Ergod. Th. & Dynam. Sys. (2015), **35**, 2587–2612 © *Cambridge University Press, 2014*
doi:10.1017/etds.2014.41

Mean equicontinuity and mean sensitivity

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(*Received* 14 *January* 2014 *and accepted in revised form* 7 *April* 2014)

Abstract. Answering an open question affirmatively it is shown that every ergodic invariant measure of a mean equicontinuous (i.e. mean-L-stable) system has discrete spectrum. Dichotomy results related to mean equicontinuity and mean sensitivity are obtained when a dynamical system is transitive or minimal. Localizing the notion of mean equicontinuity, notions of almost mean equicontinuity and almost Banach mean equicontinuity are introduced. It turns out that a system with the former property may have positive entropy and meanwhile a system with the latter property must have zero entropy.

1. *Introduction*

Let X be a compact metric space with a metric d, and let T be a continuous map from X to itself. The pair (X, T) will be called a *(topological) dynamical system.*

A dynamical system (X, T) is called *equicontinuous* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$, $d(T^n x, T^n y) < \varepsilon$ for $n = 0, 1, 2, \ldots$, that is, the family of maps $\{T^n \mid n \in \mathbb{Z}_+\}$ is uniformly equicontinuous. Equicontinuous systems have simple dynamical behaviors. It is well known that a dynamical system (X, T) with T being surjective is equicontinuous if and only if there exists a compatible metric ρ on X such that T acts on X as an isometry, i.e. $\rho(Tx, Ty) = \rho(x, y)$ for any $x, y \in X$. Moreover, a transitive equicontinuous system is conjugate to a minimal rotation on a compact abelian metric group, and (X, T, μ) has discrete spectrum, where μ is the unique Haar measure on X.

When studying dynamical systems with discrete spectrum, Fomin [**8**] introduced a notion called *stable in the mean in the sense of Lyapunov* or simply *mean-L-stable*. A dynamical system (X, T) is *mean-L-stable* if for every $\varepsilon > 0$, there is a $\delta > 0$ such

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that $d(x, y) < \delta$ implies $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$ except a set of upper density less than ε. Fomin proved that if a minimal system is mean-L-stable then it is uniquely ergodic. Mean-L-stable systems are also discussed briefly by Oxtoby in [**26**], and he proved that each transitive mean-L-stable system is uniquely ergodic. Auslander in [**2**] systematically studied mean-L-stable systems, and provided new examples. See Scarpellini [**27**] for a related work. Whether every ergodic invariant measure on a mean-L-stable system has discrete spectrum is an open question [**27**]. We will give an affirmative answer to this question (Theorem 3.8).

We introduce equicontinuity in the mean sense; more precisely, a dynamical system (X, T) is called *mean equicontinuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

We show that a dynamical system is mean equicontinuous if and only if it is mean-L-stable (Lemma 3.1). We also show each dynamical system admits a maximal mean equicontinuous factor (Theorem 3.10), and mean equicontinuity is preserved by factor maps (Theorem 3.11). We note that studying dynamical properties in the mean sense is an interesting topic, see [**25**] for the research on mean distality and [**6**, **18**] for the investigation on mean Li–Yorke chaos.

The notion of sensitivity was introduced when studying the complexity of a dynamical system, and it is a part of the known definition of chaos in the Devaney sense. We say that a dynamical system (X, T) has *sensitive dependence on initial condition* or briefly (X, T) is *sensitive* if there exists a $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ and $n \in \mathbb{N}$ such that $d(T^n x, T^n y) > \delta$.

When considering the opposite side of sensitivity, the notion of equicontinuity at a point appears naturally, see [**15**]. That is, a point $x \in X$ is called an *equicontinuous point* (or (X, T) is *equicontinuous at* x) if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $y \in X$ with $d(x, y) < \delta$, $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$. If every point in X is an equicontinuous point then by the compactness of X the dynamical system (X, T) is equicontinuous. A transitive system is called *almost equicontinuous* if there is at least one equicontinuous point. Almost equicontinuous systems have been studied intensively and have many applications. For example, the enveloping semigroup $E(X)$ is metrizable if and only if (X, T) is hereditarily almost equicontinuous [**14**].

We know that if (X, T) is almost equicontinuous then the set of equicontinuous points coincides with the set of all transitive points [**1**], it is uniformly rigid [**15**] and thus has zero topological entropy [**13**]. We have the following dichotomy results. If (X, T) is minimal, then (X, T) is either equicontinuous or sensitive [**4**]; and if (X, T) is transitive, then (X, T) is either almost equicontinuous or sensitive [**1**].

Inspired by the above ideas, we will introduce notions of almost mean equicontinuity and mean sensitivity. A point $x \in X$ is called *mean equicontinuous* if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $y \in X$ with $d(x, y) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

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A transitive system is called *almost mean equicontinuous* if there is at least one mean equicontinuous point. A dynamical system (X, T) is called *mean sensitive* if there exists a $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there exists $y \in U$ and $n \in \mathbb{N}$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

We show that if a dynamical system (X, T) is minimal, then (X, T) is either mean equicontinuous or mean sensitive (Corollary 5.5), and if (X, T) is transitive, then (X, T) is either almost mean equicontinuous or mean sensitive (Theorem 5.4). Unlike the case of almost equicontinuous systems, we show that for almost mean equicontinuous systems the set of transitive points is contained in the set of all mean equicontinuous points and there are examples in which they do not coincide. It is unexpected that there are almost mean equicontinuous systems admitting positive topological entropy (Theorem 4.7), while every almost equicontinuous system has zero topological entropy.

Thus, it is natural to seek a class of mean equicontinuous systems for which the localized systems at least have zero entropy. We find the class of *Banach mean equicontinuous systems* obtained by replacing small upper density with small Banach density in the definition of mean-L-stable systems is the right one. Namely, we show that almost Banach mean equicontinuous systems have zero topological entropy (Corollary 6.7), and this implies that the almost mean equicontinuous systems admitting positive topological entropy we construct in Theorem 4.7 are not almost Banach mean equicontinuous. The deep reason for this is that in a transitive system a transitive point can only approach a 'chaotic subsystem' for time segments, which may result in large Banach density and at the same time small upper density.

2. *Preliminaries*

In this section we recall some notions and aspects of the theory of topological dynamical systems.

2.1. *Subsets of non-negative integers.* Denote by $\mathbb{Z}_+$ ($\mathbb{N}$, $\mathbb{Z}$, respectively) the set of all non-negative integers (positive integers, integers, respectively). Let $F \subset \mathbb{Z}_+$. We say that F is an *IP-set* if there is a subsequence $\{p_i\}$ of $\mathbb{N}$ such that

$$\{p_{i_1} + \cdots + p_{i_k} \mid i_1 < \cdots < i_k, n \in \mathbb{N}\} \subset F;$$

F is *syndetic* if there is $k > 0$ such that $[i, i + k] \cap F \neq \emptyset$ for every $i \in \mathbb{N}$.

We define the *upper density* $\overline{D}(F)$ of F by

$$\overline{D}(F) = \limsup_{n\to\infty} \frac{\#(F \cap [0, n-1])}{n},$$

where $\#(\cdot)$ is the number of elements of a set. Similar, $\underline{D}(F)$, the *lower density* of F, is defined by

$$\underline{D}(F) = \liminf_{n\to\infty} \frac{\#(F \cap [0, n-1])}{n}.$$

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One may say F has *density* $D(F)$ if $\overline{D}(F) = \underline{D}(F)$, in which case $D(F)$ is equal to this common value. The *upper Banach density* $BD^*(F)$ is defined by

$$BD^*(F) = \limsup_{N-M\to\infty} \frac{\#(F \cap [M, N])}{N - M + 1}.$$

Similarly, we can define the *lower Banach density* $BD_*(F)$ and *Banach density* $BD(F)$.

2.2. *Compact metric spaces.* Let (X, d) be a compact metric space. For $x \in X$ and $\varepsilon > 0$, denote $B(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$. Denote the product space $\{(x, y) \mid x, y \in X\}$ by $X \times X$ and the diagonal $\{(x, x) \mid x \in X\}$ by Δ_X. A subset of X is called a G_δ *set* if it can be expressed as a countable intersection of open sets; a *residual set* if it contains the intersection of a countable collection of dense open sets. By the Baire category theorem, a residual set is also dense in X.

Let $C(X)$ be the set of continuous real functions on X with the supremum norm $\|f\| = \sup_{x\in X} |f(x)|$. Let $M(X)$ be the set of regular Borel probability measures on X. The *support* of a measure $\mu \in M(X)$, denoted by $\mathrm{supp}(\mu)$, is the smallest closed subset C of X such that $\mu(C) = 1$. We regard $M(X)$ as a closed convex subset of $C(X)^*$, the dual space of $C(X)$, equipped with the weak* topology. Then $M(X)$ is a compact metric space.

2.3. *Topological dynamics.* Let (X, T) be a dynamical system. The *orbit* of a point $x \in X$, $\{x, Tx, T^2x, \ldots, \}$, is denoted by $\mathrm{Orb}(x, T)$. The ω-limit set of x is the set of limit points of the orbit sequence

$$\omega(x, T) = \bigcap_{N\geq 0} \overline{\{T^n x \mid n \geq N\}}.$$

If A is a non-empty closed subset of X and $TA \subset A$, then $(A, T|_A)$ is called a *subsystem* of (X, T), where $T|_A$ is the restriction of T on A. If there is no ambiguity, we will use the notation T instead of $T|_A$.

We say that a point $x \in X$ is *recurrent* if $x \in \omega(x, T)$. The system (X, T) is called *(topologically) transitive* if $\omega(x, T) = X$ for some $x \in X$, and such a point x is called a *transitive point*. Denote by $\mathrm{Trans}(X, T)$ the set of transitive points of (X, T). With a Baire category argument, one can show that if (X, T) is transitive then $\mathrm{Trans}(X, T)$ is a dense G_δ subset of X. If the product system $(X \times X, T \times T)$ is transitive, then we say that (X, T) is *weakly mixing*.

The system (X, T) is said to be *minimal* if every point of X is a transitive point (i.e. $\mathrm{Trans}(X, T) = X$). A subset Y of X is called *minimal* if (Y, T) forms a minimal subsystem of (X, T). A point $x \in X$ is called *minimal* if it is contained in a minimal set Y or, equivalently, if the subsystem $(\overline{\mathrm{Orb}(x, T)}, T)$ is minimal.

For $x \in X$ and $A \subset X$, let $N(x, A) = \{n \in \mathbb{Z}_+ \mid T^n x \in A\}$. If U is a neighborhood of x, then the set $N(x, U)$ is called the set of *return times* of the point x to the neighborhood U. The following result is well known, see [**9**] for example.

LEMMA 2.1. *Let (X, T) be a dynamical system and $x \in X$. Then:*

(1) *x is recurrent if and only if $N(x, U)$ contains an IP-set for every neighborhood U of x;*
(2) *x is minimal if and only if $N(x, U)$ is syndetic for every neighborhood U of x.*

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A pair of points $(x, y) \in X \times X$ is said to be *proximal* if for any $\varepsilon > 0$, there exists a positive integer n such that $d(T^n x, T^n y) < \varepsilon$. Let $P(X, T)$ denote the collection of all proximal pairs in (X, T). The dynamical system (X, T) is called *proximal* if any pair of two points in X is proximal, i.e. $P(X, T) = X \times X$. If $(x, y) \in X \times X$ is not proximal, then it is said to be *distal*. A dynamical system (X, T) is called *distal* if any pair of distinct points in (X, T) is distal.

A pair of points $(x, y) \in X \times X$ is said to be *Banach proximal* if for any $\varepsilon > 0$, $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$ except a set of zero Banach density. Let $BP(X, T)$ denote the collection of all Banach proximal pairs in (X, T). See [**24**] for a detailed study on Banach proximality.

Recall that a pair of points (x, y) is called *regionally proximal* if for every $\varepsilon > 0$, there exist two points $x', y' \in X$ with $d(x, x') < \varepsilon$ and $d(y, y') < \varepsilon$, and a positive integer n such that $d(T^n x', T^n y') < \varepsilon$. Let $Q(X, T)$ be the set of all regionally proximal pairs in (X, T). Clearly, $Q(X, T) \supset P(X, T) \supset BP(X, T)$.

When (X, T) and (Y, S) are two dynamical systems and $\pi : X \to Y$ is a continuous onto map which intertwines the actions (i.e. $\pi \circ T = S \circ \pi$), one says that (Y, S) is a *factor* of (X, T) or (X, T) is an *extension* of (Y, S), and π is a *factor map*. If π is a homeomorphism, then we say that π is a *conjugacy* and that the dynamical systems (X, T) and (Y, S) are *conjugate*. By the Halmos and von Neumann theorem (see [**28**, Theorem 5.18]), a minimal system is equicontinuous if and only if it is conjugate to a minimal rotation on a compact abelian metric group.

If $\pi : (X, T) \to (Y, S)$ is a factor map, then $R_\pi = \{(x, x') \in X \times X : \pi(x) = \pi(x')\}$ is a closed $T \times T$-invariant equivalence relation, that is R_π is a closed subset of $X \times X$ and if $(x, x') \in R_\pi$, then $(Tx, Tx') \in R_\pi$. Conversely, if R is a closed $T \times T$-invariant equivalence relation on X, then the quotient space X/R is a compact metric space and T naturally induces an action on X/R by $T_R([x]) = [Tx]$. Then $(X/R, T_R)$ forms a dynamical system and the quotient map $\pi_R : X \to X/R$ is a factor map. Hence there is a one-to-one correspondence between factors and closed invariant equivalence relations; we will use them interchangeably. A factor map $\pi : (X, T) \to (Y, S)$ is called *proximal* (respectively *Banach proximal*) if whenever $\pi(x) = \pi(y)$ the pair (x, y) is proximal (respectively Banach proximal).

An equicontinuous factor of (X, T) is maximal if any other equicontinuous factor of (X, T) factors through it. It is thus unique up to conjugacy and therefore referred to as the *maximal equicontinuous factor*. Let $\pi : (X, T) \to (Y, S)$ be the factor map to the maximal equicontinuous factor. The equivalence relation R_π is called the *equicontinuous structure relation*. Similarly, we can define *the maximal distal factor* and the *distal structure relation*. It is shown in [**7**] that the equicontinuous structure relation is the smallest closed invariant equivalence relation containing the regional proximal relation, and the distal structure relation is the smallest closed invariant equivalence relation containing the proximal relation.

We refer the reader to the textbook [**28**] for information on topological entropy.

2.4. *Invariant measures.* Let (X, T) be a dynamical system and $M(X, T)$ be the set of T-invariant regular Borel probability measures on X. It is well known that

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any dynamical system (X, T) admits at least one T-invariant regular Borel probability measure. Moreover, we know that $M(X, T)$ is a compact metric space. An invariant measure is ergodic if and only if it is an extreme point of $M(X, T)$. The *support* of a dynamical system (X, T), denoted by $\mathrm{supp}(X, T)$, is the smallest closed subset C of X such that $\mu(C) = 1$ for all $\mu \in M(X, T)$.

The action of T on X induces an action on $M(X)$ in the following way: for $\mu \in M(X)$ we define $T\mu$ by

$$\int_X f(x)\, dT\mu(x) = \int_X f(Tx) d\mu(x) \quad \text{for all } f \in C(X).$$

Hence $(M(X), T)$ is also a topological dynamical system. We say (X, T) is *strongly proximal* if $(M(X), T)$ is proximal.

Let (X, T) be a dynamical system. We call (X, T) an *E-system* if it is transitive and there exists $\mu \in M(X, T)$ such that $\mathrm{supp}(\mu) = X$. We say that (X, T) is *uniquely ergodic* if $M(X, T)$ consists of a single measure. If (X, T) is a minimal rotation on a compact abelian metric group, then it is uniquely ergodic and the Haar measure is the only invariant measure.

For a dynamical system (X, T), $f \in C(X)$ and $n \in \mathbb{N}$, let $f_n(x) = (1/n) \sum_{i=0}^{n-1} f(T^i x)$. The following theorem is well known.

THEOREM 2.2. [**26**] *Let (X, T) be a dynamical system. Then the following conditions are equivalent:*

(1) *(X, T) is uniquely ergodic;*
(2) *for each $f \in C(X)$, $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X to a constant;*
(3) *for each $f \in C(X)$, there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which converges pointwise on X to a constant;*
(4) *(X, T) contains only one minimal set, and for each $f \in C(X)$, $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X.*

Let (X, T) be a dynamical system and μ be an invariant measure. A complex number λ is called an *eigenvalue* of (X, T, μ) if there is $f \in L^2(\mu)$, with f not the zero function, satisfying $f(Tx) = \lambda f(x)$ for μ-almost everywhere $x \in X$. The function f is called an *eigenfunction* of (X, T, μ) corresponding to the eigenvalue λ. The measurable dynamical system (X, T, μ) has *discrete spectrum* or *pure point spectrum* if there exists an orthonormal basis for $L^2(\mu)$ which consists of eigenfunctions of (X, T, μ). If μ is ergodic, then (X, T, μ) has discrete spectrum if and only if it is conjugate to an ergodic rotation on some compact abelian group (see [**28**, Theorem 3.6]).

3. *Mean equicontinuous systems*

In this section, we study mean equicontinuous systems. We obtain several equivalent conditions of mean equicontinuity. We also show that every ergodic invariant measure on a mean equicontinuous system has discrete spectrum, every dynamical system admits a maximal mean equicontinuous factor, and mean equicontinuity is preserved by factor maps. We first observe that mean equicontinuity is equivalent to mean-L-stability, which was first introduced in [**8**].

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LEMMA 3.1. *A dynamical system is mean equicontinuous if and only if it is mean-L-stable.*

Proof. Assume that (X, T) is mean equicontinuous. For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon^2,$$

for all $x, y \in X$ with $d(x, y) < \delta$. Let $E = \{k \in \mathbb{Z}_+ \mid d(T^k x, T^k y) \geq \varepsilon\}$. One has

$$\varepsilon^2 > \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) \geq \limsup_{n\to\infty} \frac{1}{n} (\varepsilon \#([0, n-1] \cap E)) = \varepsilon \overline{D}(E),$$

and then $\overline{D}(E) < \varepsilon$, which implies (X, T) is mean-L-stable.

Conversely, assume that (X, T) is mean-L-stable. Fix a positive number $\varepsilon > 0$ and choose a positive number $\eta < \varepsilon/(\mathrm{diam}(X) + 1)$. There is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(T^n x, T^n y) < \eta$ for all $n \in \mathbb{Z}_+$ except a set of upper density less than η. Let $F = \{k \in \mathbb{Z}_+ \mid d(T^k x, T^k y) \geq \eta\}$. One has $\overline{D}(F) < \eta$ and

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) &\leq \limsup_{n\to\infty} \frac{1}{n} (\mathrm{diam}(X) \#([0, n-1] \cap F) + \eta n) \\ &\leq \mathrm{diam}(X) \overline{D}(F) + \eta < \varepsilon, \end{aligned}$$

which implies (X, T) is mean equicontinuous. □

The following lemma is easy to verify.

LEMMA 3.2. *Let (X, T) and (Y, S) be two dynamical systems. Then $(X \times Y, T \times S)$ is mean equicontinuous if and only if both (X, T) and (Y, S) are mean equicontinuous.*

We have the following characterization of mean equicontinuous systems which is implicit in [**26**]. For completeness we include a proof.

THEOREM 3.3. *Let (X, T) be a dynamical system. Then the following conditions are equivalent:*

(1) *(X, T) is mean equicontinuous;*
(2) *for each $f \in C(X \times X)$, the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous;*
(3) *for each $f \in C(X \times X)$, the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent to a $T \times T$-invariant continuous function $f^* \in C(X \times X)$.*

Proof. To make the idea of the proof clearer, when proving (1)⇒(2) and (2)⇒(3), we assume $f \in C(X)$ instead of $f \in C(X \times X)$, because if (X, T) is mean equicontinuous then so is $(X \times X, T \times T)$.

(1)⇒(2) Fix $f \in C(X)$. To show that $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous, it suffices to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$ with $d(x, y) < \delta$ implies $|f_n(x) - f_n(y)| < \varepsilon$ for all $n \in \mathbb{N}$. By the definition of f_n, one has

$$|f_n(x) - f_n(y)| = \left| \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) - \frac{1}{n} \sum_{i=0}^{n-1} f(T^i y) \right| \leq \frac{1}{n} \sum_{i=0}^{n-1} |f(T^i x) - f(T^i y)|.$$

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Fix a positive number $\varepsilon > 0$. By uniform continuity of f, there exists $\delta_1 > 0$ such that if $x, y \in X$ with $d(x, y) < \delta_1$ then $|f(x) - f(y)| < \varepsilon/2$. As (X, T) is mean equicontinuous, it is also mean-L-stable. Choose a positive number $\eta < \min\{\delta_1, \varepsilon/(8\|f\| + 1)\}$. There is $\delta_2 \in (0, \delta_1)$ such that $d(x, y) < \delta_2$ implies $d(T^n x, T^n y) < \eta$ for all $n \in \mathbb{Z}_+$ except in a set E with $\overline{D}(E) < \eta$.

Choose N large enough such that $(1/n)\#(E \cap [0, n-1]) < 2\eta$ for all $n \geq N$. Then for every $n \geq N$ and $x, y \in X$ with $d(x, y) < \delta_2$, one has

$$\frac{1}{n}\sum_{i=0}^{n-1} |f(T^i x) - f(T^i y)| \leq \frac{1}{n}\Bigg(\sum_{i\in E\cap[0,n-1]} 2\|f\| + \sum_{i\in[0,n-1]\setminus E} |f(T^i x) - f(T^i y)|\Bigg)$$
$$\leq 4\eta\|f\| + \frac{\varepsilon}{2} < \varepsilon.$$

By the compactness of X, there exists $\delta_3 > 0$ such that for every $n \in \{1, 2, \ldots, N\}$ and $x, y \in X$ with $d(x, y) < \delta_3$, $|f_n(x) - f_n(y)| < \varepsilon$. Choose $0 < \delta < \min\{\delta_2, \delta_3\}$. Then for every $n \in \mathbb{N}$ and $x, y \in X$ with $d(x, y) < \delta$, $|f_n(x) - f_n(y)| < \varepsilon$. This shows that $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous.

(2)⇒(3) Clearly, $\|f_n\| \leq \|f\|$ for every $n \in \mathbb{N}$. Then the sequence $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded. Since $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous, by the Arzela–Ascoli theorem there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which is uniformly convergent to $f^* \in C(X)$. Then f^* is T-invariant, that is for every $x \in X$

$$|f^*(x) - f^*(Tx)| = \left|\lim_{k\to\infty} \frac{1}{n_k}\left(\sum_{i=0}^{n_k-1} f(T^i x) - \sum_{i=0}^{n_k-1} f(T^i(Tx))\right)\right|$$
$$= \lim_{k\to\infty} \frac{1}{n_k}|f(T^{n_k} x) - f(x)| = 0.$$

By the continuity of f^*, $f^*|_{\overline{\mathrm{Orb}(x,T)}}$ is constant for every $x \in X$. Then by Theorem 2.2 $(\overline{\mathrm{Orb}(x, T)}, T)$ is uniquely ergodic. Again by Theorem 2.2, $f_n \to f^*$ as $n \to \infty$ uniformly on $(\overline{\mathrm{Orb}(x, T)}, T)$. Since x is arbitrary, one has $f_n(x) \to f^*(x)$ as $n \to \infty$ pointwise for every $x \in X$. Then $f_n(x) \to f^*(x)$ as $n \to \infty$ uniformly on X, since $\{f_n\}_{n=1}^{\infty}$ is uniformly equicontinuous.

(3) ⇒ (1) Recall that $d(\cdot, \cdot)$ is the metric on X and is a continuous function on $X \times X$. Then the sequence $\{d_n\}_{n=1}^{\infty}$ is uniformly equicontinuous, that is for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_n(x, y) < \varepsilon/2$ for every $n \in \mathbb{Z}_+$ and every $x, y \in X$ with $d(x, y) < \delta$. Then for every $x, y \in X$ with $d(x, y) < \delta$, one has

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) \leq \sup_{n\in\mathbb{N}} d_n(x, y) < \varepsilon,$$

which implies that (X, T) is mean equicontinuous. □

By Theorems 2.2 and 3.3, we have the following corollary.

COROLLARY 3.4. **[26]** *Let (X, T) be a dynamical system. If (X, T) is mean equicontinuous, then for every $x \in X$, $(\overline{\mathrm{Orb}(x, T)}, T)$ is uniquely ergodic. In particular, if (X, T) is also transitive, then (X, T) is uniquely ergodic.*

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It is shown in [**2**] that in a mean equicontinuous system a pair of points is proximal if and only if it is persistently proximal (proximal with density one). We can strengthen this result as follows.

THEOREM 3.5. *Let (X, T) be a dynamical system. If (X, T) is mean equicontinuous, then $Q(X, T) = P(X, T) = BP(X, T)$ and it is a closed invariant equivalence relation.*

Proof. It is clear that $Q(X, T) \supset P(X, T) \supset BP(X, T)$. We are going to show that $P(X, T) = BP(X, T)$ and $Q(X, T) = P(X, T)$.

Assume first that $(x, y) \in P(X, T)$. Since $(X \times X, T \times T)$ is also mean-L-stable, by Corollary 3.4 $\overline{\mathrm{Orb}((x, y), T \times T)}$ is uniquely ergodic. The proximality of (x, y) implies that $\overline{\mathrm{Orb}((x, y), T \times T)} \cap \Delta_X \neq \emptyset$, which again implies that $\mathrm{supp}(\overline{\mathrm{Orb}((x, y), T \times T)}) \subset \Delta_X$ and thus $(x, y) \in BP(X, T)$ by [**24**].

Assume now that $(x, y) \in Q(X, T)$. As (X, T) is mean equicontinuous, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $z_1, z_2 \in X$ with $d(z_1, z_2) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i z_1, T^i z_2) < \frac{\varepsilon}{3}.$$

For this $\delta > 0$, there exist $x', y' \in X$ and $k \in \mathbb{N}$ such that $d(x, x') < \delta$, $d(y, y') < \delta$ and $d(T^k x', T^k y') < \delta$. Then

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) &\le \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (d(T^i x, T^i x') + d(T^i y', T^i y) + d(T^i x', T^i y')) \\
&\le \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i x') + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i y', T^i y) \\
&\quad + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i (T^k x'), T^i (T^k y')) \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since ε is arbitrary, one has

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) = 0,$$

which implies that (x, y) is proximal.

In general $Q(X, T)$ is a closed relation and $BP(X, T)$ is an invariant equivalence relation, which implies $Q(X, T)$ is a closed invariant equivalence relation when (X, T) is mean equicontinuous. □

Theorem 3.5 has the following direct corollaries.

COROLLARY 3.6. *Let (X, T) be a dynamical system.*

(1) *If (X, T) is mean equicontinuous, then:*
   (a) *it is a proximal extension of its maximal equicontinuous factor;*
   (b) *the maximal distal factor and the maximal equicontinuous factor coincide.*

(2) *If (X, T) is mean equicontinuous and distal, then it is equicontinuous.*

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COROLLARY 3.7. *Let (X, T) be a dynamical system. Assume that $P(X, T)$ is dense in $X \times X$, for example (X, T) is weakly mixing or transitive with a fixed point* [**20**]. *Then it is mean equicontinuous if and only if it is strongly proximal.*

Proof. First assume that (X, T) is mean equicontinuous. Then by Theorem 3.5 $BP(X, T) = X \times X$ which implies that (X, T) is strongly proximal by [**24**]. Now assume that (X, T) is strongly proximal. It is shown in [**24**] that (X, T) is strongly proximal if and only if $(X \times X, T \times T)$ is uniquely ergodic. Thus (X, T) is mean equicontinuous by Theorem 3.3. □

In [**27**] the author asked the following question: does every ergodic invariant measure on a mean equicontinuous system have discrete spectrum? We show that this question has a positive answer. We note that the proof is inspired by [**17**, Theorem 4.4].

THEOREM 3.8. *Let (X, T) be a dynamical system. If (X, T) is mean equicontinuous, then every ergodic invariant measure on (X, T) has discrete spectrum and hence the topological entropy of (X, T) is zero.*

Proof. Let μ be an ergodic invariant measure on (X, T). Without loss of generality, assume that $\mathrm{supp}(\mu) = X$ and then (X, T) is uniquely ergodic by Corollary 3.4. Let (Y, S) be the maximal equicontinuous factor of (X, T) and $\pi : (X, T) \to (Y, S)$ be the factor map. Then π is proximal by Theorem 3.5.

Let ν be the unique invariant measure on (Y, S). We have $\pi(\mu) = \nu$. We consider the disintegration of μ over ν. That is, for almost everywhere $y \in Y$ we have a measure μ_y on X such that $\mathrm{supp}(\mu_y) \subset \pi^{-1}(y)$ and $\mu = \int_{y\in Y} \mu_y d\nu$. Let $W = \{(x, y) \in X \times X \mid \pi(x) = \pi(y)\}$. As $\mathrm{supp}(\mu_y) \subset \pi^{-1}(y)$, for almost everywhere $y \in Y$ we have $\mathrm{supp}(\mu_y \times \mu_y) \subset \pi^{-1}(y) \times \pi^{-1}(y) \subset W$, for almost everywhere $y \in Y$. Let $\mu \times_Y \mu = \int_{y\in Y} \mu_y \times \mu_y d\nu$. Then $\mu \times_Y \mu$ is an invariant measure on $(X \times X, T \times T)$. Moreover,

$$\mu \times_Y \mu(W) = \int_{y\in Y} \mu_y \times \mu_y(W) d\nu = 1,$$

then $\mathrm{supp}(\mu \times_Y \mu) \subset W$. By Theorem 3.5 every point in W is Banach proximal. Thus we have $\mathrm{supp}(\mu \times_Y \mu) \subset \Delta_X$ (otherwise, there exists a point z in $W \setminus \Delta_X$ which returns to any neighborhood U of z with positive Banach density), and $\mu \times_Y \mu(\Delta_X) = 1$. Since

$$\mu \times_Y \mu(\Delta_X) = \int_{y\in Y} \mu_y \times \mu_y(\Delta_X) d\nu = 1,$$

we have $\mu_y \times \mu_y(\Delta_X) = 1$ for almost everywhere $y \in Y$. By Fubini's theorem we get that for almost everywhere $y \in Y$, μ_y is a combination of countably many atomic measures. Using the fact $\mu_y \times \mu_y(\Delta_X) = 1$, we conclude that for almost everywhere $y \in Y$, there exists a point $c_y \in \pi^{-1}(y)$ such that $\mu_y = \delta_{c_y}$.

Let Z_0 be the collection of $y \in Y$ such that μ_y is not equal to δ_x for any $x \in X$. Then $\nu(\bigcup_{i\in\mathbb{Z}_+} S^{-i} Z_0) = 0$. Let $Y_0 = Y \setminus \bigcup_{i\in\mathbb{Z}_+} S^{-i} Z_0$ and $X_0 = \{c_y \mid y \in Y_0\}$. Then $\nu(Y_0) = 1$. Now we show that X_0 is a measurable set. In fact, the map $y \mapsto \mu_y$ from Y_0 to $M(X)$ is measurable and $x \mapsto \delta_x$ is an embedding. Since Y_0 is a measurable set and maps are

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one-to-one, it follows from Souslin's theorem that X_0 is a measurable set, and it is clear that $\mu(X_0) = \mu(\pi^{-1}Y_0) = \nu(Y_0) = 1$. By the same argument, $\pi : X_0 \to Y_0$ is an isomorphism. This shows that μ is isomorphic to ν, and thus has discrete spectrum. □

Remark 3.9. We have the following remarks.

(1) If $(X \times X, T \times T)$ is uniquely ergodic, then by Theorems 2.2 and 3.3 (X, T) is mean equicontinuous. It is shown in [**24**] that a dynamical system is strongly proximal if and only if $(X \times X, T \times T)$ is uniquely ergodic. There exist some strongly mixing systems which are strongly proximal, and thus they are also mean equicontinuous. Since there exists a uniquely ergodic minimal system with positive entropy [**16**], unique ergodicity does not imply mean equicontinuity by Theorem 3.8.

(2) The authors in [**19**] have studied equicontinuous subsets and equicontinuity with respect to an invariant measure. Let (X, T) be a dynamical system and K be a subset of X. We say that K is *equicontinuous* if for any $\varepsilon > 0$ there is $\delta > 0$ such that when $x, y \in K$ with $d(x, y) < \delta$, then $d(T^n x, T^n y) < \varepsilon$ for all $n \geq 0$. Let $\mu \in M(X, T)$. We say that (X, T) is *μ-equicontinuous* if for any $\tau > 0$ there is a T-equicontinuous compact subset K of X satisfying $\mu(K) > 1 - \tau$. It is shown in [**19**, Corollary 5.6] that if (X, T) is μ-equicontinuous then μ has discrete spectrum. Using the same idea, García-Ramos in a recent preprint [**10**] defines the notion of *μ-mean equicontinuous*, and shows that (X, T, μ) is μ-mean equicontinuous if and only if μ has discrete spectrum.

(3) The notion of *tightness* was defined by Furstenberg, see [**25**]. Since tightness is an isomorphic invariant [**25**, Proposition 3], it is easy to see that if (X, T) is mean equicontinuous and $\mu \in M(X, T)$ is ergodic, then (X, T, μ) is tight.

In [**7**], Ellis and Gottschalk proved the existence of a maximal equicontinuous factor in a dynamical system. It is easy to see that mean equicontinuity satisfies Remarks 6 and 7 in [**7**], i.e. mean equicontinuity is preserved under subsystems and products, so we have the following result.

THEOREM 3.10. *Each dynamical system admits a maximal mean equicontinuous factor.*

To end the section we show that mean continuity is preserved by factor maps.

THEOREM 3.11. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between two dynamical systems. If (X, T) is mean equicontinuous, then so is (Y, S).*

Proof. Assume the contrary, that (Y, S) is not mean equicontinuous. Then there is $\delta > 0$ and two sequences $\{y_k\}$, $\{z_k\}$ of points in Y with $d(y_k, z_k) < 1/k$ such that for each $k \geq 1$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(S^i(y_k), S^i(z_k)) \geq 2\delta.$$

By the compactness of Y, we may assume that $\lim_{k\to\infty} y_k = \lim_{k\to\infty} z_k = y \in Y$.

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For each $k \geq 1$,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(S^i(y_k), S^i(y)) + \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(S^i(z_k), S^i(y)) \geq \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(S^i(y_k), S^i(z_k)).$$

Then either

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(S^i(y_k), S^i(y)) \geq \delta \quad \text{or} \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(S^i(z_k), S^i(y)) \geq \delta.$$

Without loss of generality, assume that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(S^i(y_k), S^i(y)) \geq \delta$$

holds for all $k \geq 1$. Choose a sequence $\{x_k\}$ in X with $\pi(x_k) = y_k$. By the compactness of X, we can assume that $\lim_{k\to\infty} x_k = x$. Then $\pi(x) = y$.

Choose an $\eta > 0$ such that $\eta < \delta/(1 + \operatorname{diam}(Y))$. By the continuity of π, there exists $\theta \in (0, \eta)$ such that if $d(u, v) < \theta$ then $d(\pi(u), \pi(v)) < \eta$. Since (X, T) is mean equicontinuous, there is $\varepsilon > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i w, T^i x) < \theta^2$$

for all $w \in B(x, \varepsilon)$. Choose $x_j \in B(x, \varepsilon)$. Let $E = \{k \in \mathbb{Z}_+ \mid d(T^k x_j, T^k x) \geq \theta\}$. One has

$$\theta^2 > \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x_j, T^i x) \geq \limsup_{n\to\infty} \frac{1}{n}(\theta \#([0, n-1] \cap E)) = \theta \overline{D}(E),$$

and then $\overline{D}(E) < \theta$. Let $F = \{k \in \mathbb{Z}_+ \mid d(T^k y_j, T^k y) \geq \eta\}$. By the choice of θ, we have $F \subset E$. Then $\overline{D}(F) < \theta$ and

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(S^i(y_j), S^i(y)) &\leq \limsup_{n\to\infty} \frac{1}{n}(\operatorname{diam}(Y) \#([0, n-1] \cap F) + \eta n) \\ &\leq \operatorname{diam}(Y)\overline{D}(F) + \eta \leq \operatorname{diam}(Y)\theta + \eta \\ &\leq (\operatorname{diam}(Y) + 1)\eta < \delta,\end{aligned}$$

which is a contradiction. Thus (Y, S) is mean equicontinuous. □

4. *Almost mean equicontinuity*

In this section, we study the localization of mean equicontinuity. Let (X, T) be a dynamical system. A point $x \in X$ is called *mean equicontinuous* if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $y \in B(x, \delta)$,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon.$$

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By the compactness of X, (X, T) is mean equicontinuous if and only if every point in X is mean equicontinuous. A transitive system (X, T) is called *almost mean equicontinuous* if there is at least one mean equicontinuous point. We show that for a transitive system (X, T), the set of mean equicontinuous points is either empty or residual. If in addition (X, T) is almost mean equicontinuous, then every transitive point is mean equicontinuous. While almost equicontinuous systems must have zero topological entropy, we construct many almost mean equicontinuous systems which have positive topological entropy.

Similarly to Theorem 3.3, we have the following characterization of mean equicontinuous points.

THEOREM 4.1. *Let (X, T) be a dynamical system and $x \in X$. Then the following conditions are equivalent:*

(1) *x is a mean equicontinuous point in (X, T);*
(2) *for each $f \in C(X \times X)$, the sequence $\{f_n\}_{n=1}^{\infty}$ is equicontinuous at (x, x);*
(3) *for each $f \in C(X \times X)$, the sequence $\{f_n\}_{n=1}^{\infty}$ is convergent at (x, x);*
(4) *for each $f \in C(X \times X)$, $\bar{f}$ is continuous at (x, x), where $\bar{f}(y) = \limsup_{n\to\infty} f_n(y)$.*

Let $\mathcal{E}$ denote the set of all mean equicontinuous points. For every $\varepsilon > 0$, let

$$\mathcal{E}_\varepsilon = \left\{ x \in X \;\middle|\; \exists \delta > 0, \forall y, z \in B(x, \delta), \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i y, T^i z) < \varepsilon \right\}.$$

PROPOSITION 4.2. *Let (X, T) be a dynamical system and $\varepsilon > 0$. Then $\mathcal{E}_\varepsilon$ is open and inversely invariant, that is $T^{-1}\mathcal{E}_\varepsilon \subset \mathcal{E}_\varepsilon$. Moreover, $\mathcal{E} = \bigcap_{m=1}^{\infty} \mathcal{E}_{1/m}$ is a G_δ subset of X.*

Proof. Let $x \in \mathcal{E}_\varepsilon$. Choose $\delta > 0$ satisfying the condition from the definition of $\mathcal{E}_\varepsilon$ for x. Fix $y \in B(x, \delta/2)$. If $z, w \in B(y, \delta/2)$, then $z, w \in B(x, \delta)$, so

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i z, T^i w) < \varepsilon.$$

This shows that $B(x, \delta/2) \subset \mathcal{E}_\varepsilon$ and thus $\mathcal{E}_\varepsilon$ is open.

Let $x \in X$ with $Tx \in \mathcal{E}_\varepsilon$. Choose $\delta > 0$ satisfying the condition from the definition of $\mathcal{E}_\varepsilon$ for Tx. By the continuity of T, there exists $\eta > 0$ such that $d(Ty, Tz) < \delta$ for any $y, z \in B(x, \eta)$. If $y, z \in B(x, \eta)$, then $Ty, Tz \in B(Tx, \delta)$. Thus

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i y, T^i z) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(Ty), T^i(Tz)) < \varepsilon.$$

This implies $x \in \mathcal{E}_\varepsilon$.

If $x \in X$ belongs to all $\mathcal{E}_{1/m}$, then clearly $x \in \mathcal{E}$. Conversely, if $x \in \mathcal{E}$ and $m > 0$, then there exists $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \frac{1}{2m}$$

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for all $y \in B(x, \delta)$. If $y, z \in B(x, \delta)$,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i y, T^i z) &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} (d(T^i x, T^i y) + d(T^i x, T^i z)) \\ &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) + \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i z)) \\ &< \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}.\end{aligned}$$

Thus $x \in \mathcal{E}_{1/m}$. This ends the proof. □

PROPOSITION 4.3. *Let (X, T) be a transitive system.*

(1) *The set of mean equicontinuous points is either empty or residual. If in addition (X, T) is almost mean equicontinuous, then every transitive point is mean equicontinuous.*
(2) *If (X, T) is minimal and almost mean equicontinuous, then it is mean equicontinuous.*

Proof. By the transitivity of (X, T), every $\mathcal{E}_\varepsilon$ is either empty or dense, since $\mathcal{E}_\varepsilon$ is open and inversely invariant. Then $\mathcal{E}$ is either empty or residual by the Baire Category theorem.

If $\mathcal{E}$ is residual, then every $\mathcal{E}_\varepsilon$ is open and dense. Let $x \in X$ be a transitive point and $\varepsilon > 0$. Then there exists some $k \in \mathbb{Z}_+$ such that $T^k x \in \mathcal{E}_\varepsilon$, and since $\mathcal{E}_\varepsilon$ is inversely invariant, $x \in \mathcal{E}_\varepsilon$. Thus $x \in \mathcal{E}$. □

Remark 4.4. We have the following remarks.

(1) It should be noticed that the set of mean equicontinuous points may not coincide with the set of transitive points. For example, a weakly mixing strongly proximal system is mean equicontinuous, but the set of non-transitive points is dense in the space.
(2) Every almost equicontinuous system is almost mean equicontinuous. There exist some almost equicontinuous systems which have more than one fixed point (see [**21**]) and are thus not uniquely ergodic. So those systems are almost equicontinuous but not mean equicontinuous.
(3) It is shown in [**15**] that factors of almost equicontinuous systems may not be almost equicontinuous. In fact, factors of almost equicontinuous systems may not even be almost mean equicontinuous. Let (Y, S) be a uniformly rigid weakly mixing minimal system. By Corollary 3.7 (Y, S) is not almost mean equicontinuous. By [**15**, Proposition 1.5] there exists an extension (X, T) of (Y, S) which is almost equicontinuous.

Let $\pi : X \to Y$ be a map. The map π is called *open* if for every non-empty open subset U of X, $\pi(U)$ is open in Y, and *semi-open* if $\pi(U)$ has non-empty interior in Y. We say that π is *open at a point* $x \in X$ if for every neighborhood U of x, $\pi(U)$ is a neighborhood of $\pi(x)$.

LEMMA 4.5. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between two dynamical systems. Let $x \in X$ be a mean equicontinuous point and suppose that π is open at x. Then $y = \pi(x)$ is also a mean equicontinuous point.*

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Proof. If y is not a mean equicontinuous point, then there exists $\delta > 0$ and a sequence $y_k \to y$ such that for any k,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(S^i y_k, S^i y) > \delta.$$

Then there exists $t > 0$ such that for any $k \geq 1$, the upper density of $\{n \in \mathbb{Z}_+ \mid d(S^i y_k, S^i y) > t\}$ is greater than t.

By the openness of π at x, there is a sequence $x_k \to x$ with $\pi(x_k) = y_k$. Since x is mean equicontinuous, for every $\varepsilon > 0$, for large enough k, one has

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x_k, T^i x) < \varepsilon.$$

By the uniform continuity of π, there exists $s > 0$ such that if $d(u, v) < s$, then $d(\pi(u), \pi(v)) < t/2$. For this s, there exists $k \geq 1$ such that the upper density of $\{n \in \mathbb{Z}_+ \mid d(T^i x_k, T^i x) > s\}$ is less than t, then the upper density of $\{n \in \mathbb{Z}_+ \mid d(S^i y_k, S^i y) > t\}$ is less than t. This is a contradiction. Thus y is a mean equicontinuous point. □

THEOREM 4.6. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between transitive systems. Suppose that (X, T) is almost mean equicontinuous and π is semi-open. Then (Y, S) is also almost mean equicontinuous.*

Proof. Since π is semi-open, by [**12**, Lemma 2.1] the set $\{x \in X \mid \pi \text{ is open at } x\}$ is residual in X. Pick a transitive point $x \in X$ such that π is open at x. Note that $\pi(x)$ is also a transitive point in Y. Since (X, T) is almost mean equicontinuous, x is a mean equicontinuous point. By Lemma 4.5, $\pi(x)$ is a mean equicontinuous point, which implies that (Y, S) is almost mean equicontinuous. □

Recall that an almost equicontinuous system is uniformly rigid and thus has zero topological entropy. Theorem 4.7 shows that an almost mean equicontinuous system behaves quite differently.

To start with we need some preparation. Let $\Sigma_2 = \{0, 1\}^{\mathbb{N}}$ with the product topology. A metric inducing the topology is given by $d(x, y) = 0$ if $x = y$, and $d(x, y) = 1/i$ if $x \neq y$ and $i = \min\{i \mid x_i \neq y_i\}$ when $x = x_1 x_2 \cdots$ and $y = y_1 y_2 \cdots$. For $n \in \mathbb{N}$, we call $A \in \{0, 1\}^n$ a *finite block with length n* and denote $|A| = n$. For two blocks $A = x_1, \ldots, x_n$ and $B = y_1, \ldots, y_m$ define $AB = x_1, \ldots, x_n y_1, \ldots, y_m$. For a block A, let $[A]$ be the collection of $x \in \Sigma_2$ starting from A. For $x \in \Sigma_2$ and $i < j$, $x_{[i,j]}$ stands for the finite block $x_i x_{i+1} \cdots x_j$.

The *shift map* $\sigma : \Sigma_2 \to \Sigma_2$ is defined by the condition that $\sigma(x)_n = x_{n+1}$ for $n \in \mathbb{N}$. It is clear that σ is a continuous surjection. The dynamical system (Σ_2, σ) is called the *full shift*. If X is non-empty, closed, and σ-invariant (i.e. $\sigma(X) \subset X$), then the dynamical system (X, σ) is called a *subshift*.

THEOREM 4.7. *In the full shift (Σ_2, σ), every minimal subshift (Y, σ) is contained in an almost mean equicontinuous subshift (X, σ).*

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Proof. If $Y = \{0^\infty\}$, the result is obvious. Now assume that $Y \neq \{0^\infty\}$. Choose a point $y = y_1 y_2 \cdots \in Y$ starting with 1. For every $n \geq 1$, set $B_n = y_1 \cdots y_n$.

Set $A_1 = 11$, $A_2 = A_1 0^{k_1} B_1 0^{k_1} A_1$ and

$$A_{n+1} = A_n 0^{k_n} B_n 0^{k_n} A_n$$

for $n \geq 2$, where $k_n \gg 3|A_n|$. We will require the sequence $\{k_n\}$ to satisfy some proper properties later.

Let $x = \lim_{n\to\infty} A_n 0^\infty$ and $X = \overline{\mathrm{Orb}(x, \sigma)}$. It is clear that (Y, σ) is a subsystem of (X, σ). We are going to show that (X, σ) is almost mean equicontinuous.

For a finite block A over $\{0, 1\}$, we denote by $\mathfrak{o}(A)$ the function which counts the number of ones in A. Since y is minimal, by Lemma 2.1 the word 1 appears in y syndetically. Then there is $N \in \mathbb{N}$ such that A_N does not appear in y. For any $n \geq N$, we can express x as

$$x = \underline{A_n 0^{k_n} B_n 0^{k_n}}\ \underline{A_n 0^{k_{n+1}} B_{n+1} 0^{k_{n+1}}} \cdots \underline{A_n 0^{k_m} B_m 0^{k_m}}\ \underline{A_n 0^{k_q} B_q 0^{k_q}} \cdots .$$

CLAIM 1. *For any $m \geq n$ and $1 \leq i \leq |A_n| + 2k_m + m$, we have*

$$\mathfrak{o}(A_n 0^{k_m} B_m 0^{k_m}[1,i]) \leq \max\left\{1, \frac{i}{|A_n| + 2k_n + n}\right\} \cdot (|A_n| + n).$$

Proof. It is clear that we only need to prove the case $m > n$. The result is obvious for $i \in [1, |A_n| + k_m]$. If we require the sequence $\{k_n\}$ to satisfy

$$\frac{k_q}{|A_p| + 2k_p + p} > \frac{q}{p} \quad \text{for all } 1 \leq p < q,$$

then $k_m > 2k_n$ and $m < k_m \cdot n/(|A_n| + 2k_n + n)$. So for $i > |A_n| + k_m$, we have

$$\begin{aligned}\mathfrak{o}(A_n 0^{k_m} B_m 0^{k_m}[1,i]) &\leq |A_n| + m \leq |A_n| + \frac{k_m \cdot n}{|A_n| + 2k_n + n} \\ &\leq \frac{(|A_n| + k_m)(|A_n| + n)}{|A_n| + 2k_n + n} \\ &\leq \frac{i}{|A_n| + 2k_n + n}(|A_n| + n).\end{aligned}$$

□

CLAIM 2. *If $j - i > |A_n|$ and $x_{[i,i+|A_n|-1]} = A_n$, then we have*

$$\mathfrak{o}(x_{[i,j-1]}) \leq \left(\frac{j-i}{|A_n| + 2k_n + n} + 1\right)(|A_n| + n).$$

Proof. Since A_N does not appear in y and $x_{[i,i+|A_n|-1]}$ is equal to A_n, $x_{[i,j-1]}$ can be expressed as

$$x_{[i,j-1]} = \underline{A_n 0^{k_{i_1}} B_{i_1} 0^{k_{i_1}}}\ \underline{A_n 0^{k_{i_2}} B_{i_2} 0^{k_{i_2}}} \cdots \underline{A_n 0^{k_{i_m}} B_{i_m} 0^{k_{i_m}}}\ \underline{A_n 0^{k_q} B_q 0^{k_q}[1, p]}$$

for $i_l \geq n$, $l = 1, \ldots, m$ and for some $p < |A_n| + 2k_q + q$.

Then by Claim 1 we have

$$\mathfrak{o}(A_n 0^{k_{i_l}} B_{i_l} 0^{k_{i_l}}) \leq \frac{|A_n 0^{k_{i_l}} B_{i_l} 0^{k_{i_l}}|}{|A_n| + 2k_n + n}(|A_n| + n)$$

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for $l = 1, \ldots, m$ and

$$\mathfrak{o}(A_n 0^{k_q} B_q 0^{k_q}{}_{[1,p]}) \le \frac{p}{|A_n| + 2k_n + n}(|A_n| + n) + |A_n| + n.$$

Thus

$$\begin{aligned}\mathfrak{o}(x_{[i,j-1]}) &\le \mathfrak{o}(A_n 0^{k_{i_1}} B_{i_1} 0^{k_{i_1}}) + \cdots + \mathfrak{o}(A_n 0^{k_{i_m}} B_{i_m} 0^{k_{i_m}}) + \mathfrak{o}(A_n 0^{k_q} B_q 0^{k_q}{}_{[1,p]}) \\ &\le \left(\frac{j-i}{|A_n| + 2k_n + n} + 1\right)(|A_n| + n). \qquad \square\end{aligned}$$

Fix a positive number ε. There is $K \in \mathbb{N}$ such that for any $a, b \in \Sigma$ if $a[0, K-1] = b[0, K-1]$ then $d(a, b) < \varepsilon/5$. If we require the sequence $\{k_n\}$ to satisfy

$$\frac{|A_n| + n}{|A_n| + 2k_n + n} \searrow 0 \text{ as } n \to \infty,$$

then we can choose a large enough integer n such that

$$\frac{2K(|A_n| + n)}{|A_n| + 2k_n + n} < \varepsilon/4. \tag{1}$$

Fix a point $z \in [A_n] \cap X$. If there exists some $k \in \mathbb{Z}_+$ such that $T^k z = 0^\infty$, then it is clear that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} d(T^i z, 0^\infty) = 0.$$

Now we assume that $T^k z \neq 0^\infty$ for all $k \in \mathbb{Z}_+$. For any $i \ge 1$, there is $m_i \in \mathbb{N}$ such that $z_{[0,i]} = x_{[m_i, m_i+i]}$. Since z starts with A_n, by Claim 2 we have

$$\mathfrak{o}(z_{[0,i-1]}) \le \left(\frac{i}{|A_n| + 2k_n + n} + 1\right)(|A_n| + n) \tag{2}$$

for all $i > |A_n|$. By the choice of K, we have

$$\begin{aligned}\limsup_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} d(T^i z, 0^\infty) &\le \frac{\varepsilon}{5} \cdot \limsup_{N\to\infty} \frac{1}{N} \#\{0 \le i \le N-1 \mid (T^i z)_{[0,K-1]} = 0^K\} \\ &\quad + \limsup_{N\to\infty} \frac{1}{N} \#\{0 \le i \le N-1 \mid (T^i z)_{[0,K-1]} \neq 0^K\} \\ &< \frac{\varepsilon}{4} + \limsup_{N\to\infty} \frac{1}{N} 2K \cdot \mathfrak{o}(z_{[0,N-1]}) \le \varepsilon/2,\end{aligned}$$

by (1) and (2). This implies that for any $z \in [A_n] \cap X$

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} d(T^i z, T^i x) < \varepsilon,$$

and thus x is a mean equicontinuous point.

Since it is well known that there are many minimal subshifts of (Σ_2, σ) with positive topological entropy, an immediate corollary of Theorem 4.7 is the following result.

COROLLARY 4.8. *There exist many almost mean equicontinuous systems which have positive topological entropy.*

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5. *Mean sensitivity*

In this section we study the opposite side of mean equicontinuity: mean sensitivity. It turns out that if a dynamical system (X, T) is minimal then (X, T) is either mean equicontinuous or mean sensitive, and if (X, T) is transitive then (X, T) is either almost mean equicontinuous or mean sensitive.

A dynamical system (X, T) is *mean sensitive* if there exists $\delta > 0$ such that for every $x \in X$ and every $\varepsilon > 0$ there is $y \in B(x, \varepsilon)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

PROPOSITION 5.1. *Let (X, T) be a dynamical system. Then the following conditions are equivalent:*

(1) *(X, T) is mean sensitive;*

(2) *there exists $\delta > 0$ such that for every non-empty open subset U of X there are $x, y \in U$ satisfying*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta;$$

(3) *there exists $\eta > 0$ such that for every $x \in X$, the set*

$$D_\eta(x) = \left\{ y \in X \;\middle|\; \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) \geq \eta \right\}$$

is a dense G_δ subset of X;

(4) *there exists $\eta > 0$ such that*

$$D_\eta = \left\{ (x, y) \in X \times X \;\middle|\; \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) \geq \eta \right\}$$

is a dense G_δ subset of $X \times X$.

Proof. (1) $\Rightarrow$ (4) Since (X, T) is mean sensitive, there exists $\delta > 0$ such that for every $x \in X$ and every $\varepsilon > 0$ there is $y \in B(x, \varepsilon)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

Let $\eta = \delta/2$. It is not hard to check that

$$D_\eta = \bigcap_{m=1}^{\infty} \bigcap_{\ell=1}^{\infty} \left(\bigcup_{n\geq\ell} \left\{ (x, y) \in X \times X \;\middle|\; \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \eta - \frac{1}{m} \right\} \right).$$

Then D_η is a G_δ subset of $X \times X$. If D_η is not dense in $X \times X$, then there exist two non-empty open subsets U and V of X such that

$$U \times V \subset \left\{ (x, y) \in X \times X \;\middle|\; \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \eta \right\}.$$

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Pick $x \in U$, $z \in V$ and $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$. Then for every $y \in B(x, \varepsilon)$, one has

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} (d(T^i x, T^i z) + d(T^i y, T^i z)) \\ &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i z) + \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i y, T^i z) \\ &< \eta + \eta = \delta,\end{aligned}$$

which is a contradiction. Thus D_η is a dense G_δ subset of $X \times X$.

(4) $\Rightarrow$ (3) Assume that there exists $\eta > 0$ such that $D_{2\eta}$ is a dense G_δ subset of $X \times X$. Note that $D_\eta(x)$ is a G_δ subset of X. Thus it suffices to show that $D_\eta(x)$ is dense. For every non-empty open subset U of X, there exist two points $y, z \in U$ with $(y, z) \in D_{2\eta}$, that is

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i y, T^i z) > 2\eta.$$

Then

$$\begin{aligned}2\eta < \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i y, T^i z) &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} (d(T^i x, T^i y) + d(T^i x, T^i z)) \\ &\le \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) + \limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i z).\end{aligned}$$

Thus, either

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) > \eta$$

or

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i z) > \eta,$$

that is either $y \in D_\eta(x)$ or $z \in D_\eta(x)$.

(3)$\Rightarrow$ (2) This is obvious.

(2)$\Rightarrow$ (1) For any $x \in X$ and $\varepsilon > 0$, there are $y, z \in B(x, \varepsilon)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i y, T^i z) > \delta.$$

Then either

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) > \frac{\delta}{2}$$

or

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i z) > \frac{\delta}{2},$$

which implies that (X, T) is mean sensitive. □

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A point $x \in X$ is *mean sensitive* if there exists $\delta > 0$ such that for every $\varepsilon > 0$ there is $y \in B(x, \varepsilon)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

It is clear that a point is either mean equicontinuous or mean sensitive. The following example indicates that although in a dynamical system every point is mean sensitive, the dynamical system may not be sensitive.

Example 5.2. Let $T : [0, 1] \to [0, 1]$ be the standard tent map, that is $T(x) = 1 - |1 - 2x|$. Let

$$Y = \{(x, 0) \mid x \in [0, 1]\} \cup \bigcup_{k=1}^{\infty} \left\{-\frac{1}{k}\right\} \times \left[0, \frac{1}{k}\right],$$

be endowed with a metric induced by the Euclidean metric. For $x \in [0, 1]$, put $S(x, 0) = S(Tx, 0)$, and for $k \geq 1$ and $x \in [0, 1/k]$, put $S(-1/k, x) = (-1/k, (1/k)T(kx))$. It is not hard to verify that every point in (Y, S) is mean sensitive, but (Y, S) is not sensitive.

PROPOSITION 5.3. *Let (X, T) be a transitive system. If there exists a transitive point which is mean sensitive, then (X, T) is mean sensitive.*

Proof. Let $x \in X$ be a mean sensitive transitive point, that is there exists $\delta > 0$ such that for every $\varepsilon > 0$ there is $y \in B(x, \varepsilon)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

Fix a non-empty open subset U of X. As x is a transitive point, there exist $\varepsilon > 0$ and $k \in \mathbb{Z}_+$ such that $T^k(B(x, \varepsilon)) \subset U$. There exists $y \in B(x, \varepsilon)$ satisfying

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.$$

Let $u = T^k x$ and $v = T^k y$. Then $u, v \in U$ and

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i u, T^i v) &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i(T^k x), T^i(T^k y)) \\ &= \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \delta.\end{aligned}$$

Therefore (X, T) is mean sensitive by Proposition 5.1. □

Now we have the following dichotomy for transitive systems.

THEOREM 5.4. *Let (X, T) be a dynamical system. If (X, T) is transitive, then (X, T) is either almost mean equicontinuous or mean sensitive.*

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Proof. Let $x \in X$ be a transitive point. If x is mean sensitive, then (X, T) is mean sensitive by Proposition 5.3. If x is not mean sensitive, then it is mean equicontinuous. So (X, T) is almost mean equicontinuous by Proposition 4.3. □

COROLLARY 5.5. *Let (X, T) be a minimal system. Then (X, T) is either mean equicontinuous or mean sensitive.*

Remark 5.6. In fact, from the proof of Theorem 3.11 we have that if $\pi : (X, T) \to (Y, S)$ is a factor map and $y \in Y$ is a mean sensitive point, then there exists a mean sensitive point $x \in \pi^{-1}(y)$.

By Lemma 4.5, we have the following result.

LEMMA 5.7. *Let $\pi : (X, T) \to (Y, S)$ be a factor map. Assume that $y \in Y$ is a mean sensitive point and $x \in \pi^{-1}(y)$. If π is open at x, then x is also a mean sensitive point.*

Similarly to Theorem 4.6, we have the following theorem.

THEOREM 5.8. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between transitive systems. Assume that (Y, S) is mean sensitive. If π is semi-open, then (X, T) is mean sensitive.*

PROPOSITION 5.9. *Let $\pi : (X, T) \to (Y, S)$ be a factor map between transitive systems. If (Y, S) is mean sensitive, then there is a mean sensitive subsystem (Z, T) of (X, T).*

Proof. Using Zorn's lemma and the compactness of X, we can find a subsystem (Z, T) of (X, T) such that $\pi(Z) = Y$ and Z is minimal with respect to this property. Let $y \in Y$ be a transitive point which is also a mean sensitive point. By Remark 5.6 there is a mean sensitive point $x \in (Z, T)$ with $\pi(x) = y$. Let $Z' = \omega(x, T)$. Then (Z', T) is a subsystem of (Z, T) with $\pi(Z') = Y$. The minimality of Z implies that $Z' = Z$. Then x is a transitive point of (Z, T) and (Z, T) is mean sensitive. □

6. *Banach mean equicontinuity and almost Banach mean equicontinuity*

Globally speaking a mean equicontinuous system is 'simple', since it is a Banach proximal extension of an equicontinuous system and each of its ergodic measures has discrete spectrum. Unfortunately, the local version does not behave so well, as Theorem 4.7 shows. In this section we introduce the notion of Banach mean equicontinuity, whose local version has the better behavior that we are looking for.

Let (X, T) be a dynamical system. We say that (X, T) is *Banach mean equicontinuous* (BME) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$,

$$\limsup_{M-N\to\infty} \frac{1}{M-N} \sum_{i=N}^{M-1} d(T^i x, T^i y) < \varepsilon.$$

A point $x \in X$ is called BME if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $y \in B(x, \delta)$,

$$\limsup_{M-N\to\infty} \frac{1}{M-N} \sum_{i=N}^{M-1} d(T^i x, T^i y) < \varepsilon.$$

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We say that a transitive system (X, T) is *almost Banach mean equicontinuous* (ABME) if there is at least one BME point.

By the compactness of X, (X, T) is BME if and only if every point in X is BME. Following the proofs of Propositions 4.2 and 4.3 we know that the set of BME points is a G_δ set, and if (X, T) is ABME, then every transitive point is BME.

A dynamical system (X, T) is *Banach mean sensitive* (BMS) if there exists $\delta > 0$ such that for every $x \in X$ and every $\varepsilon > 0$ there is $y \in B(x, \varepsilon)$ satisfying

$$\limsup_{M-N\to\infty} \frac{1}{M-N} \sum_{i=N}^{M-1} d(T^i x, T^i y) > \delta.$$

We list several properties whose proofs are similar to ones in the previous sections.

PROPOSITION 6.1. *Let (X, T) be a dynamical system.*

(1) *The system (X, T) is BME if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$ implies $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$ except a set of upper Banach density less than ε.*
(2) *The system (X, T) is BMS if and only if there is a $\delta > 0$ such that for every $x \in X$ and every neighborhood U of x, there is $y \in U$ such that $d(T^n x, T^n y) > \delta$ in a set of upper Banach density larger than δ.*
(3) *Every dynamical system has a maximal BME factor.*
(4) *Let $\pi : (X, T) \to (Y, S)$ be a factor map. If (X, T) is BME, then so is (Y, S).*
(5) *If (X, T) is a transitive system, then (X, T) is either ABME or BMS; and if (X, T) is a minimal system, then (X, T) is either BME or BMS.*
(6) *Let $\pi : (X, T) \to (Y, S)$ be a factor map between transitive systems. Suppose that π is semi-open.*
   (a) *If (X, T) is ABME, then so is (Y, S).*
   (b) *If (Y, S) is BMS, then so is (X, T).*

The main result in this section is as follows.

THEOREM 6.2. *Let (X, T) be a transitive system. If the topological entropy of (X, T) is positive, then (X, T) is Banach mean sensitive.*

To prove the above theorem we need some preparation. The following result is implicit in [**11**], see also [**19**, Proposition 5.8] for this version.

PROPOSITION 6.3. *Let $(X, \mathcal{B}, \mu)$ be a probability space, and $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets with $\mu(E_i) \geq a > 0$ for some constant a and any $i \in \mathbb{N}$. Then for any $k \geq 1$ and $\varepsilon > 0$, there is $N = N(a, k, \varepsilon)$ such that for any tuple $\{s_1 < s_2 < \cdots < s_n\}$ with $n \geq N$ there exist $1 \leq t_1 < t_2 < \cdots < t_k \leq n$ with*

$$\mu(E_{s_{t_1}} \cap E_{s_{t_2}} \cap \cdots \cap E_{s_{t_k}}) \geq a^k - \varepsilon.$$

By the well-known Furstenberg corresponding principle [**9**], we have the following proposition.

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PROPOSITION 6.4. *Let S be a subset of $\mathbb{Z}_+$ with $BD^*(S) > 0$. Then for any $k \geq 1$ and $\varepsilon > 0$, there is $N = N(a, k, \varepsilon)$ such that for any tuple $\{s_1 < s_2 < \cdots < s_n\}$ with $n \geq N$ there exist $1 \leq t_1 < t_2 < \cdots < t_k \leq n$ with*

$$BD^*((S - s_{t_1}) \cap (S - s_{t_2}) \cap \cdots \cap (S - s_{t_k})) \geq (BD^*(S))^k - \varepsilon.$$

LEMMA 6.5. *If S has positive upper Banach density and W is an IP-set, then there are $q, q_1, q_2 \in S$ and $l_1, l_2 \in W$ with $l_i = q_i - q$ for each $i = 1, 2$ and*

$$BD^*((S - l_1) \cap (S - l_2)) \geq \tfrac{1}{4}(BD^*(S))^4.$$

Proof. Let W be the IP-set generated by $\{p_i\}_{i=1}^{\infty}$. Without loss of generality, assume that $\sum_{i=1}^{n} p_i < p_{n+1}$. For the sequence $\{\sum_{i=1}^{n} p_i\}_{i=1}^{\infty}$, by Proposition 6.4, there exist $n_1 < n_2$ such that

$$BD^*\left(\left(S - \sum_{i=1}^{n_1} p_i\right) \cap \left(S - \sum_{i=1}^{n_2} p_i\right)\right) \geq \frac{1}{2} BD^*(S)^2.$$

Let $r_1 = \sum_{i=n_1}^{n_2} p_i$ and $S_1 = S \cap (S - r_1)$. Then $r_1 \in W$ and $BD^*(S_1) \geq \frac{1}{2} BD^*(S)^2$. Let W_1 be the IP-set generated by $\{p_i\}_{i=n_2+1}^{\infty}$. Again by Proposition 6.4, there is $r_2 \in W$ such that $S_2 = S_1 \cap (S_1 - r_2)$ satisfying $BD^*(S_2) \geq \frac{1}{2} BD^*(S_1)^2$. Let $l_1 = r_1$ and $l_2 = r_1 + r_2$. Choose $q \in S_2$. Then $q_1 = q + r_1 \in (S_1 + r_1) \subset S$,

$$q_2 = q + r_1 + r_2 \in (S_2 + r_2) + r_1 \subset (S_1 + r_1) \subset S$$

and

$$BD^*((S - l_1) \cap (S - l_2)) \geq BD^*(S_2) \geq \tfrac{1}{2} BD^*(S_1)^2 \geq \tfrac{1}{4}(BD^*(S))^4. \qquad \square$$

The notion of entropy pair was introduced by Blanchard in [**5**]. It is known that if the topological entropy of a dynamical system is positive then there exist some entropy pairs. We have the following characterization of entropy pairs.

PROPOSITION 6.6. [**22**, **23**] *Let (X, T) be a dynamical system. Then $(x_1, x_2) \in X \times X$ is an entropy pair if and only if for each neighborhood U_i of x_i for $i = 1, 2$, there is an independence set of positive density for (U_1, U_2), i.e. there is a subset $J \subset \mathbb{Z}_+$ with positive density such that for any non-empty finite subset $I \subset J$, we have*

$$\bigcap_{i \in I} T^{-i} U_{s(i)} \neq \emptyset$$

for any $s \in \{1, 2\}^I$.

Now we are ready to give a proof of Theorem 6.2. We note that some ideas in the proof are inspired by [**29**].

Proof of Theorem 6.2. Since the topological entropy of (X, T) is positive, there exists an entropy pair (x_1, x_2) in (X, T) [**5**]. Let U_i be a neighborhood of x_i for $i = 1, 2$ with $d(U_1, U_2) > 2\delta$. Let S be a positive density subset of $\mathbb{N}$ associated with (U_1, U_2) in Proposition 6.6.

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Let $x \in X$ be a transitive point and U be a neighborhood of x. By Lemma 2.1, $N(x, U)$ contains an IP-set. Then by Lemma 6.5 there are $q, q_1, q_2 \in S$ and $l_1, l_2 \in N(x, U)$ with $l_1 = q_1 - q$, $l_2 = q_2 - q$ and

$$BD^*((S-q_1)\cap(S-q_2)) = BD^*((S-l_1)\cap(S-l_2)) \geq \tfrac{1}{4}(BD^*(S))^4.$$

For each $n \in (S-q_1)\cap(S-q_2)$, there exist $s_{n,1} < s_{n,2} \in S$ such that $n = s_{n,1} - q_1 = s_{n,2} - q_2$. If $n_1 < n_2$, then $s_{n_1,1} < s_{n_2,1}$ and $s_{n_1,2} < s_{n_2,2}$. So $\{s_{n_1,1}, s_{n_1,2}\} \cap \{s_{n_2,1}, s_{n_2,2}\} \neq \emptyset$ if and only if $s_{n_1,2} = s_{n_2,1}$. Then for every $n \in (S-q_1)\cap(S-q_2)$, there is at most one $m \in (S-q_1)\cap(S-q_2)$ such that $\{s_{n,1}, s_{n,2}\} \cap \{s_{m,1}, s_{m,2}\} \neq \emptyset$. Let H be a maximal subset of $(S-q_1)\cap(S-q_2)$ with the property that for every $m \neq n \in H$, $\{s_{n,1}, s_{n,2}\} \cap \{s_{m,1}, s_{m,2}\} = \emptyset$. Then

$$BD^*(H) \geq \tfrac{1}{2}BD^*((S-q_1)\cap(S-q_2)) \geq \tfrac{1}{8}(BD^*(S))^4.$$

Let $S_1 = \{s_{n,1} \mid n \in H\}$ and $S_2 = \{s_{n,2} \mid n \in H\}$. Then $S_1 \cup S_2 \subset S$ and $S_1 \cap S_2 = \emptyset$.

Since $BD^*(H) \geq \frac{1}{8}(BD^*(S))^4$, we can find integers $M_j, N_j \in \mathbb{N}$ with $N_j - M_j \to \infty$ as $j \to \infty$, and

$$\frac{|H \cap [M_j, N_j - 1]|}{N_j - M_j} \geq \frac{1}{8}(BD^*(S))^4 - \frac{1}{j}.$$

Since S is an independent set for (U_1, U_2), for any $j \geq 1$ there is $y \in X$ with $T^q y \in U_1$, $T^{q_i} y \in U_i$ for $i = 1, 2$ and $T^{s_{n,i}}(y) \in U_i$ for $i = 1, 2$ and $n \in H \cap [M_j, N_j - 1]$. Then $H \cap [M_j, N_j - 1] \subset N(T^{q_1} y, U_1) \cap N(T^{q_2} y, U_2)$.

Since x is a transitive point, there is a sequence $\{m_i\}$ with $\lim_{i\to\infty} T^{m_i} x = T^q y$. Then $T^{l_1} x, T^{l_2} x \in U$ and

$$\lim_{i\to\infty} T^{m_i}(T^{l_1} x) = T^{q_1} y \in U_1, \quad \lim_{i\to\infty} T^{m_i}(T^{l_2} x) = T^{q_2} y \in U_2.$$

Since T is continuous, we can find $\eta > 0$ such that if $u, v \in X$ with $d(u, v) < \eta$ then $d(T^k u, T^k v) < \delta/2$. Then there exists $N \in \mathbb{N}$, such that if $t \geq N$, then

$$d(T^{m_t}(T^{l_1} x), T^{q_1} y) < \eta, \quad d(T^{m_t}(T^{l_2} x), T^{q_2} y) < \eta.$$

For such m_t, we have

$$d(T^{m_t+k}(T^{l_1} x), T^{q_1+k} y) < \frac{\delta}{2}, \quad d(T^{m_t+k}(T^{l_2} x), T^{q_2+k} y) < \frac{\delta}{2},$$

if $k \in H \cap [M_j, N_j - 1]$ since then $T^{q_1+k} y \in U_1$ and $T^{q_2+k} y \in U_2$. From the fact that $d(U_1, U_2) > 2\delta$ we have that

$$d(T^{m_t+k}(T^{l_1} x), T^{m_t+k}(T^{l_2} x)) > \delta$$

and so

$$\frac{|\{n \in \mathbb{Z}_+ \mid d(T^n(T^{l_1} x), T^n(T^{l_2} x)) > \delta\} \cap [m_t + M_j, m_t + N_j - 1]|}{N_j - M_j} \geq \frac{1}{8}(BD^*(S))^4 - \frac{1}{j}.$$

Since j is arbitrary, we have that the set

$$\{n \in \mathbb{Z}_+ \mid d(T^n(T^{l_1} x), T^n(T^{l_2} x)) > \delta\}$$

has upper Banach density no less than $\frac{1}{8}(BD^*(S))^4$, which implies that (X, T) is Banach mean sensitive by Proposition 6.1(5).

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The following corollary is a direct consequence of Theorem 6.2.

COROLLARY 6.7. *If (X, T) is almost Banach mean equicontinuous then the topological entropy of (X, T) is zero.*

To end the paper we discuss minimal examples which are Banach mean equicontinuous. It is easy to check that the Denjoy example or Sturmian minimal systems are minimal Banach equicontinuous. In [**2**] Auslander gave an example which is minimal mean equicontinuous by modifying an example of Floyd. We will refer to this as the Auslander–Floyd example. We can show that the Auslander–Floyd example in [**2**] is also Banach mean equicontinuous. In fact, we can prove a slightly more general result, and it is easy to see that the Auslander–Floyd example satisfies the condition of the following proposition.

PROPOSITION 6.8. *Let (X, T) be a dynamical system. If for every $\varepsilon > 0$ there exists an open cover $\{U_1, \ldots, U_n\}$ of X such that:*
(1) $T(U_i) \subset U_{i+1 \,(\mathrm{mod}\, n)}$ *for* $i = 1, \ldots, n$;
(2) $\#\{i \mid \mathrm{diam}(U_i) \geq \varepsilon\} < \varepsilon n$;
then (X, T) is Banach mean equicontinuous.

Proof. Let $\delta > 0$ be a Lebesgue number of the open cover. For every $x, y \in X$ with $d(x, y) < \delta$, there is $i_0 \in \{1, \ldots, n\}$ such that $x, y \in U_{i_0}$. Without loss of generality, we assume that $i_0 = 1$. For $0 < N < M$ and $M - N > n$

$$\frac{1}{M-N}\sum_{i=N}^{M-1} d(T^i x, T^i y) \leq \frac{1}{M-N}\sum_{i=N}^{M-1} \mathrm{diam}(U_{i \,(\mathrm{mod}\, n)})$$
$$< \varepsilon + \frac{\mathrm{diam}(X)}{M-N} \cdot \frac{M-N+n}{n} \cdot \varepsilon n < (1 + 2\mathrm{diam}(X))\varepsilon.$$

Thus (X, T) is Banach mean equicontinuous. □

7. *Further discussions*

By Corollary 4.8, there are many almost mean equicontinuous systems which have positive entropy. Then by Theorem 6.2 they are not almost Banach mean equicontinuous. But the following question is still open.

Question 7.1. Is there a minimal system which is mean equicontinuous but not Banach mean equicontinuous?

We know that a mean equicontinuous system is a proximal extension of its maximal equicontinuous factor. It would be interesting to know whether the following question has a positive answer.

Question 7.2. Is a minimal Banach equicontinuous system an almost one-to-one extension of its maximal equicontinuous factor?

We note that the same question is asked for minimal mean equicontinuous systems in [**2**].

Acknowledgements. The first author was supported in part by Scientific Research Fund of Shantou University (YR13001), Guangdong Natural Science Foundation

https://doi.org/10.1017/etds.2014.41 Published online by Cambridge University Press

(S2013040014084) and NNSF of China (11326135). The second and third authors were supported in part by NNSF of China (11371339). The authors would like to thank Wen Huang, Song Shao for very useful discussions; and Felipe García-Ramos, Benjy Weiss for very useful comments. The authors would also like to thank the anonymous referee for his/her helpful suggestions concerning this paper.

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