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INVERSE z TRANSFORM AND MOMENT PROBLEM

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Numerical inversion of a one-sided z transform, corresponding to causal positive sequence, is considered. The numerical inversion requires the availability of a finite number of the transform's derivatives. The approximate analytical form is obtained by resorting to the maximum entropy principle. Entropy and then L_1-norm convergence are proved.

1. INTRODUCTION

We consider a positive discrete function f_n (typically a probability distribution), which takes on nonzero values only for the non-negative integers (i.e., for $n = 0,1,2,\ldots$ and $f_n = 0$ for $n < 0$), so that $\sum_{n=0}^{\infty} f_n = 1$. The corresponding one-sided z transform (also known as the probability generating function) is given by

$$F(z) = \sum_{n=0}^{\infty} f_n z^n \tag{1.1}$$

where z is a complex variable and $F(z)$ is an analytic function. Once $F(z)$ is assigned, the inverse problem consists in recovering f_n. From the adopted hypothesis about f_n, $F(z)$ is defined at least for $|z| \leq 1$ and is infinitely differentiable for $|z| < 1$.

Classical methods for carrying out the inversion, as power series, inspection method, partial-fraction expansion, or the inversion formula

$$f_n = \frac{1}{2\pi j} \oint_C \frac{F(z)}{z^{n+1}}\, dz, \qquad n = 0,1,2,\ldots, \tag{1.2}$$

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where C is a circle, defined in the counterclockwise sense, large enough to enclose the poles of $F(z)$, are well known. I would quote even the method, based on the intermediate value theorem, giving each term of the sequence f_n through the relation

$$f_n = \frac{1}{n!}\frac{d^n}{dz^n}F(z)\bigg|_{z=0}, \quad n = 0,1,2,\ldots.$$

Efficient evaluation of $d^nF(z)/dz^n|_{z=0}$ is provided by the Lyness algorithm [5], for instance.

When the analytical form of $F(z)$ does not allow us to resort to the previous methods, then numerical methods based on the discrete inverse Fourier transform are available; the discrete Fourier transform of f_n is simply the set of $F(z)$ samples at uniform spacing around a circle in the z plane. There are some cases in which the $F(z)$ analytical form is not immediately available, as only the functional equation of $F(z)$ is known. We are referring, for instance, to the queueing systems (Kleinrock [4]). If f_n indicates the probability distribution of the number of customers served in a busy period, then its z transform $F(z)$ satisfies the functional equation

$$F(z) = zB^*[\lambda - \lambda F(z)], \tag{1.3}$$

where λ represents the average arrival rate and $B^*(\cdot)$ is the Laplace transform of the service-time distribution. Except in the cases where $B^*(\cdot)$ assumes a simple form, $F(z)$ is not available from the solution of (1.3). Nevertheless, in such a case, it is possible to obtain the moments μ_k of f_n given by

$$\mu_k = \sum_{n=0}^{\infty} n^k f_n, \qquad k = 1,2,\ldots, \qquad \mu_0 = 1, \tag{1.4}$$

and related to $F(z)$ derivatives, evaluated at $z = 1$, through the relationship

$$\sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-k+1)f_n = z^k\frac{d^k}{dz^k}F(z)\bigg|_{z=1} \tag{1.5}$$

by assigning to k the successive values $1,2,\ldots$. In particular,

$$\mu_0 = \sum_{n=0}^{\infty} f_n = F(1), \qquad \mu_1 = \sum_{n=0}^{\infty} nf_n = F'(1),$$

$$\mu_2 = \sum_{n=0}^{\infty} n^2 f_n = F''(1) + F'(1),$$

$$\mu_3 = \sum_{n=0}^{\infty} n^3 f_n = F'''(1) + 3F''(1) + F'(1). \tag{1.6}$$

Thus, the problem of recovering an infinite positive discrete sequence from the knowledge of its $F(z)$ transform, or equivalently, of its moments μ_k, $k = 1,2,\ldots$, is related to the classical Stieltjes moment problem. For practical purposes, only a finite number of moments μ_k, $k = 1,2,\ldots,M$, is assigned. The problem of recovering

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f_n is then underdetermined and it is necessary to resort to choice criterion. The maximum entropy (ME) principle (Jaynes [2]) indicates a way to select one solution of an underdetermined inverse problem without improperly adding information.

Define the Shannon entropy $H[\,f_n]$ of f_n sequence by

$$H[\,f_n] = -\sum_{n=0}^{\infty} f_n \ln f_n. \tag{1.7}$$

The use of Lagrange's multipliers method to maximize the entropy (1.7), subject to constraints (1.6), provides the Lagrangian

$$L = -\sum_{n=0}^{\infty} f_n \ln f_n - (\lambda_0 - 1)\left(\sum_{n=0}^{\infty} f_n - \mu_0\right) - \sum_{r=1}^{M} \lambda_r \left(\sum_{n=0}^{\infty} n^r f_n - \mu_r\right), \tag{1.8}$$

from which we obtain the analytical form of the $f_n^{(M)}$ approximant (Kesavan and Kapur [3])

$$f_n^{(M)} = \exp\left(-\sum_{j=0}^{M} \lambda_j n^j\right) \tag{1.9}$$

(the positivity of the approximant is thus guaranteed), where $\lambda_j, j = 0,1,\ldots,M$, are the Lagrange multipliers satisfying the constraints of the moments

$$\sum_{n=0}^{\infty} n^k f_n^{(M)} = \mu_k, \qquad k = 0,\ldots,M \tag{1.10}$$

(obviously $\lambda_M \geq 0$). Then, the entropy $H[\,f_n^{(M)}]$ is given by

$$H[\,f_n^{(M)}] = -\sum_{n=0}^{\infty} f_n^{(M)} \ln f_n^{(M)} = \sum_{j=0}^{M} \lambda_j \mu_j. \tag{1.11}$$

From (1.10) (setting $k = 0$) and (1.9), λ_0 can be obtained as a function of $\lambda_1,\ldots,\lambda_M$,

$$\lambda_0 = \ln\left\{\frac{1}{\mu_0}\sum_{n=0}^{\infty} \exp\left(-\sum_{j=1}^{M} \lambda_j n^j\right)\right\}; \tag{1.12}$$

λ_0 is a convex function of $\lambda_1,\ldots,\lambda_M$ (Kesavan and Kapur [3]) and

$$\mu_k = \mu_0 \frac{\displaystyle\sum_{n=0}^{\infty} n^k \exp\left(-\sum_{j=1}^{M} \lambda_j n^j\right)}{\displaystyle\sum_{n=0}^{\infty} \exp\left(-\sum_{j=1}^{M} \lambda_j n^j\right)}, \qquad k = 1,\ldots,M. \tag{1.13}$$

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Consequently, taking into account (1.12),

$$H[\, f_n^{(M)}] = \mu_0 \ln\left\{ \frac{1}{\mu_0} \sum_{n=0}^{\infty} \exp\left(-\sum_{j=1}^{M} \lambda_j n^j \right) \right\} + \sum_{j=1}^{M} \lambda_j \mu_j \qquad \textbf{(1.14)}$$

is a convex function of $\lambda_1, \ldots, \lambda_M$. Then, for practical purposes, Lagrange's multipliers are obtained by calculating the minimum of $H[\, f_n^{(M)}]$, as $\partial H[\, f_n^{(M)}]/\partial \lambda_k = 0$, $k = 1, \ldots, M$, provides (1.13).

Thus, the unknown f_n sequence is substituted by an approximant $f_n^{(M)}$ given by (1.9), where the Lagrange's multipliers $\lambda_j, j = 1, \ldots, M$, are obtained by minimizing the entropy $H[\, f_n^{(M)}]$ given by (1.14), whereas λ_0 is obtained by (1.12).

It will be proved that the $f_n^{(M)}$ approximant converges in entropy to f_n, in the sense that

$$\lim_{M \to \infty} H[\, f_n^{(M)}] = H[\, f_n] \qquad \textbf{(1.15)}$$

by assuming an infinite sequence of moments exists. The entropy convergence (1.15) implies L_1-norm convergence (Cover and Thomas [1]):

$$H[\, f_n^{(M)}] - H[\, f_n] \geq \frac{1}{2 \ln 2} \| f_n^{(M)} - f_n \|_1^2 \geq \frac{1}{2 \ln 2} \| f_n^{(M)} - f_n \|_\infty^2. \qquad \textbf{(1.16)}$$

Thus, (1.15) and (1.16) justify the choice of $f_n^{(M)}$ as an approximant.

2. GENERAL RESULTS OF THE STIELTJES MOMENT PROBLEM

We recall some classical results concerning the Stieltjes moment problem which will be used in the entropy convergence proof. Such results are drawn essentially from the classic book of Shohat and Tamarkin [6].

Given a set of moments $\mu_j, j = 0,1, \ldots$, the Hankel's determinants are defined as

$$\Delta_0 = \mu_0, \Delta_1 = \begin{vmatrix} \mu_0 & \mu_1 \\ \mu_1 & \mu_2 \end{vmatrix}, \ldots, \Delta_n = \begin{vmatrix} \mu_0 & \ldots & \mu_n \\ \vdots & \vdots & \vdots \\ \mu_n & \ldots & \mu_{2n} \end{vmatrix}, \qquad \textbf{(2.1)}$$

$$\Delta_0^{(1)} = \mu_1, \Delta_1^{(1)} = \begin{vmatrix} \mu_1 & \mu_2 \\ \mu_2 & \mu_3 \end{vmatrix}, \ldots, \Delta_n^{(1)} = \begin{vmatrix} \mu_1 & \ldots & \mu_{n+1} \\ \vdots & \vdots & \vdots \\ \mu_{n+1} & \ldots & \mu_{2n+1} \end{vmatrix}. \qquad \textbf{(2.2)}$$

2.1 $f_n^{(M)}$ Existence

Existence conditions of an f_n sequence are given by the following theorem.

THEOREM 2.1: *In order that there exists an f_n solution not reducible to a finite set of points, it is necessary and sufficient that*

$$\Delta_n > 0, \qquad \Delta_n^{(1)} > 0 \qquad n = 0,1, \ldots. \qquad \textbf{(2.3)}$$

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In order that there exists an f_n solution consisting of precisely $(k+1)$ points distinct from $n=0$, it is necessary and sufficient that

$$\begin{aligned} &\Delta_0 > 0, \ldots, \Delta_k > 0, \qquad \Delta_{k+1} = \Delta_{k+2} = \cdots = 0, \\ &\Delta_0^{(1)} > 0, \ldots, \Delta_k^{(1)} > 0, \qquad \Delta_{k+1}^{(1)} = \Delta_{k+2}^{(1)} = \cdots = 0. \end{aligned} \tag{2.4}$$

In order that there exists f_n solution consisting of precisely $(k+1)$ points, one of them at $n=0$, it is necessary and sufficient that

$$\begin{aligned} &\Delta_0 > 0, \ldots, \Delta_k > 0, \qquad \Delta_{k+1} = \Delta_{k+2} = \cdots = 0, \\ &\Delta_0^{(1)} > 0, \ldots, \Delta_{k-1}^{(1)} > 0, \qquad \Delta_k^{(1)} = \Delta_{k+1}^{(1)} = \cdots = 0. \end{aligned} \tag{2.4a}$$

From Theorem 2.1, concerning f_n existence conditions, $f_n^{(M)}$ existence conditions are drawn as follows. If $M=1$ or $M \geq 4$, then $f_n^{(M)}$ existence conditions are identical to those for f_n (Tagliani [8]). If $M=2$ or $M=3$, upper bounds for the moments μ_2 and μ_3 increase: When $M=2$, $\mu_2 \leq \mu_1^2(2+1/\mu_1)$, and when $M=3$ and $\mu_2 < \mu_1^2(2+1/\mu_1)$, the μ_3 upper bound is graphically provided.

For convergence purposes, if an infinite sequence of moments is considered, then the moment problem determinacy must be taken into account.

2.2 Determinacy

Moment problem determinacy concerns one-to-one correspondence between f_n and its infinite moments sequence. We bound ourselves to quote a necessary and sufficient condition guaranteeing the moment problem determinacy. Let us associate to the following sequence (Shohat and Tamarkin [6]) the moments sequence μ_j ($j = 0,1,\ldots$):

$$\rho_n(0) = \frac{\Delta_n}{\begin{vmatrix} \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots \\ \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}}, \qquad n = 1,2,\ldots, \tag{2.5}$$

and, similarly, $\rho_n^{(1)}(0)$ corresponding to the $\{\mu_1, \mu_2, \ldots\}$ sequence. The following theorem holds (Shohat and Tamarkin [6]).

THEOREM 2.2: *A necessary and sufficient condition for the Stieltjes moment problem to be determined is that at least one of the quantities*

$$\rho(0) = \lim_{n\to\infty} \rho_n(0), \qquad \rho^{(1)}(0) = \lim_{n\to\infty} \rho_n^{(1)}(0) \tag{2.6}$$

be equal to zero.

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2.3 Geometrical Interpretation

Once $\mu_1,\ldots,\mu_M$ are assigned with $M = 2n$ without loss of generality, let $\mu_{0,n}^{-}$ be the μ_0 value such that $\Delta_n = 0$. From

$$\Delta_n = \mu_0 \begin{vmatrix} \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots \\ \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} + C, \tag{2.7}$$

where $C = C(\mu_1,\ldots,\mu_{2n})$ and

$$0 = \mu_{0,n}^{-} \begin{vmatrix} \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots \\ \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} + C, \tag{2.8}$$

we obtain

$$\rho_n(0) = \frac{\Delta_n}{\begin{vmatrix} \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots \\ \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}} = \mu_0 - \mu_{0,n}^{-}, \qquad n = 1,2,\ldots, \tag{2.9}$$

which represents the distance between μ_0 and its lower bound $\mu_{0,n}^{-}$. A similar relation for $\rho^{(1)}(0)$ with $M = 2n+1$ holds. Then,

$$\rho_n(0) = \mu_0 - \mu_{0,n}^{-}, \qquad \rho_n^{(1)}(0) = \mu_1 - \mu_{1,n}^{-}. \tag{2.10}$$

3. ENTROPY CONVERGENCE

Entropy convergence implies the moment problem determinacy. We will prove that $\lim_{M\to\infty} H[\, f_n^{(M)}] = H[\, f_n]$. From (2.6), it is necessary to consider two cases separately.

3.1. $\rho(0) = \lim_{n\to\infty} \rho_n(0) = 0$.

Let $M = 2N$ without loss of generality. Let $\mu_1,\ldots,\mu_M$ be fixed and μ_0 vary continuously. Then, from (1.9), the Lagrange multipliers λ_j are functions of μ_0. By differentiating (1.10) with respect to μ_0, the differential system

$$\Delta_M \cdot \begin{bmatrix} d\lambda_0/d\mu_0 \\ \vdots \\ d\lambda_M/d\mu_0 \end{bmatrix} = -\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{3.1}$$

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where unknown moments $\mu_{M+1},\dots,\mu_{2M}$ are calculated by (1.13), is obtained. From (1.11), by taking into account the first equation of (3.1), we get

$$\frac{d}{d\mu_0} H[\, f_n^{(M)}] = \lambda_0 + \sum_{j=0}^{M} \mu_j \frac{d\lambda_j}{d\mu_0} = \lambda_0 - 1, \tag{3.2}$$

and, by (2.6) and (3.1),

$$\frac{d^2}{d\mu_0^2} H[\, f_n^{(M)}] = \frac{d\lambda_0}{d\mu_0} = -\frac{1}{\rho_M(0)} < 0 \tag{3.3}$$

holds. Thus, $H[\, f_n^{(M)}]$ is a differentiable concave function of μ_0. Let us suppose that the relationship

$$\lim_{M\to\infty} H[\, f_n^{(M)}] = H_{\lim} > H[\, f_n] \tag{3.4}$$

holds. At $\mu_0 = 1$, $H[\, f_n^{(M)}] > H[\, f_n]$ holds, where $H[\, f_n^{(M)}]$ is the maximum entropy sequence consistent with the assigned constraints $\mu_0,\dots,\mu_M$. As $\mu_0 \to \mu_{0,N}^-$, all the sequences having $\mu_1,\dots,\mu_M$ degenerate in the unique sequence having a finite number of positive values. Such a sequence has an entropy, indicated by $H_{\min}$, with $H_{\min} < H[\, f_n]$, for all the sequences f_n having $\mu_1,\dots,\mu_M$ assigned as a consequence of $H[\, f_n]$ concavity. Thus, at the edges of the interval $[\mu_{0,N}^-, \mu_0 = 1]$, $H[\, f_n^{(M)}]$ assumes respectively the value $H_{\min} < H[\, f_n]$ and a proper value $H[\, f_n^{(M)}] > H[\, f_n]$ for each sequence f_n satisfying the constraints $\mu_0,\dots,\mu_M$, being $f_n^{(M)}$ a maximum entropy sequence.

By increasing M, the moment problem determinacy hypothesis implies $\lim_{M\to\infty} \mu_{0,N}^- = \mu_0$. Thus, because of $H[\, f_n^{(M)}]$ continuity as a function of μ_0, $H_{\lim} = H[f_n]$ holds and so the result

$$\lim_{M\to\infty} H[\, f_n^{(M)}] = H[\, f_n] \tag{3.5}$$

is proved.

3.2. $\rho(0) = c > 0$ with c a Constant and $\rho^{(1)}(0) = \lim_{n\to\infty} \rho_n^{(1)}(0) = 0$

Let $\mu_0, \mu_2 \dots, \mu_M$ be fixed and μ_1 vary continuously. Then, $\lambda_j = \lambda_j(\mu_1), j = 0,\dots,M$. By differentiating (1.10) with respect to μ_1, the differential system

$$\Delta_M \cdot \begin{bmatrix} d\lambda_0/d\mu_1 \\ \vdots \\ d\lambda_M/d\mu_1 \end{bmatrix} = -\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.6}$$

is obtained. From (1.11), by taking into account the first equation of (3.6), we get

$$\frac{d}{d\mu_1} H[\, f_n^{(M)}] = \lambda_1 + \sum_{j=0}^{M} \mu_j \frac{d\lambda_j}{d\mu_1} = \lambda_1; \tag{3.7}$$

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then,

$$\frac{d^2}{d\mu_1^2} H[\, f_n^{(M)}] = \frac{d\lambda_1}{d\mu_1} = -\frac{\Delta_{M,1}}{\Delta_M} \tag{3.8}$$

holds. Here, $\Delta_{M,1}$ is obtained from Δ_M by deleting the second row and column. $\Delta_{M,1}$ is positive and is the determinant of the principal matrix $P_{M,1}^T[\Delta_M]P_{M,1}$ of M order, where $P_{M,1}$ is the permutation matrix interchanging, starting from the second row, every row with the successive one. Thus, $H[\, f_n^{(M)}]$ is a differentiable concave function of μ_1. By taking into account the $\rho^{(1)}(0)$ geometrical meaning, the same conclusions as in the previous case can be drawn. The obtained result can be formulated by means of the following theorem.

THEOREM 3.1: *Whenever the moments determine uniquely the f_n sequence then the maximum entropy approximant $f_n^{(M)}$ converges in entropy to f_n.*

Define the directed divergence

$$D(\, f_n \| \, f_n^{(M)}) = \sum_{n=0}^{\infty} f_n \ln \frac{f_n}{f_n^{(M)}} \tag{3.9}$$

for $f_n^{(M)}$ and f_n having the same moments $\mu_0, \ldots, \mu_M$; then the following inequalities hold (Cover and Thomas [1]):

$$H[\, f_n^{(M)}] - H[\, f_n] = D(\, f_n \| \, f_n^{(M)}) \geq \frac{1}{2 \ln 2} \| \, f_n^{(M)} - f_n \|_1^2$$

$$\geq \frac{1}{2 \ln 2} \| \, f_n^{(M)} - f_n \|_\infty^2. \tag{3.10}$$

Thus, entropy convergence (or, equivalently, directed divergence) implies L_1-norm and then L_∞-norm convergence.

4. NUMERICAL RESULTS

We illustrate through a few examples, borrowed from queueing systems, the accuracy of the approximate sequence $f_n^{(M)}$ as the number M of assigned moments is increased. For each example, the following will be reported:

1. $H[\, f_n^{(M)}] - H[\, f_n]$
2. $\| \, f_n^{(M)} - f_n \|_1$
3. $\| \, f_n^{(M)} - f_n \|_\infty$
4. $f_n^{(M)}$ and f_n graphical comparison.

For practical purposes, we recall that the moments μ_j, $j = 1,2,\ldots$, are obtained by successive derivatives of $F(z)$ at $z = 1$ [(1.6)]. λ_j multipliers are calculated by min-

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imizing the entropy $H[\, f_n^{(M)}]$ [(1.14)] as a function of $\lambda_1, \ldots, \lambda_M$, whereas λ_0 is obtained from (1.12).

Example 1 [$M/D/1$ queue (Poissonian arrival, deterministic service distribution)]: Let f_n be the distribution of the number served in a busy period. $F(z)$ satisfies the functional equation (Kleinrock [4])

$$F(z) = z \exp[\, \rho(F(z) - 1)], \tag{4.1}$$

where $0 < \rho < 1$. By (4.1), successive differentiations, $F^{(j)}(1)$, $j = 1,2,\ldots$, are obtained and then $\mu_j, j = 1,2,\ldots$. Fortunately, for comparison reasons, an analytical solution of (4.1) is available,

$$F(z) = \sum_{n=1}^{\infty} \frac{(n\rho)^{n-1}}{n!}\, e^{-n\rho} z^n \tag{4.2}$$

(the $F(z)$ analyticity region includes the unit disk $|z| = 1$) from which we obtain the exact sequence

$$f_n = \frac{(n\rho)^{n-1}}{n!}\, e^{-n\rho}, \qquad n = 1,2,\ldots. \tag{4.3}$$

Graphical and numerical comparisons corresponding to the $\rho = 0.25$ value are reported in Figure 1 and Table 1. Here, $H[\, f_n] \simeq 0.740467$.

Example 2 [$M/H_2/1$ (Poissonian arrival, hyperexponential service distribution)]: f_n represents the queue size probability distribution (Kleinrock [4]). Then,

$$F(z) = \frac{3}{32}\left(\frac{1}{1 - \frac{2}{5}z} + \frac{3}{1 - \frac{2}{3}z}\right) \tag{4.4}$$

TABLE 1. Comparison between $f_n^{(M)}$ and f_n: Entropy, L_1-Norm, and L_∞-Norm for Example 1

M	$H[\, f_n^{(M)}] - \mathrm{H}[\, f_n]$	$\| f_n^{(M)} - f_n \|_1$	$\| f_n^{(M)} - f_n \|_\infty$
1	$0.9313E-02$	$0.7691E-01$	$0.3586E-01$
2	$0.7446E-02$	$0.6940E-01$	$0.3266E-01$
3	$0.2675E-03$	$0.9760E-02$	$0.4310E-02$
4	$0.2331E-03$	$0.9044E-02$	$0.3965E-02$
5	$0.1440E-04$	$0.2127E-02$	$0.8094E-03$

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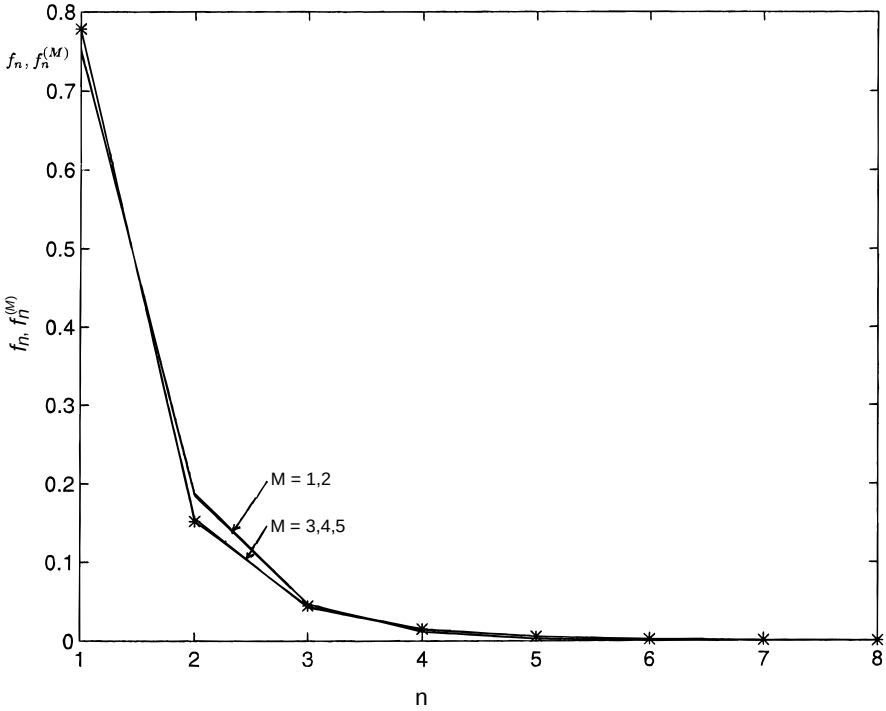


FIGURE 1. Exact sequence (∗) and approximate sequence (solid line) for Example 1.

from which

$$f_n = \frac{3}{32}\left[\left(\frac{2}{5}\right)^n + 3\left(\frac{2}{3}\right)^n\right], \qquad n = 0,1,2,\ldots, \tag{4.5}$$

and $H[\,f_n] \simeq 1.819922$, $\mu_1 \simeq 1.79166$, and $\mu_2 \simeq 8.68055$. As $\mu_2 > \mu_1^2(2 + 1/\mu_1)$, $f_n^{(2)}$ does not exist (example already reported in Kesavan and Kapur [3] and Shore [7], including $f_n^{(2)}$). Graphical and numerical comparisons are reported in Figure 2 and Table 2.

5. CONCLUSIONS

One-sided z transform numerical inversion, corresponding to a casual positive infinite sequence (typically a probability distribution), has been illustrated. The approximant analytical form has been resorted to the maximum entropy principle, guaranteeing the approximant positivity. Exact and approximate sequences are characterized by the same moments. The approximate sequence converges in entropy

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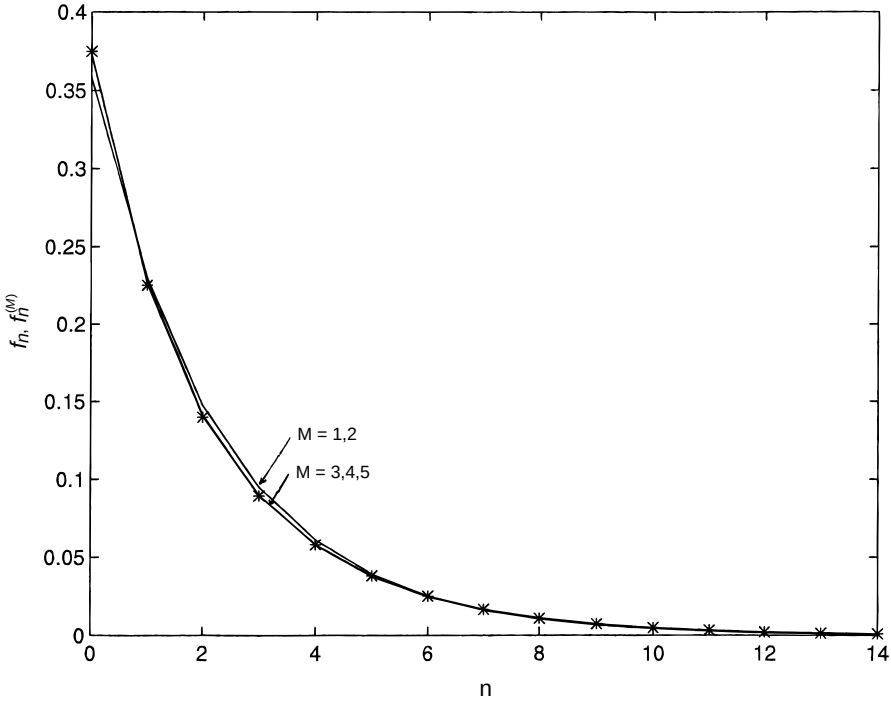


FIGURE 2. Exact sequence (∗) and approximate sequence (solid line) for Example 2.

and then in the L_1-norm. For practical purposes, the method requires, uniquely, the first moments of the unknown sequence; such moments can be obtained by z transform successive differentiation or by successive differentiation of the functional equation satisfying the z transform itself.

TABLE 2. Comparison between $f_n^{(M)}$ and f_n: Entropy, L_1-Norm, and L_∞-Norm for Example 2

M	$H[\, f_n^{(M)}] - H[\, f_n]$	$\| f_n^{(M)} - f_n \|_1$	$\| f_n^{(M)} - f_n \|_\infty$
1	$0.1307E-02$	$0.4324E-01$	$0.1679E-01$
3	$0.7725E-04$	$0.9285E-02$	$0.2537E-02$
4	$0.5545E-04$	$0.8107E-02$	$0.2294E-02$
5	$0.7650E-05$	$0.3657E-02$	$0.1421E-02$

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