Pitot's theorem, dynamic geometry and conics

A. F. BEARDON

1. *Introduction*

It is well known that a convex quadrilateral is a cyclic quadrilateral if, and only if, the sum of each pair of opposite angles is π. This result (which gives a necessary and sufficient condition for the existence of a circle which *circumscribes* a given quadrilateral) is beautifully complemented by Pitot's theorem which says that a given convex quadrilateral has an *inscribed* circle if, and only if, the sum of the lengths of one pair of opposite edges is the same as the sum for the other pair. Henri Pitot, a French engineer, noticed the easy part of this result in 1725 (see Figure 1), and the converse was first proved by J-B Durrande in 1815. Accordingly, we shall say that a convex quadrilateral is a *Pitot quadrilateral* if, and only if, the sum of the lengths of one pair of opposite edges is the same as the sum for the other pair.

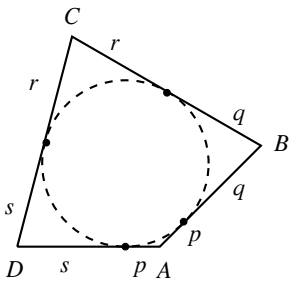


FIGURE 1: An inscribed circle

We are familiar with statements in number theory that can easily be understood, but difficult to prove, and similar situations arise in geometry. Indeed, sometimes an argument that uses dynamic geometry (for example, by using Geogebra) can be justified at an intuitive level, but, nevertheless, is difficult to justify completely at the level of school mathematics. Often such arguments can only be justified by the use of more advanced mathematics, an issue which we address briefly at the end of this article. In this article we discuss various aspects of Pitot quadrilaterals, including a short proof of the converse result, based on the use of dynamic geometry, and we comment on how the theorem fits into a wider context. The author is grateful for correspondence with the author of the Note [1] in the *Gazette* (which led to this article), and hopes that this article will encourage readers to use dynamic geometry as a tool to understand mathematics regardless of the validity of proof.

2. *Two constructions of Pitot quadrilaterals*

Given a Pitot quadrilateral $ABCD$, by definition we have $AB + CD = AD + BC$ so that $BC - BA = DC - DA$. This last condition

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is one of the many classical ways to define a hyperbola, so Pitot quadrilaterals are intimately connected to conics. Consider Figure 2. In this, we have drawn a hyperbola with foci at A and C, and marked two points B and D_1 on the same branch of the hyperbola. The defining property of the hyperbola is that $BC - BA = D_1C - D_1A$, so that $ABCD_1$ is a Pitot quadrilateral. In fact, this is the hyperbola given by $3x^2 - y^2 = 12$, and A, B, C and D_1 are the points $(4, 0)$, $(4, 6)$, $(-4, 0)$ and $(3, -\sqrt{15})$, respectively, with edge lengths given by $AB = 6$, $BC = 10$, $CD_1 = 8$ and $D_1A = 4$.

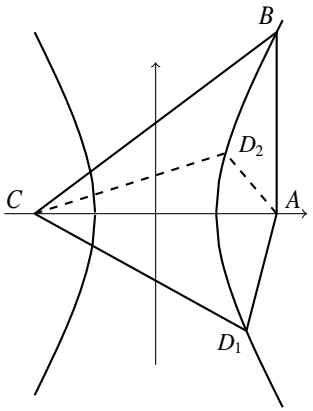


FIGURE 2: $BC - AB = DC - AD$ if, and only if, $AD + BC = AB + DC$

From the perspective of dynamic geometry, we can keep A, B and C fixed, and move D_1 along the one branch of the hyperbola; then, regardless of the position of D_1, the resulting quadrilateral will be a Pitot quadrilateral. The hyperbola, then, enables us to create a one-parameter family of Pitot quadrilaterals (all of which have an inscribed circle). If D_1 is moved so as to lie on the segment CA, our quadrilateral becomes a triangle. Better still, we can move B as well as D_1, and if we move these points so that the segment that joins them passes through A, then the Pitot quadrilateral will again be a triangle; in this case A is still a vertex of the quadrilateral, and the interior angle at A is π. In this sense, the existence of a circle inscribed in a triangle can be considered as a limiting case of Pitot's theorem (where the points of tangency on the edges AB and AD have coincided at A). When D_1 has moved to D_2, we will have constructed a *non-convex* Pitot quadrilateral. This idea alone suggests that we should examine to what extent the result remains true when we continue this deformation in the obvious way to obtain a quadrilateral that is *not* convex. This case has been discussed in the literature, and we leave the interested readers to explore these ideas for themselves. We simply remark that a natural extension of the idea of an inscribed circle is that there is a circle that is tangent to the four lines obtained by extending the four edges (without restricting the points of tangency to lie within the four edges of the quadrilateral).

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There is a second construction of a Pitot quadrilateral which has an interesting connection with other parts of mathematics. Suppose that we can construct a chain of four circles, say $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$, where each $\mathscr{C}_j$ is externally tangent to its two neighbours $\mathscr{C}_{j-1}$ and $\mathscr{C}_{j+1}$ (where, of course, we are taking $\mathscr{C}_5$ to be the same as $\mathscr{C}_1$). Obviously (see Figure 3) the centres of these circles form a Pitot quadrilateral. From a dynamical point of view, the main features of this picture will be preserved if we increase the radii of the circles centred at A and C, and decrease the radii of the circles centered at B and D_1, all by the same (sufficiently small) amount. Thus once again we have constructed a one-parameter family of Pitot qudrilaterals.

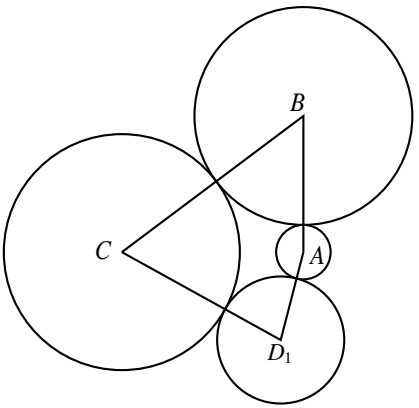


FIGURE 3: A chain of four circles

The algebra behind Figure 3 is worth exploring. If we let the lengths of the four edges of a Pitot quadrilateral be ℓ_1, ℓ_2, ℓ_3 and ℓ_4, then, in order to construct a set of circles with radii r_1, r_2, r_3 and r_4, as illustrated in Figure 3, we have to solve the system of equations

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{pmatrix}. \tag{1}$$

where, of course, we are seeking solutions with each r_j positive. As the determinant of this matrix is 0, either there are no solutions, or there are infinitely many solutions. Now the rank of this matrix is three, and the linear map α of $\mathbb{R}^4$ into itself that is defined by this matrix actually maps $\mathbb{R}^4$ onto the three-dimensional subspace W, where

$$\alpha\left(\mathbb{R}^4\right) = W = \left\{(x_1, x_2, x_3, x_4) : x_1 + x_3 = x_2 + x_4\right\}$$

of $\mathbb{R}^4$, with kernel K, where $K = \left\{(t, -t, t, -t) : t \in \mathbb{R}\right\}$. Now, from the standard ideas about the solutions of such linear equations, we see that a system of circles such as those in Figure 3 exists if, and only if,

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$(\ell_1, \ell_2, \ell_3, \ell_4) \in W$ or, equivalently, $\ell_1 + \ell_3 = \ell_2 + \ell_4$. Thus we have confirmed, through linear algebra, that such a set of radii exists if, and only if, the quadrilateral is a Pitot quadrilateral. Of course, the solutions r_j may not be positive, and the convexity of the quadrilateral plays no role in this analysis. The reader may wish to study a similar problem for a chain of n circles, and discover how the outcomes depend on the parity of n.

3. *The search for positive solutions*

The search for positive solutions of the matrix equation (1) is closely related to a proof of the converse part of Pitot's theorem. For the system of equations (1) to have a solution, we must have $\ell_1 + \ell_3 = \ell_2 + \ell_4$. In this case, it is easy to check that

$$(r_1, r_2, r_3, r_4) = (\ell_4, \ell_2 - \ell_3, \ell_3, 0)$$

is one solution, so the general solution of (1) is

$$(r_1, r_2, r_3, r_4) = (\ell_4 - s, \ell_1 - \ell_4 + s, \ell_3 - s, s), \qquad s \in \mathbb{R}. \quad (2)$$

Thus there is a solution with $r_j \geqslant 0$ for $j = 1, 2, 3, 4$ if, and only if, there is a solution of the simultaneous inequalities

$$\ell_4 \geqslant s \geqslant 0, \ell_3 \geqslant s, s \geqslant \ell_4 - \ell_1 = \ell_3 - \ell_2. \quad (3)$$

Suppose that Q is a Pitot quadrilateral. We may assume that ℓ_4 is the longest edge. Then ℓ_2 is the shortest edge, so the next longest edge is ℓ_3 or ℓ_1; by relabelling, we may assume that it is ℓ_3. Then $\ell_4 \geqslant \ell_3 \geqslant \ell_1 \geqslant \ell_2$, so there is a solution to (3); in fact, in general, there is a range of acceptable solutions with each r_j positive found by choosing any s in the range $\min\{\ell_3, \ell_4\} \geqslant s \geqslant \ell_4 - \ell_1 = \ell_3 - \ell_2 \geqslant 0$.

4. *A dynamic proof of the converse*

We now give a simple proof of the converse of Pitot's theorem using dynamic geometry which we believe is of interest in its own right. For more formal proofs see [1]. We begin with a quadrilateral $ABCD$ such that $AB + CD = BC + DA$. Next, we subdivide each edge of the quadrilateral such that the lengths p, q, r and s are as in Figure 4. The linear algebra in the previous section shows that this is possible, and the fact that the general solution (2) of (1) contains the one real parameter corresponds to the fact that once we have one solution *we can obtain other solutions simply by sliding the points along the edges while retaining the configuration of lengths*, as shown in Figure 4.

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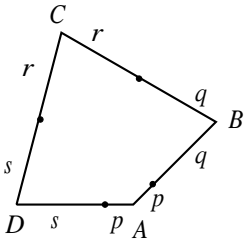


FIGURE 4: Subdivision of the edges

Now, with these four points in a generic position as in Figure 4, we join these points so as to form a new (dashed) quadrilateral as in Figure 5.

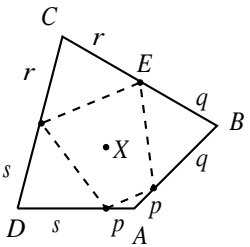


FIGURE 5: Many cyclic quadrilaterals

It is easy to see that the angle of the vertex E of the inner quadrilateral is $\frac{1}{2}(\angle B + \angle C)$, and similarly for the other division points. Thus, because $\angle A + \angle B + \angle C + \angle D = 2\pi$, *the inner (dashed) quadrilateral is a cyclic quadrilateral, regardless of the values of p, q, r and s*, and this means that the perpendicular bisectors of the four dashed edges are concurrent at the centre, say X, of the corresponding circle. However, these perpendicular bisectors are the four angle bisectors of the outer quadrilateral *ABCD*; so we now see that *the four angle bisectors of ABCD are concurrent at X* and therefore that the location of *X is independent of the choice of p, q, r and s*. Now the four vertices of the inner quadrilateral are at the same distance, say R, from X, and as we slide the four points along the edges of the quadrilateral, R varies continuously. If we choose these vertices so as to minimise R then it is clear that the circle through them must be the inscribed circle of the quadrilateral *ABCD*. Indeed, when R is minimal, this circle will be tangent to, say, *BC*, and then, by symmetry, it must be tangent to *CD*. As it is tangent to *CD* it must also (by the same argument) be tangent to *DA*, and then to *AB*. This completes our dynamic proof of Pitot's theorem. This is not a rigorous proof but perhaps it does help us understand why the result is true.

5. *A more general perspective*

There are many known results of the type we are considering in projective geometry; for example there are results on the possibility of inscribing an ellipse in a convex polygon, and also finding a circumscribing

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ellipse for the polygon (for example, Brianchon's theorem for an ellipse inscribed in a hexagon). We shall not delve further into this extensive subject, but we do wish to alert the reader to its existence, and the interested reader can easily search the web for more information. Nevertheless, we will illustrate the ideas with two beautiful (and simply stated) results. First, given any triangle, and any two points on two of the edges, there is a unique inscribed ellipse that is tangent to the triangle at these two points; thus there is a two-parameter family of ellipses that are inscribed in a given triangle. Even more impressive is Marden's theorem, which was apparently first proved by Siebeck in 1864 [2], and which is as follows. Let $p(z)$ be the cubic, complex polynomial $(z - z_1)(z - z_2)(z - z_3)$, where the z_j are not collinear, and which therefore form the vertices of a triangle T. Let w_1 and w_2 be the zeros of the (quadratic) derivative $p'(z)$. Then, by the Gauss-Lucas theorem (see below), w_1 and w_2 lie inside T, and Marden's theorem is that *w_1 and w_2 are the foci of an ellipse, called the Steiner ellipse, which is inscribed in the triangle T, and which is tangent to each edge of T at its mid-point* (see Figure 6).

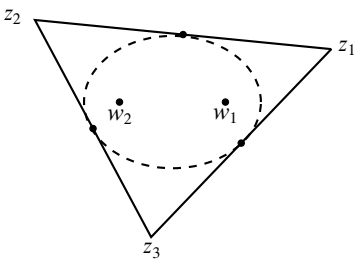


FIGURE 6: The Steiner ellipse

The Gauss-Lucas theorem holds for all polynomials, and here is a short proof for the case of a triangle T with vertices z_1, z_2 and z_3. Let $p(z) = (z - z_1)(z - z_2)(z - z_3)$. Then

$$p'(z) = \frac{p(z)}{z - z_1} + \frac{p(z)}{z - z_2} + \frac{p(z)}{z - z_3}.$$

Now let w be a root of $p'(z)$. If $p(w) = 0$ then w is some z_j so that w lies in T as required; thus we may now assume that $p(w) \neq 0$. Then

$$0 = \frac{p'(w)}{p(w)} = \sum_{j=1}^{3} \frac{1}{w - z_j} = \sum_{j=1}^{3} \frac{\overline{w} - \overline{z_j}}{|w - z_j|^2}.$$

If we now take the complex conjugate of each side of this equation, and write $\lambda_j = \dfrac{1}{|w - z_j|^2}$, we find that

$$w = \frac{1}{\lambda_1 + \lambda_2 + \lambda_3}(\lambda_1 z_1 + \lambda_2 z_2 + \lambda_3 z_3),$$

which (since each λ_j is positive) implies that w is in the convex hull T of the

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points z_1, z_2 and z_3. In fact, the same proof works for any polynomial p and so proves the Gauss-Lucas theorem. For more details on these ideas we refer the reader to [3, 4, 5, 6].

6. *The midpoint problem*

Consider the following exercise. Suppose that I draw a quadrilateral, and then mark the midpoints of the four edges. I now copy the location of the midpoints (but *not* the quadrilateral) onto another sheet of paper, and give you this new sheet of paper. Can you reconstruct my original quadrilateral? What if I give you the midpoints of a triangle, or a pentagon or, more generally, an n-gon?

If we consider this problem for a quadrilateral with edges of lengths ℓ_j, and define a, b, c and d by

$$2a = \ell_1, \qquad 2b = \ell_2, \qquad 2c = \ell_3, \qquad 2d = \ell_4$$

then this system of equations is equivalent to the system

$$\begin{aligned} a + b &= \frac{1}{2}(\ell_1 + \ell_2), \\ b + c &= \frac{1}{2}(\ell_2 + \ell_3), \\ c + d &= \frac{1}{2}(\ell_3 + \ell_4), \\ d + a &= \frac{1}{2}(\ell_4 + \ell_1), \end{aligned}$$

which is essentially the same as the linear system of equations (1). Thus, algebraically, there is no difference between the two problems. In one case we consider a quadrilateral whose edges have lengths $a + a$, $b + b$, $c + c$ and $d + d$, whereas in the other problem the edges of the quadrilateral have lengths $a + b$, $b + c$, $c + d$ and $d + a$. The problem concerning the midpoints of an n-gon is related to that of finding a closed chain of circles $\mathscr{C}_j$, where each $\mathscr{C}_j$ is externally tangent to its two immediate neighbouring circles, and for each of these problems the parity of n is important. For more details on these and other problems of a similar nature, see [7, 8, 9, 10], together with many articles on the web.

7. *Chains of circles*

We now consider yet another digression. We have considered a chain of four circles $\mathscr{C}_j$ in which each of the circles is externally tangent to its two neighbouring circles. Suppose that for a given circle $\mathscr{C}_j$ we apply the transformation I_j of inversion across $\mathscr{C}_j$; then the three circles exterior to $\mathscr{C}_j$ will be 'inverted' into three circles interior to $\mathscr{C}_j$, and the relation of tangency will be preserved. We will then obtain a chain of twelve image

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circles, as illustrated in Figure 7. Note that this figure contains the four original circles $\mathscr{C}_j$, and the twelve smaller circles $I_s(\mathscr{C}_t)$, where $s, t = 1, 2, 3, 4$ and $s \neq t$.

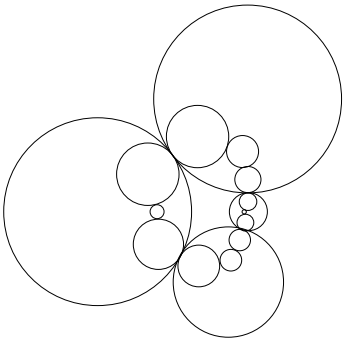


FIGURE 7: The circles $I_j(\mathscr{C}_k)$

Suppose that we now repeat the operation in the following sense. To draw the next stage we apply I_r to each $I_s(\mathscr{C}_t)$ providing that $r \neq s$; thus we obtain 4×3^2 new circles which are $I_r(I_s(\mathscr{C}_t))$, where $r \neq s$ and $s \neq t$. It is not difficult to imagine that if we were to continue this process indefinitely then, in the limit, we would obtain a closed curve $\mathscr{C}$ which will be a fractal curve. In fact, it can be shown that, providing that there is no circle that is orthogonal to the four original circles (for in this exceptional case $\mathscr{C}$ would be this circle) the curve $\mathscr{C}$ will 'wind about itself' so much that it does *not* have a tangent at any of a 'dense' set of points on the curve. This construction goes back to Fricke and Klein; see [11, pp. 136, 193, 197] for more details.

8. *Concluding remarks*

We end with a comment on the possible justification of arguments in dynamic geometry. In general, there are two types of results in mathematics; there are results which describe properties of objects that are known to exist, and there are results which proclaim the existence of objects. Of these, the former tend to be the easier to prove, and the existence theorems are invariably the deeper results. For example, the Intermediate Value theorem (which says that if a continuous function f takes positive and negative values in an interval, then it also takes the value 0 in the interval) is an existence theorem. Often dynamic geometry is involved with existence theorems (for example, Pitot's theorem), and the ultimate justification of a typical argument of this type usually lies in continuity (obviously) and some kind of compactness argument such as the Bolzano-Weierstrass theorem (that every bounded sequence has a convergent subsequence) which is itself an existence theorem.

Dynamic geometry is inherent in Felix Klein's famous Erlangen address in 1872 in which, roughly speaking, he defined geometry as the collection of

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properties that are invariant under a group G of transformations acting on a space X (for example, the properties of a figure that remain invariant under the rigid motions of the Euclidean plane). In a loose sense, dynamic geometry is available when we are allowed to vary the elements in G continuously as, for example, in the case of the rotations of a satellite as it travels through space.

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A. F. BEARDON

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Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB
e-mail: *afb@dpmms.cam.ac.uk*

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