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MORE ON RESTRICTED BOUNDARIES

SONJA MOUTON and ROBIN HARTE

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Abstract

In our paper ['Linking the boundary and exponential spectra via the restricted topology', *J. Math. Anal. Appl.* **454** (2017), 730–745], we defined and used the restricted topology to establish certain connections among the boundary spectrum, the exponential spectrum, the topological boundary of the spectrum and the connected hull of the spectrum, and in ['The restricted connected hull: filling the hole', *Bull. Aust. Math. Soc.* **109** (2024), 388–392], we presented further properties of the restricted connected hull. We now continue our investigation of the restricted boundary. In particular, we prove a number of mapping and regularity-type properties of the restricted boundary. In addition, we use this concept to provide a new characterisation of the Jacobson radical of a Banach algebra and a variation of Harte's theorem. Finally, we establish spectral continuity results, in particular, in ordered Banach algebras.

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1. Introduction

Given a complex unital Banach algebra A, we showed in [8] that it was possible to define a topology (different from the norm topology) on A by using closed subalgebras B. This was done in such a way that, given an element $a \in A$, a whole range of 'boundaries' $\partial_B(a)$ and accompanying 'connected hulls' $\eta_B(a)$ could be obtained. It was further shown in [8] that

$$\partial_{\mathbb{C}}(a) \subseteq \partial_B(a) \subseteq \partial_A(a)$$

and

$$\eta_A(a) \subseteq \eta_B(a) \subseteq \eta_{\mathbb{C}}(a),$$

with $\partial_{\mathbb{C}}(a) = \partial\sigma(a)$, $\partial_A(a) = S_\partial(a)$ and $\eta_A(a) = \varepsilon(a)$ (see Theorems 2.1 and 2.2). In addition, if $a \notin \mathbb{C}$, then $\eta_{\mathbb{C}}(a) = \mathbb{C}$, while if $a \in \mathbb{C}$, then $\eta_{\mathbb{C}}(a) = \eta\sigma(a)$. (Here, $\partial\sigma(a)$, $S_\partial(a)$, $\varepsilon(a)$ and $\eta\sigma(a)$ indicate the topological boundary of the spectrum, the

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boundary spectrum, the exponential spectrum and the connected hull of the spectrum, respectively, of a.)

Further properties of η_B were established in [9]. In this note, we prove a number of mapping and regularity-type properties of ∂_B (Propositions 3.4 and 3.5, Theorem 3.10, and Propositions 3.11 and 3.12). We also use this concept to provide a new characterisation of the Jacobson radical of a Banach algebra (see (3.2)), to exhibit a variation of Harte's theorem [2, Theorem 3] (see Theorem 3.13) and to establish spectral continuity results, in particular, in ordered Banach algebras (Theorem 4.8).

2. Preliminaries

We will denote the set of all nonnegative real numbers by $\mathbb{R}^+$. If A is a complex Banach algebra with unit 1, then the set of all invertible elements will be indicated by A^{-1} and elements of the form $\lambda 1$ in A will be denoted by λ. If a is an element of A, then the *spectrum* $\{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1}\}$ of a in A will be denoted by $\sigma(a)$ (or by $\sigma(a, A)$, if necessary, to avoid confusion). We will use the symbols $\partial\sigma(a)$ and $\eta\sigma(a)$ for the boundary and the connected hull, respectively, of $\sigma(a)$, and $r(a)$ will indicate the spectral radius of a.

The *boundary spectrum* of a in A was introduced in [4] as

$$S_\partial(a) := \{\lambda \in \mathbb{C} : a - \lambda \in \partial_A(A\backslash A^{-1})\},$$

and the *exponential spectrum* of a in A was introduced in [2] as

$$\varepsilon(a) := \{\lambda \in \mathbb{C} : a - \lambda \notin \mathrm{Exp}(A)\},$$

where $\partial_A(A\backslash A^{-1})$ denotes the norm boundary of $A\backslash A^{-1}$ in A and $\mathrm{Exp}(A)$ the set $\{e^{a_1}e^{a_2}\cdots e^{a_k} : k \in \mathbb{N},\ a_1, \ldots, a_k \in A\}$. Note that $\mathrm{Exp}(A)$ equals the component $\mathrm{Comp}_A(1, A^{-1})$ of 1 in A^{-1} (see, for instance, [1, Theorem 3.3.7]).

Both the boundary spectrum and the exponential spectrum of a are nonempty compact subsets of the complex plane, for every $a \in A$, and

$$\partial\sigma(a) \subseteq S_\partial(a) \subseteq \sigma(a) \subseteq \varepsilon(a) \subseteq \eta\sigma(a).$$

The boundary spectrum plays a role in spectral continuity if A is an ordered Banach algebra (see [6]) and the exponential spectrum is of importance in the study of Fredholm theory (see [2]).

Let $K(\mathbb{C})$ denote the metric space of all compact subsets of $\mathbb{C}$ with the Hausdorff metric (see [1, page 48] and also [6, page 1778]). A mapping $\omega : A \to K(\mathbb{C})$ is said to be a *Mobius spectrum* [3] on A if $\omega(f(a)) = f(\omega(a))$ for all $a \in A$, for all functions f of the form $f(\lambda) = \alpha\lambda + \beta$ ($\alpha, \beta \in \mathbb{C}$) and for $f(\lambda) = 1/\lambda$, such that f is well defined on $\omega(a) \cup \sigma(a)$. Note that σ, ε, $\partial\sigma$, $\eta\sigma$ [3] and S_∂ [5] are Mobius spectra on any Banach algebra.

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Let A be a complex Banach algebra with unit 1 and B a closed subalgebra of A such that $1 \in B$. If $K \subseteq A$ is arbitrary, then (see [8, Definition 3.2]) the *restricted closure* of K in A relative to B is given by

$$\mathrm{cl}^B(K) = \{a \in A : \text{for all } U \in \mathrm{Nbd}_B(0), (a - U) \cap K \neq \emptyset\},$$

where $\mathrm{Nbd}_B(0)$ indicates the set of all neighbourhoods of 0 in B in the relative (norm) topology. The restricted closure does, in fact, define a topology (see [8, Theorem 3.5]), called the *restricted topology* or the *B-topology*. It is also shown in [8] that, in the case where $K \subseteq B$, the restricted closure of K in A coincides with the relative closure of K in B, and if $B = A$, then the B-topology coincides with the norm topology. The restricted boundary and interior of K in A will be denoted by $\partial^B(K)$ and $\mathrm{Int}^B(K)$, respectively, and the norm closure, boundary and interior of K in A by $\mathrm{cl}_A(K)$, $\partial_A(K)$ (as above with $K = A\backslash A^{-1}$) and $\mathrm{Int}_A(K)$, respectively. We recall the following from [8] (where more general versions were given). If $K \subseteq A$, then

$$\mathrm{Int}^B(K) = \{a \in A : \text{there exists } U \in \mathrm{Nbd}_B(0), a - U \subseteq K\}, \tag{2.1}$$

$$\partial^B(K) = \{a \in A : \text{for all } U \in \mathrm{Nbd}_B(0), (a - U) \cap K \neq \emptyset \text{ and } (a - U) \cap A\backslash K \neq \emptyset\} \tag{2.2}$$

and, with the set of all neighbourhoods of a in A in the B-topology denoted by $\mathrm{Nbd}^B(a)$,

$$\mathrm{Nbd}^B(a) = \{U \subseteq A : a - U \in \mathrm{Nbd}_B(0)\} \tag{2.3}$$

for all $a \in A$.

In [8], we then defined

$$\eta^B(A\backslash A^{-1}) = A\backslash \mathrm{Comp}^B(1, A^{-1}),$$

where $\mathrm{Comp}^B(1, A^{-1})$ is the component of A^{-1} containing 1 in the B-topology.

Finally, the *restricted boundary* and the *restricted connected hull of an element* $a \in A$ were defined in [8] as

$$\partial_B(a) := \partial_B(a, A) = \{\lambda \in \mathbb{C} : a - \lambda \in \partial^B(A\backslash A^{-1})\} \tag{2.4}$$

and

$$\eta_B(a) := \eta_B(a, A) = \{\lambda \in \mathbb{C} : a - \lambda \in \eta^B(A\backslash A^{-1})\},$$

respectively. The following results regarding these sets have been obtained in [8], where we recall that $\sigma(a) := \sigma(a, A)$.

THEOREM 2.1 [8, Corollary 6.4]. *For any $a \in A$,*

$$\partial_{\mathbb{C}}(a) \subseteq \partial_B(a) \subseteq \partial_A(a),$$

with $\partial_{\mathbb{C}}(a) = \partial\sigma(a)$ and $\partial_A(a) = S_\partial(a)$.

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THEOREM 2.2 [8, Corollary 6.9]. *For any $a \in A$,*

$$\eta_A(a) \subseteq \eta_B(a) \subseteq \eta_{\mathbb{C}}(a),$$

with $\eta_A(a) = \varepsilon(a)$. In addition, if $a \notin \mathbb{C}$, then $\eta_{\mathbb{C}}(a) = \mathbb{C}$, while if $a \in \mathbb{C}$, then $\eta_{\mathbb{C}}(a) = \eta\sigma(a)$.

In conclusion, we recall that an *ordered Banach algebra (OBA)* is a complex unital Banach algebra which contains an *algebra cone*, that is, a subset C with the properties that C contains the unit and is closed under addition, nonnegative real scalar multiplication and multiplication (see [11]). An algebra cone C induces a partial order on A: $a \leq b$ if and only if $b - a \in C$, and C is *normal* if there exists a constant $\alpha > 0$ such that $\|a\| \leq \alpha\|b\|$ whenever $0 \leq a \leq b$.

THEOREM 2.3 [11, Proposition 5.1]. *Let A be an OBA with a closed and normal algebra cone C. If $a \in C$, then $r(a) \in \sigma(a)$.*

Throughout the rest of this paper, A will be a complex Banach algebra with unit 1 and B a closed subalgebra of A such that $1 \in B$.

If $a \in B$, then we denote the open ball in B with centre a and radius ε by $B_B(a, \varepsilon)$. We note that, if $a \in A$, then

$$a - B_B(0, \varepsilon) = B_A(a, \varepsilon) \cap (a - B), \tag{2.5}$$

and so, if $a \in B$, then

$$a - B_B(0, \varepsilon) = B_B(a, \varepsilon). \tag{2.6}$$

3. Mapping and regularity-type properties

Similarly to the definition of the spectrum of an element a, we can equivalently write $a - \lambda$ or $\lambda - a$ in the definition of $\partial_B(a)$ (see (2.4)). To see this, we note that if $U \subseteq A$, then clearly $U \in \mathrm{Nbd}_B(0)$ if and only if $-U \in \mathrm{Nbd}_B(0)$ (and hence U is open in A in the B-topology if and only if $-U$ is open in A in the B-topology). Therefore, by (2.3), $U \in \mathrm{Nbd}^B(-a)$ if and only if $-a - U \in \mathrm{Nbd}_B(0)$ if and only if $a - (-U) \in \mathrm{Nbd}_B(0)$ if and only if $-U \in \mathrm{Nbd}^B(a)$.

LEMMA 3.1. *If $K \subseteq A$ is such that $K = -K$, then $\partial^B(K) = -\partial^B(K)$.*

PROOF. Let $a \in \partial^B(K)$. If $U \in \mathrm{Nbd}^B(-a)$, then $-U \in \mathrm{Nbd}^B(a)$, so that $-U$ contains a point b of K as well as a point $c \in A\backslash K$. If $K = -K$, it then follows that $-b \in U \cap K$ and $-c \in U \cap (A\backslash K)$, and so $-a \in \partial^B(K)$. □

Since $A^{-1} = -A^{-1}$, it now follows that $\partial_B(a)$ can, indeed, equivalently be written as

$$\{\lambda \in \mathbb{C} : \lambda - a \in \partial^B(A\backslash A^{-1})\}$$

for any $a \in A$. (This also follows from [8, Lemma 6.12].)

In [9], a number of mapping properties of η_B were established. In particular, it was shown that η_B is a Mobius spectrum on B [9, Proposition 3.1]. Now, if $a \in B$, then since

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$\partial_B(a,B) = S_\partial(a,B)$ and S_∂ is a Mobius spectrum on any Banach algebra, an argument similar to that for η_B shows that ∂_B is a Mobius spectrum on B.

In [8], the following mapping property of ∂_B was presented.

THEOREM 3.2 [8, Theorem 6.15]. *Let $a \in B$ and let f be a complex valued function which is analytic and one-to-one on a neighbourhood of $\sigma(a,B)$. Then $\partial_B(f(a)) = f(\partial_B(a))$.*

In Propositions 3.4 and 3.5, as well as in Theorem 3.10, we provide a few more simple mapping properties of ∂_B, which extend results from [4, 5]. We need the following lemma.

LEMMA 3.3. *If $a \in A$ and $0 \in \partial\sigma(a)$, then $a \in \partial^B(A\backslash A^{-1})$.*

PROOF. If $0 \in \partial\sigma(a)$, then $0 \in \sigma(a)$ and for every $\varepsilon > 0$, there exists $\lambda \notin \sigma(a)$ such that $|\lambda| < \varepsilon$. It follows that $a - U$ contains a point of A^{-1} as well as a point of $A\backslash A^{-1}$ for every $U \in \mathrm{Nbd}_{\mathbb{C}}(0)$, and so, by (2.2), $a \in \partial^{\mathbb{C}}(A\backslash A^{-1})$. Then $a \in \partial^B(A\backslash A^{-1})$, by [8, Corollary 3.12]. □

Note that Lemma 3.3 implies that $0 \in \partial^B(A\backslash A^{-1})$.

For $a \in B$, most of the next result follows immediately from Theorem 3.2. We also provide a direct short proof which establishes the property for all $a \in A$.

PROPOSITION 3.4. *Let $a \in A$ and $f(\lambda) = \alpha\lambda + \beta$ with $\alpha, \beta \in \mathbb{C}$. Then $\partial_B(f(a)) = f(\partial_B(a))$.*

PROOF. By [8, Corollary 3.12], $\partial^B(A\backslash A^{-1}) \subseteq \partial^A(A\backslash A^{-1})$, and since $\partial^A(A\backslash A^{-1}) \cap \mathbb{C} = \{0\}$, it follows from Lemma 3.3 that the case where $\alpha = 0$ is established. Now, let $\alpha \neq 0$ and let $\lambda \in \partial_B(f(a))$, so that $\alpha a + \beta - \lambda \in \partial^B(A\backslash A^{-1})$. If $\mu = (1/\alpha)(\lambda - \beta)$, then $\lambda = f(\mu)$ and, since products of elements of $\partial^B(A\backslash A^{-1})$ with elements of $A^{-1} \cap B$ stay in $\partial^B(A\backslash A^{-1})$ (see [8, Lemma 6.12]), we have $a - \mu = (1/\alpha)(\alpha a + \beta - \lambda) \in \partial^B(A\backslash A^{-1})$, so that $\mu \in \partial_B(a)$. The other inclusion is proved in a similar way. □

PROPOSITION 3.5. *If $a \in A^{-1} \cap B$, then $\partial_B(a^{-1}) \subseteq (\partial_B(a))^{-1}$. Hence, if $a \in B^{-1}$, then $(\partial_B(a))^{-1} = \partial_B(a^{-1})$.*

PROOF. For any $\lambda \neq 0$, we have $a^{-1} - \lambda = \lambda(1/\lambda - a)a^{-1}$ and since $\partial_B(a^{-1}) \subseteq \sigma(a^{-1})$, zero is not in $\partial_B(a^{-1})$. Hence, if $\lambda \in \partial_B(a^{-1})$, then $\lambda(1/\lambda - a)a^{-1} \in \partial^B(A\backslash A^{-1})$, so that $1/\lambda - a \in \partial^B(A\backslash A^{-1})$, by [8, Lemma 6.12]. It follows from the remark following the proof of Lemma 3.1 that $1/\lambda \in \partial_B(a)$, and so $\lambda \in (\partial_B(a))^{-1}$. □

Note that the second part of Proposition 3.5 also follows from Theorem 3.2.

We now establish a theorem which will lead to two useful corollaries.

THEOREM 3.6. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $a_1, \ldots, a_n \in A$.*

(1) *If $a_1 a_2 \cdots a_n \in \mathrm{Int}^A(A\backslash A^{-1})$, then $a_j \notin \partial^B(A\backslash A^{-1})$ for at least one $j = 1, \ldots, n$.*
(2) *If $a_1, \ldots, a_n \in B$ and $a_1 a_2 \cdots a_n \in \mathrm{Int}^B(A\backslash A^{-1})$, then $a_j \notin \partial^B(A\backslash A^{-1})$ for at least one $j = 1, \ldots, n$.*

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PROOF. (1) If $a_j = 0$ for at least one $j = 1, \dots, n$, then the assumption implies that $0 \in \mathrm{Int}^A(A\backslash A^{-1})$. By applying [8, Corollary 3.12], we then obtain a contradiction from the remark following Lemma 3.3. Therefore, our assumption implies that $a_j \neq 0$ for all $j = 1, \dots, n$.

If $a_1 a_2 \cdots a_n \in \mathrm{Int}^A(A\backslash A^{-1})$, say $B_A(a_1 a_2 \cdots a_n, \varepsilon_0) \subseteq A\backslash A^{-1}$, then it follows from [5, Corollary 3.3] that there exist $\varepsilon_j > 0$ $(j = 1, \dots, n)$ such that

$$B_A(a_1, \varepsilon_1) B_A(a_2, \varepsilon_2) \cdots B_A(a_n, \varepsilon_n) \subseteq A\backslash A^{-1},$$

and so it follows from (2.5) that

$$(a_1 - B_B(0, \varepsilon_1))(a_2 - B_B(0, \varepsilon_2)) \cdots (a_n - B_B(0, \varepsilon_n)) \subseteq A\backslash A^{-1}. \tag{3.1}$$

Now suppose $a_j \in \partial^B(A\backslash A^{-1})$ for all $j = 1, \dots, n$. Then, for each $j = 1, \dots, n$, there exists $c_j \in (a_j - B_B(0, \varepsilon_j)) \cap A^{-1}$, by (2.2). However, then $c_1 c_2 \cdots c_n \in A\backslash A^{-1} \cap A^{-1}$, by (3.1). This contradiction yields the result.

(2) As in part (1), we have $a_j \neq 0$ for all $j = 1, \dots, n$. If $a_1, \dots, a_n \in B$ and $a_1 a_2 \cdots a_n \in \mathrm{Int}^B(A\backslash A^{-1})$, then, by (2.2) and (2.6), there exists $\varepsilon_0 > 0$ such that

$$B_B(a_1 a_2 \cdots a_n, \varepsilon_0) \subseteq A\backslash A^{-1}.$$

As before, by [5, Corollary 3.3], there exists $\varepsilon_j > 0$ $(j = 1, \dots, n)$ such that

$$B_B(a_1, \varepsilon_1) B_B(a_2, \varepsilon_2) \cdots B_B(a_n, \varepsilon_n) \subseteq A\backslash A^{-1}.$$

Now suppose that $a_j \in \partial^B(A\backslash A^{-1})$ for all $j = 1, \dots, n$. Then, for each $j = 1, \dots, n$, there exists $c_j \in B_B(a_j, \varepsilon_j) \cap A^{-1}$ and, as before, we obtain a contradiction. □

Theorem 3.6 yields the next two corollaries which can be used to establish certain mapping and regularity-type properties of ∂_B (see Theorem 3.10 and Proposition 3.11).

COROLLARY 3.7. *Let $n \in \mathbb{N}$ with $n \geq 2$ and let $a_1, \dots, a_n \in A$ be mutually commutative elements.*

(1) *If $a_j \in \partial^B(A\backslash A^{-1})$ for all $j = 1, \dots, n$, then $a_1 a_2 \cdots a_n \in \partial^A(A\backslash A^{-1})$.*
(2) *If $a_1, \dots, a_n \in B$ and $a_j \in \partial^B(A\backslash A^{-1})$ for all $j = 1, \dots, n$, then $a_1 a_2 \cdots a_n \in \partial^B(A\backslash A^{-1})$.*

PROOF. Suppose that $a_j \in \partial^B(A\backslash A^{-1})$ for all $j = 1, \dots, n$. Then, by [8, Corollary 3.12], $a_j \in A\backslash A^{-1}$ for all $j = 1, \dots, n$ and, therefore, $a_1 a_2 \dots, a_n \in A\backslash A^{-1}$. By applying Theorem 3.6(1), the assumption also yields $a_1 a_2 \dots, a_n \notin \mathrm{Int}^A(A\backslash A^{-1})$, from which the conclusion follows. This proves part (1), and a similar proof, applying Theorem 3.6(2), yields part (2). □

COROLLARY 3.8. *Let $a \in A$.*

(1) *If $a \in \partial^B(A\backslash A^{-1})$, then $a^n \in \partial^A(A\backslash A^{-1})$ for all $n \in \mathbb{N}$. Hence, if $0 \in \partial_B(a)$, then $0 \in \partial_A(a^n)$ for all $n \in \mathbb{N}$.*
(2) *If $a \in B$ and $a \in \partial^B(A\backslash A^{-1})$, then $a^n \in \partial^B(A\backslash A^{-1})$ for all $n \in \mathbb{N}$. Hence, if $a \in B$ and $0 \in \partial_B(a)$, then $0 \in \partial_B(a^n)$ for all $n \in \mathbb{N}$.*

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Using Corollary 3.8(2) and building on [5, Example 3.7], we can now show that there exists a complex Banach algebra A such that A has a (proper) closed subalgebra B containing the unit and an element $a \in B$ such that $\partial\sigma(a) \subsetneq \partial_B(a)$.

EXAMPLE 3.9. Let $K = \{re^{i\theta} \in \mathbb{C} : r \in [0,1] \text{ and } \theta \in [0, \pi/2]\}$. Also, let A be the Banach algebra $C(K)$ of all continuous complex valued functions on K and let B be the closed subalgebra $\mathcal{A}(K)$ of A consisting of all functions which are continuous on K and analytic on its interior. If $g(z) = z^2$ for all $z \in K$ and $a = g^2$, then $a \in B$ and $\partial\sigma(a) \subsetneq \partial_B(a)$.

PROOF. Since $\sigma(a) := \sigma(a, A)$ is the closed unit disk, $0 \notin \partial\sigma(a)$. However, since $\sigma(g) = \{re^{i\theta} \in \mathbb{C} : r \in [0,1] \text{ and } \theta \in [0,\pi]\}$, we have $0 \in \partial\sigma(g)$, so that $0 \in \partial_B(g)$. Finally, it follows from Corollary 3.8(2) that $0 \in \partial_B(a)$, which yields the result. □

We write $K^2 = f(K)$, where $f(\lambda) = \lambda^2$ and $K \subseteq \mathbb{C}$.

THEOREM 3.10. *Let $a \in A$ such that $\partial_B(a) = -\partial_B(a)$ is symmetric with respect to the origin. Then:*

(1) $(\partial_B(a))^2 \subseteq \partial_A(a^2)$;
(2) *if, in addition, $a \in B$, then $(\partial_B(a))^2 \subseteq \partial_B(a^2)$.*

PROOF. (1) If $\lambda \in \partial_B(a)$, then $-\lambda \in \partial_B(a)$, and so $\lambda^2 - a^2 = (\lambda - a)(\lambda + a) \in \partial^A(A\backslash A^{-1})$, by Lemma 3.1 and Corollary 3.7(1), which yields $\lambda^2 \in \partial_A(a^2)$.

(2) This follows by a similar argument using Corollary 3.7(2). □

Now let $R_B = A^{-1} \cup \mathrm{Int}^B(A\backslash A^{-1})$. Then $R_B = A\backslash\partial^B(A\backslash A^{-1})$, by [8, Corollary 3.12], and we have

$$\partial_B(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin R_B\} = \{\lambda \in \mathbb{C} : a - \lambda \notin A^{-1} \cup \mathrm{Int}^B(A\backslash A^{-1})\}.$$

Note that R_A coincides with the set R_∂ in [5].

Next, we will discuss a few regularity-type properties of R_B. We refer to [10], where the definitions and properties of upper and lower semiregularities are presented. In particular, by [10, Remark 3, page 212], a nonempty subset R of A satisfying

(P1) if $a, b \in A$ such that $ab = ba$ and $ab \in R$, then $a, b \in R$

is a lower semiregularity. Also, it clearly follows from the definition and [10, Lemma 2(iii), page 212], that a lower semiregularity satisfies the following property:

(P2) if $a \in A$ and $b \in A^{-1}$ such that $ab = ba$, then $ab \in R$ if and only if $a \in R$.

We first note that although R_B is in general neither a lower semiregularity (see [12, Example 2.1(1)]), nor an upper semiregularity (see [12, Example 3.2]) on B, R_B does satisfy the following properties, of which part (2) is a weaker version of property (P1).

PROPOSITION 3.11. *Let $a, b \in B$.*

(1) *If $n \in \mathbb{N}$ and $a^n \in R_B$, then $a \in R_B$.*
(2) *If $ab = ba$ and $ab \in R_B$, then $a \in R_B$ or $b \in R_B$.*

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Parts (1) and (2) in the above result follow easily from Corollaries 3.8(2) and 3.7(2), respectively.

The set R_B also satisfies the following property, which is in some sense weaker, and in another sense stronger, than property (P2).

PROPOSITION 3.12. *Let $a \in A$ and $b \in B^{-1}$. Then $ab \in R_B$ if and only if $a \in R_B$ if and only if $ba \in R_B$.*

PROOF. If $b \in B^{-1}$, then it follows from [8, Lemma 6.12] that $ab \notin R_B$ if and only if $ab \in \partial^B(A\backslash A^{-1})$ if and only if $a \in \partial^B(A\backslash A^{-1})$ if and only if $a \notin R_B$. Similarly, $ba \in R_B$ if and only if $a \in R_B$. □

We note that since $\mathrm{Exp}(A) \subseteq A^{-1} \subseteq R_B$, [5, Theorem 4.4] and its obvious analogue give the following characterisation of the (Jacobson) radical:

$$\mathrm{Rad}(A) = \{a \in A : R_B a \subseteq \mathrm{QN}(A)\} = \{a \in A : aR_B \subseteq \mathrm{QN}(A)\} \tag{3.2}$$

for any closed subalgebra B of A such that $1 \in B$.

Finally, we exhibit a variation of Harte's theorem [2, Theorem 3] involving the restricted boundary.

THEOREM 3.13. *Let A_1 and A_2 be Banach algebras with the same unit 1, B a closed subalgebra of both A_1 and A_2 such that $1 \in B$, and $T : A_1 \to A_2$ a homomorphism which is either bounded with closed range or satisfies the strong Riesz property. Then, for all $a \in A_1$,*

$$\bigcap_{Tb=0} \partial_B(a+b, A_1) \subseteq \eta\partial_B(Ta, A_2).$$

Theorem 3.13 follows directly from [7, Theorem 5.4], which generalises [2, Theorem 3], via the use of abstract mappings. (Here, T has the *strong Riesz property* if $\sigma(a) \subseteq \eta\sigma(Ta) \cup \mathrm{iso}\,\sigma(a)$ for all $a \in A_1$, with $\mathrm{iso}\,\sigma(a)$ denoting the set of all isolated points of $\sigma(a)$.)

4. Continuity properties

For a set K in $\mathbb{C}$, let $|K|$ indicate the set $\{|\lambda| : \lambda \in K\}$ in this section.

We first note that $\partial_B(a)$ is a compact set in $\mathbb{C}$ (see the remark following Corollary 4.2), and then show that the function $a \mapsto \partial_B(a)$ from A into $K(\mathbb{C})$ is upper semicontinuous on B in the norm topology (see Theorem 4.5). We start with the following lemma.

LEMMA 4.1. *Let $a \in A$, let F be a subset of A which is closed in A in the B-topology and $K = \{\lambda \in \mathbb{C} : a - \lambda \in F\}$. Then let K is closed in $\mathbb{C}$.*

PROOF. Let $\lambda \in \mathbb{C}\backslash K$. Then, $a - \lambda \in A\backslash F$, and since $A\backslash F$ is open in A in the B-topology, it follows from (2.1) that there exists $\varepsilon > 0$ such that $a - \lambda - B_B(0, \varepsilon) \subseteq A\backslash F$. It follows that $B_{\mathbb{C}}(\lambda, \varepsilon) \subseteq \mathbb{C}\backslash K$, and so $\mathbb{C}\backslash K$ is open in $\mathbb{C}$. □

Since $\partial^B(A\backslash A^{-1})$ is closed in A in the B-topology, we have the following corollary.

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COROLLARY 4.2. *If $a \in A$, then the set $\partial_B(a)$ is closed in $\mathbb{C}$.*

Since $\partial\sigma(a) \subseteq \partial_B(a) \subseteq \sigma(a)$, it follows that $\partial_B(a)$ is a nonempty, compact set in $\mathbb{C}$, for all $a \in A$.

To establish the upper semicontinuity of certain maps, we start with the following two lemmas (recalling that $\sigma(x) := \sigma(x, A)$ for any $x \in A$ (or B)).

LEMMA 4.3. *Let $\phi : A \to K(\mathbb{C})$ be a function with the property that either $\phi(x) \subseteq \sigma(x)$ for all $x \in A$ or $|\phi(x)| \subseteq \sigma(x)$ for all $x \in A$. In addition, let $a \in B$ and let U be an open set in $\mathbb{C}$ containing $\phi(a)$. If there is no $\delta > 0$ such that U contains $\phi(x)$ for all $x \in B$ such that $\|x - a\| < \delta$, then there exist a sequence (a_n) in B such that $a_n \to a$ as $n \to \infty$ and a convergent sequence (α_n) in $\mathbb{C}$ such that $\alpha_n \in \phi(a_n)$ for all $n \in \mathbb{N}$, but $\lim_{n\to\infty} \alpha_n \notin \phi(a)$.*

PROOF. If there is no such δ, then there exist a sequence (a_n) in B such that $a_n \to a$ as $n \to \infty$ and a sequence (α_n) in $\mathbb{C}$ such that $\alpha_n \in \phi(a_n) \cap \mathbb{C}\backslash U$ for all $n \in \mathbb{N}$. It follows from the assumption that $|\alpha_n| \leq r(a_n) \leq \|a_n\|$ for all $n \in \mathbb{N}$, and since (a_n) is a bounded sequence, we may assume without loss of generality that (α_n) is convergent, say $\alpha_n \to \alpha$ as $n \to \infty$. The fact that U is an open set implies that $\alpha \notin U$, and hence $\alpha \notin \phi(a)$. □

LEMMA 4.4. *Let (a_n) be a sequence in B and $a \in B$ such that $a_n \to a$ as $n \to \infty$. If $a \notin \partial^B(A\backslash A^{-1})$, then there exists $N \in \mathbb{N}$ such that $a_n \notin \partial^B(A\backslash A^{-1})$ for all $n \geq N$.*

PROOF. Since $A\backslash\partial^B(A\backslash A^{-1})$ is open in A in the B-topology, it follows from $a \notin \partial^B(A\backslash A^{-1})$ that $a \in \mathrm{Int}^B(A\backslash\partial^B(A\backslash A^{-1}))$. Hence, by (2.1), there exists $\varepsilon > 0$ such that $a - B_B(0, \varepsilon) \subseteq A\backslash\partial^B(A\backslash A^{-1})$, that is, $B_B(a, \varepsilon) \subseteq A\backslash\partial^B(A\backslash A^{-1})$, by (2.6). However, since $a_n \to a$ as $n \to \infty$, we have $\|a_n - a\| < \varepsilon$ for all n large enough, which yields the result. □

THEOREM 4.5. *If $a \in B$ and U is an open set in $\mathbb{C}$ containing $\partial_B(a)$, then there exists $\delta > 0$ such that if $x \in B$ and $\|x - a\| < \delta$, then U contains $\partial_B(x)$, that is, the map $a \mapsto \partial_B(a)$ from A into $K(\mathbb{C})$ is upper semicontinuous on B in the norm topology.*

PROOF. If this is not the case, then, taking $\phi(x) = \partial_B(x)$ $(x \in A)$, it follows from Lemma 4.3 that there exist a sequence (a_n) in B such that $a_n \to a$ as $n \to \infty$, and a convergent sequence (α_n) in $\mathbb{C}$, say $\alpha_n \to \alpha$ as $n \to \infty$, such that $\alpha_n \in \partial_B(a_n)$ for all $n \in \mathbb{N}$, but $\alpha \notin \partial_B(a)$, that is, $a - \alpha \notin \partial^B(A\backslash A^{-1})$. By Lemma 4.4, $a_n - \alpha_n \notin \partial^B(A\backslash A^{-1})$ for all n large enough. This contradicts the fact that $\alpha_n \in \partial_B(a_n)$ for all $n \in \mathbb{N}$. □

Finally, we develop a spectral continuity result in OBAs (Theorem 4.8) using ∂_B. We need some preliminary theory and results.

For any $a \in A$, let

$$|\partial|_B(a) := \{\lambda \in \mathbb{C} : a - |\lambda| \in \partial^B(A\backslash A^{-1})\}.$$

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By Lemma 3.1, $|\partial|_B(a) = \{\lambda \in \mathbb{C} : |\lambda| - a \in \partial^B(A\backslash A^{-1})\}$, $|\partial|_A(a)$ equals the set $T(a)$ of [6] and

$$|\partial|_B(a) = \{\lambda \in \mathbb{C} : |\lambda| \in \partial_B(a)\} = \{\lambda \in \mathbb{C} : |\lambda| \in \partial_B(a) \cap \mathbb{R}^+\},$$

so that $||\partial|_B(a)| \subseteq \partial_B(a)$. Clearly, $|\partial|_B(a)$ is a compact subset of $\mathbb{C}$.

We now prove an important property of the map $a \mapsto |\partial|_B(a)$.

THEOREM 4.6. *Define $\phi : A \to K(\mathbb{C})$ by $\phi(a) = |\partial|_B(a)$ for all $a \in A$. Then ϕ is upper semicontinuous on B (in the norm topology).*

PROOF. If not, then there exist $a \in B$ and an open set U in $\mathbb{C}$ containing $|\partial|_B(a)$ such that there is no $\delta > 0$ with the property that U contains $|\partial|_B(x)$ for all $x \in B$ such that $\|x - a\| < \delta$. It follows from Lemma 4.3 that there exist a sequence (a_n) in B such that $a_n \to a$ as $n \to \infty$ and a convergent sequence (α_n) in $\mathbb{C}$, say $\alpha_n \to \alpha$ as $n \to \infty$, such that $\alpha_n \in |\partial|_B(a_n)$ for all $n \in \mathbb{N}$, but $\alpha \notin |\partial|_B(a)$, that is, $a - |\alpha| \notin \partial^B(A\backslash A^{-1})$. As in the proof of Theorem 4.5, it follows from Lemma 4.4 that $a_n - |\alpha_n| \notin \partial^B(A\backslash A^{-1})$ for all n large enough, which contradicts the fact that $\alpha_n \in |\partial|_B(a_n)$ for all $n \in \mathbb{N}$. □

By applying Theorem 2.3, we have the following result in OBAs.

PROPOSITION 4.7. *If A is an OBA with a closed and normal algebra cone C, B a closed subalgebra of A such that $1 \in B$ and $a \in C$, then $r(a) \in \partial_B(a)$. Hence $r(a) \in |\partial|_B(a)$ and $|\partial|_B(a) \neq \emptyset$.*

We now arrive at our spectral continuity result for OBAs, which extends [6, Theorem 4.6].

THEOREM 4.8. *Let A be an OBA with a closed and normal algebra cone C, B a closed subalgebra of A such that $1 \in B$ and $a \in C \cap B$ such that $\partial_B(a) \cap \mathbb{R}^+ = \{r(a)\}$. If (a_n) is in $C \cap B$ such that $a_n \to a$ as $n \to \infty$ (in the norm topology), then $r(a_n) \to r(a)$.*

PROOF. Note that $\partial_B(a) \cap \mathbb{R}^+ = \{r(a)\}$ if and only if $|\partial|_B(a)$ equals the circle with centre 0 and radius $r(a)$, and hence $d(\lambda, |\partial|_B(a)) = |r(a) - \lambda|$ for all $\lambda \in \mathbb{R}^+$. Let $\varepsilon > 0$ and let $U = \{\lambda \in \mathbb{C} : d(\lambda, |\partial|_B(a)) < \varepsilon\}$. Then, U is an open set and $|\partial|_B(a) \subseteq U$. Since $a_n \to a$ as $n \to \infty$, it follows from Theorem 4.6 that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then $|\partial|_B(a_n) \subseteq U$. Now, Proposition 4.7 implies that $r(a_n) \in |\partial|_B(a_n)$ for all $n \in \mathbb{N}$, and hence $r(a_n) \in U$ for all $n \geq N$. Therefore, $d(r(a_n), |\partial|_B(a)) < \varepsilon$, that is, $|r(a) - r(a_n)| < \varepsilon$ for all $n \geq N$. □

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SONJA MOUTON, Department of Mathematical Sciences,
Stellenbosch University, Stellenbosch, South Africa
e-mail: smo@sun.ac.za

ROBIN HARTE, School of Mathematics,
Trinity College, Dublin, Ireland
e-mail: rharte@maths.tcd.ie

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