LEMMA 2. *Let θ and ϕ be any real numbers and N any positive integer. Then there exists an integer n satisfying*

$$\left.\begin{aligned} &1 \leqslant n \leqslant N, \\ &\|\theta n\| < N^{-1/2} \quad \|\phi n\| < N^{-1/2}. \end{aligned}\right\} \tag{2·3}$$

Proof. See Theorem 200 in ((5), p. 170).

LEMMA 3. *Let $T = \sum_{n=1}^{N} e(tn^k)$, where $k = 2, 3, \ldots$ and t is any real number. For any $\epsilon > 0$ we have*

$$|T|^K \ll N^{K-1} + N^{K-k+\epsilon} \sum_{j=1}^{k!N^{k-1}} \min(N, 1/\|jt\|), \tag{2·4}$$

where $K = 2^{k-1}$.

Proof. See Satz 266 in ((8), p. 255).

LEMMA 4. *For any real t if there are integers a, q with $q > 0$ such that $(a, q) = 1$ and $|t - a/q| \leqslant q^{-2}$, then for any positive integers P and N,*

$$\sum_{j=p+1}^{p+q} \min(N, 1/\|jt\|) \ll N + q \log q. \tag{2·5}$$

https://doi.org/10.1017/S030500410004826X Published online by Cambridge University Press

Proof. Lemma 4 is well known. See, for example, Lemma 3·5 in (7).

LEMMA 5. *Let* $K = 2^{k-1}$ $(k = 2, 3, \ldots)$. *For any* $\epsilon > 0$ *there exists some positive constant* $C(k, \epsilon)$ *depending only on* k *and* ϵ, *such that for any real number* θ *and integer* $N \geqslant 1$ *there exists an integer* n *satisfying*

$$1 \leqslant n \leqslant N, \quad \|\theta n^k\| < C(k, \epsilon) N^{-(1/K)+\epsilon}. \tag{2·6}$$

Proof. Lemma 5 is due to H. Davenport ((4), p. 344).

3. *Proof of the theorem.* Suppose that for some positive λ there is no integral solution $X = (x_1, \ldots, x_s)$ of the following inequalities

$$\left.\begin{aligned} 1 \leqslant \max_{1 \leqslant i \leqslant s} |x_i| \leqslant N, \\ \|f(X)\| \leqslant N^{-\lambda}, \quad \|g(X)\| \leqslant N^{-\lambda}. \end{aligned}\right\} \tag{3·1}$$

I.e. for each X with integers $x_1, \ldots, x_s$ satisfying $1 \leqslant \max_{1 \leqslant i \leqslant s} |x_i| \leqslant N$ we have either

$$\|f(X)\| > N^{-\lambda} \quad \text{or} \quad \|g(X)\| > N^{-\lambda}.$$

Putting $\Delta = N^{-\lambda}$ in Lemma 1 we have

$$\sum_X \psi(f(X)\, \psi(g(X))) = 0,$$

where the summation is taken over all x_i with $1 \leqslant x_i \leqslant N$. Then from $(2·1)_2$, $(2·1)_1$

$$N^{s-2\lambda} + \sum_{(u,v) \neq (0,0)} c_u c_v \prod_{i=1}^{s} S_{uv}(i) = 0, \tag{3·2}$$

where

$$S_{uv}(i) = \sum_{u=1}^{N} e((u\theta_i + v\phi_i)\, x^k). \tag{3·3}$$

On the other hand, from $(2·2)_2$, (3·3)

$$\sum_{|u|,\, |v| \geqslant N^{\lambda+\epsilon}} |c_u c_v| \prod_{i=1}^{s} |S_{uv}(i)| \ll N^{s-2\epsilon r},$$

and

$$\Sigma |c_u c_v| \prod_{i=1}^{s} |S_{uv}(i)| \ll N^{s-\lambda-\epsilon r},$$

where the summation is taken over $|u| < N^{\lambda+\epsilon}$ and $|v| \geqslant N^{\lambda+\epsilon}$. If we put $r = [\lambda/\epsilon] + 1$, then from $(2·2)_2$, (3·2)

$$\begin{aligned} N^s &\ll \Sigma^* \prod_{i=1}^{s} |S_{uv}(i)| \\ &\ll \sum_{i=1}^{s} \Sigma^* |S_{uv}(i)|^s, \end{aligned}$$

where the above summations Σ^* (and also the following ones) are taken over $|u|$, $|v| < N^{\lambda+\delta}$ and $(u, v) \neq (0, 0)$. Thus there is some i for which we write $\theta = \theta_i$, $\phi = \phi_i$ and $S_{uv} = S_{uv}(i)$, such that

$$\Sigma^* |S_{ur}|^s \gg N^s \quad (\text{or } \geqslant CN^s, \text{ say}). \tag{3·4}$$

https://doi.org/10.1017/S030500410004826X Published online by Cambridge University Press

So there exists a ρ,
$$0 \leqslant \rho \leqslant \lambda+\epsilon \tag{3·5}$$
such that
$$(\tfrac{1}{8}C)^{1/s} N^{1-2p} \leqslant |S_{uv}|, \quad p = \rho/s \tag{3·6}$$
for at least $[N^{2\rho-\epsilon}]+1$ pairs (u, v) in
$$|u|, |v| < N^{\lambda+\epsilon}. \tag{3·7}$$

Suppose that such a ρ satisfying (3·5), (3·6), (3·7) does not exist. For integer m with $\epsilon m > 4(\lambda+\epsilon)$ write
$$\Sigma^*|S_{uv}|^s = \Sigma|S_{uv}|^s + \sum_{j=0}^{m-1} T_j,$$
where the first summation in the right of the above equality is taken over $|u|, |v| < N^{\lambda+\epsilon}$, $(u, v) \neq (0, 0)$, and $|S_{uv}|^s < (\frac{1}{8}C) N^{s-2(\lambda+\epsilon)}$, while $T_j = \Sigma|S_{uv}|^s$ which are taken over $|u|, |v| < N^{\lambda+\epsilon}$, $(u, v) \neq (0, 0)$, and
$$(\tfrac{1}{8}C) N^{s-(2j+1)(\lambda+\epsilon)/m} \leqslant |S_{uv}|^s < (\tfrac{1}{8}C) N^{s-(2j(\lambda+\epsilon)/m)}.$$
Then it follows that $\Sigma^*|S_{uv}|^s < CN^s$ for every large N. This contradicts (3.4).

In what follows, we shall confine our attention to pairs (u, v) satisfying (3·5), (3·6), (3·7). From (2·4), (3·3) we have
$$|S_{uv}|^K \ll N^{K-1} + N^{K-k+\epsilon} \sum_{j=1}^{k!N^{k-1}} \min(N, 1/\|jt\|), \tag{3·8}$$
where $t = u\theta + v\phi$. Define
$$Q = N^{k-1+\alpha K\lambda - Kp}, \tag{3·9}$$
where $p = \rho/s$ and
$$\alpha = \begin{cases} \frac{5}{2} & \text{if } s = 1, k = 2, \\ 2+\dfrac{1}{kK} & \text{if } s = 1, k \geqslant 3, \\ 2+\dfrac{1}{K}-\dfrac{k-1}{ks} & \text{if } s \geqslant 2. \end{cases} \tag{3·10}$$

By Dirichlet's theorem, there are integers a, q such that
$$(a, q) = 1, \quad 1 \leqslant q \leqslant Q, \quad |qt-a| < Q^{-1}. \tag{3·11}$$
Fix such a q and divide the sum in (3·8) into blocks of q terms. It follows from Lemma 4 and (3·8) that
$$|S_{uv}|^K \ll N^{K-1} + N^{K-k+\epsilon}(N^{k-1}q^{-1}+1)(N+q\log q).$$
Then by (3·6)
$$N^{1-2Kp-\epsilon} \ll Nq^{-1} + N^{\epsilon} + qN^{1-k+\epsilon}. \tag{3·12}$$
Suppose that λ satisfies
$$\begin{cases} 7\lambda < 1-A\epsilon & \text{if } s = 1, k = 2, \\ \lambda(3K+(1/k)) < 1-A\epsilon & \text{if } s = 1, k \geqslant 3, \\ \lambda(2K+1+(K/ks)) < 1-A\epsilon & \text{if } s \geqslant 2, \end{cases} \tag{3·13}$$
where A is some positive constant which depends on k and s only. The value of A is

https://doi.org/10.1017/S030500410004826X Published online by Cambridge University Press

so defined that all following inequalities (3·15), (3·16), (3·20), (3·21) will be satisfied. In view of (3·9), $(3\cdot11)_2$ and (3·13), (3·12) gives

$$q \ll N^{2Kp+\epsilon}, \tag{3·14}$$

since by (3·5), (3·10), (3·13)

$$\alpha K\lambda + pK + 2\epsilon < 1 \quad \text{and} \quad 2Kp + 2\epsilon < 1. \tag{3·15}$$

By Lemma 2, for given θ, ϕ and integer $[QN^{-\epsilon}]$ there are integers y, z, w such that

$$1 \leqslant w \leqslant [QN^{-\epsilon}], \quad |w\theta - y| \leqslant [QN^{-\epsilon}]^{-\frac{1}{2}}, \quad |w\phi - z| \leqslant [QN^{-\epsilon}]^{-\frac{1}{2}}.$$

But now by (3·7), $(3\cdot11)_3$, (3·14)

$$\begin{aligned} |q(uy+vz)-aw| &\leqslant q|u|\,|w\theta-y| + q|v|\,|w\phi-z| + w|q(u\theta+v\phi)-a| \\ &\ll N^{2Kp+\epsilon}N^{\lambda+\epsilon}[QN^{-\epsilon}]^{-\frac{1}{2}} + [QN^{-\epsilon}]Q^{-1} \\ &\ll N^{-\epsilon}, \end{aligned}$$

since by (3·5), (3·9), (3·10), (3·13)

$$2Kp + 2\epsilon + \lambda - \tfrac{1}{2}\{k - 1 + \alpha K\lambda - Kp - \epsilon\} < -\epsilon. \tag{3·16}$$

So for large N, $|q(uy+vz)-aw| < 1$. That is, $q(uy+vz) = aw$. Since $(a, q) = 1$, q divides w. Then the number of possibilities for q is $O(N^{\epsilon})$. So if $[N^{2\rho-\epsilon}]+1$ (> 1) pairs (u, v) in (3·7) satisfy (3·6) then $\gg N^{2(\rho-\epsilon)}$ of them have the same q. Divide the square $|u|, |v| < N^{\lambda+\epsilon}$ into subsquares by straight lines

$$|u|, |v| = bjN^{\lambda+\epsilon-(\rho-\epsilon)}, \tag{3·17}$$

where $j = 0, 1, 2, \ldots$ and b is some positive constant which is suitably chosen such that the following Dirichlet's pigeonhole argument holds. Totally there are $\ll N^{2(\rho-\epsilon)}$ subsquares in the square $|u|, |v| < N^{\lambda+\epsilon}$. Hence there is at least one subsquare containing two pairs (u_1, v_1), (u_2, v_2), say, such that these two pairs have the same q. Suppose that $u_1 \neq u_2$, otherwise we consider v_1 and v_2. Let $u_1 > u_2$ and put

$$u = u_1 - u_2 \,(\geqslant 1), \quad v = v_1 - v_2.$$

By $(3\cdot11)_3$, (3·17) we have

$$\left.\begin{aligned} |u|, |v| &\ll N^{\lambda+2\epsilon-\rho}, \\ \|(u\theta+v\phi)\,q\| &= \|(u_1\theta+v_1\phi)\,q - (u_2\theta+v_2\phi)\,q\| < 2Q^{-1}. \end{aligned}\right\} \tag{3·18}$$

In what follows we shall confine our attention to the pair (u, v) satisfying (3·18). For the real number $\phi u^{k-1}q^k$ and integer $[N^{(2\lambda+4\epsilon-\rho)K}]$ by Lemma 5 there is an integer n such that

$$1 \leqslant n \leqslant N^{(2\lambda+4\epsilon-\rho)K}, \quad \|(\phi u^{k-1}q^k)\,n^k\| \ll N^{-(2\lambda+4\epsilon-\rho)+\epsilon}. \tag{3·19}$$

Taking $x = nqu$ by (3·14), (3·18), (3·19) we have

$$\begin{aligned} \|\phi x^k\| &\leqslant u\|(\phi u^{k-1}q^k)\,n^k\| \ll N^{(\lambda+2\epsilon-\rho)+(\rho-2\lambda-3\epsilon)} \ll N^{-\lambda-\epsilon}, \\ \|\theta x^k\| &\leqslant |v|\,\|(\phi u^{k-1}q^k)\,n^k\| + n^k u^{k-1}q^{k-1}\|q(u\theta+v\phi)\| \\ &\ll N^{-\lambda-\epsilon} + N^{\sigma_1} \ll N^{-\lambda-\epsilon}, \end{aligned}$$

https://doi.org/10.1017/S030500410004826X Published online by Cambridge University Press

since by (3·5), (3·10), (3·13)

$$\left.\begin{aligned}\sigma_1 \leqslant kK(2\lambda+4\epsilon-\rho)+(k-1)(\lambda+2\epsilon-\rho)\\ +(k+1)(2Kp+\epsilon)-\{(k-1)+\alpha K\lambda-Kp\}<-\lambda-\epsilon.\end{aligned}\right\} \tag{3·20}$$

Also by (3·5), (3·13)

$$1 \leqslant x = nqu \ll N^{\sigma_2} \ll N^{1-\epsilon},$$

since
$$\sigma_2 = K(2\lambda+4\epsilon-\rho)+(2Kp+\epsilon)+(\lambda+2\epsilon-\rho) < 1-\epsilon. \tag{3·21}$$

This contradicts (3·1). Thus by (3·13) with a suitable choice of ϵ used in section 3 the proof of the Theorem is complete.

REFERENCES

(1) COOK, R. J. The fractional parts of an additive form. *Proc. Cambridge Philos. Soc.* **72** (1972), 209–212.

(2) DANICIC, I. An extension of a theorem of Heilbronn. *Mathematika* **5** (1958), 30–37.

(3) DANICIC, I. The distribution (mod 1) of pairs of quadratic forms with integer variables. *J. London Math. Soc.* **42** (1967), 618–623.

(4) DAVENPORT, H. On a theorem of Heilbronn. *Quart. J. Math. Oxford, Ser.* 2, **18** (1967), 339–344.

(5) HARDY, G. H. and WRIGHT, E. M. *An Introduction to the Theory of Numbers* (Oxford, 4th ed., 1960).

(6) HEILBRONN, H. On the distribution of the sequence θn^2 (mod 1). *Quart. J. Math. Oxford* **19** (1948), 249–256.

(7) HUA, L. K. *Additive Theory of Prime Numbers.* Translations of Mathematical Monographs, vol. 13 (Amer. Math. Soc. Providence, R.I., 1965).

(8) LANDAU, E. *Vorlesungen über Zahlentheorie,* Bd. I (Leipzig, Hirzel, 1927).

(9) LIU, M. C. On the fractional parts of θn^k and ϕn^k. *Quart. J. Math. Oxford, Ser.* 2 ,**21** (1970), 481–486.

(10) LIU, M. C. On a theorem of Heilbronn concerning the fractional part of θn^2. *Canad. J. Math.* **22** (1970), 784–788.

(11) VINOGRADOV, I. M. Analytischer Beweis des Satzes über die Verteilung der Bruchteile eines ganzen Polynoms. *Bull. Acad. Sci. USSR* (6) **21** (1927), 567–578.

(12) VINOGRADOV, I. M. *The Method of Trigonometric Sums in the Theory of Numbers* (New York, Interscience, 1954).

https://doi.org/10.1017/S030500410004826X Published online by Cambridge University Press