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The core model for sequences of measures. I

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The model $K(\mathscr{F})$ presented in this paper is a new inner model of ZFC which can contain measurable cardinals of high order. Like the model $L(\mathscr{F})$ of [**14**], from which it is derived, $K(\mathscr{F})$ is constructed from a sequence $\mathscr{F}$ of filters such that $K(\mathscr{F})$ satisfies for each $(\alpha,\beta)\in\text{domain}(\mathscr{F})$ that $\mathscr{F}(\alpha,\beta)$ is a measure of order β on α and the only measures in $K(\mathscr{F})$ are the measures $\mathscr{F}(\alpha,\beta)$. Furthermore $K(\mathscr{F})$, like $L(\mathscr{F})$, has many of the basic properties of L: the GCH and $\diamondsuit$ hold and there is a definable well ordering which is Δ^1_3 on the reals. The model $K(\mathscr{F})$ is derived from $L(\mathscr{F})$ by using techniques of Dodd and Jensen [**2–5**] to build in absoluteness for measurability and related properties. The following theorem summarizes some of the basic properties of $K(\mathscr{F})$:

THEOREM. *If there is no model of* $\exists\kappa\, o(\kappa)=\kappa^{++}$ *then there is a unique sequence* $\mathscr{F}_M$ *such that*

(i) *If* U *is any ultrafilter on* $P(\alpha)\cap K(\mathscr{F}_M)$ *such that* $K(\mathscr{F}_M)^\alpha/U$ *is well founded and* U *is normal for functions in* $K(\mathscr{F}_M)$ *then* $U=\mathscr{F}_M(\alpha,\beta)$ *for some* $\beta<o^{\mathscr{F}_M}(\alpha)$.

(ii) *Any elementary embedding* $i\colon K(\mathscr{F}_M)\to N$ *into a transitive class* N *is an iterated ultrapower of* $K(\mathscr{F}_M)$.

(iii) *If* κ *is a singular cardinal then* κ^+ *in* $K(\mathscr{F}_M)$ *is the same as in the real world.*

Part I of this paper will include the definition and basic properties of $K(\mathscr{F})$, and will conclude with the proof that $K(\mathscr{F})$ is a model of $ZFC+GCH$ with basic properties similar to those of L. Sections 4–7 will be in part II, to appear later.

Part II will develop the fine structure, prove the covering lemma, and give a number of applications. The rest of this introduction will cover both parts.

Before listing some applications we will outline enough of the basic notation to summarize our results more precisely. The notation will be given in more detail in Section 1 and in the body of the paper.

The ordering $\lhd$ of normal measures on κ is defined by $U\lhd U'$ if $U\in V^\kappa/U'$. It is shown in [**14**] that $\lhd$ is a well founded partial order. The order function o is defined on ultrafilters by recursion: $o(U)=\{o(U')\colon U'\lhd U\}$ and $o(\kappa)$ is defined to be $\{o(U)\colon U$ is a normal ultrafilter on $\kappa\}$. Then $o(\kappa)\leqslant(2^\kappa)^+$ or, with the GCH, $o(\kappa)\leqslant\kappa^{++}$. Thus $(\exists\kappa\, o(\kappa)=\kappa^{++})$ is the largest interesting large cardinal property which only uses normal measures. The sequences $\mathscr{F}$ are defined so that in $L(\mathscr{F})$ or $K(\mathscr{F})$ the only normal ultrafilters are those in the sequence $\mathscr{F}$, $\lhd$ is a well ordering on the ultrafilters on a cardinal κ, and $o(\kappa)$ may be as large as κ^{++} of $K(\mathscr{F})$ or $L(\mathscr{F})$.

The covering lemma is proved in part II under the assumption that there is no model of $\exists\kappa\, o(\kappa)=\kappa^{++}$. The version given in this paper is only used to show that there is a sequence $\mathscr{F}$ such that $(\kappa^+)^{(K(\mathscr{F}))}=\kappa^+$ for all sufficiently large singular cardinals.

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Stronger forms of the covering lemma are possible and will be discussed in later papers, but the strongest form, that obtained by Dodd and Jensen for $L(U)$, may be false for $K(\mathscr{F})$ if there is an inaccessible limit of measurable cardinal. A proof will appear in [**15**].

The version proved in this paper is used to adapt arguments of Kunen [**10**]. Thus we prove (as Kunen proved for $L(U)$) that if $\mathscr{F}$ and $\mathscr{F}'$ are sequences of measures in $K(\mathscr{F})$ and $K(\mathscr{F}')$ respectively and they have the same domain then $\mathscr{F} = \mathscr{F}'$. We also prove that this result is false if $K(\mathscr{F})$ and $K(\mathscr{F}')$ are replaced by $L(\mathscr{F})$ and $L(\mathscr{F}')$.

The maximal sequence $\mathscr{F}_M$ is defined by recursion over its domain (Theorem 7·8) and the covering lemma is used again to show that any strong sequence $\mathscr{F}$ is an initial segment of an iterated ultrapower of $\mathscr{F}_M$. These results are then used to show that any of the following imply the existence of a model of $\exists\kappa\, o(\kappa) = \kappa^{++}$. The references are to proofs of consistency with ZFC, using the indicated hypothesis.

(i) κ and κ^+ are both weakly compact ([**1**], from κ^+-supercompactness).

(ii) κ is κ^+ strongly compact ([**18**], from κ^+-supercompactness).

(iii) κ is measurable and $2^\kappa > \kappa^+$ (Woodin [**20**], from a $P^2(\kappa)$-measurable cardinal).

(iv) κ is measurable and $2^\kappa > (\kappa^+)^{(K(\mathscr{F}_M))}$ (Woodin [**20**], from a $P^2(\kappa)$-measurable cardinal.)

(v) There is a normal ultrafilter U on κ such that $i^U(\kappa) = \kappa^+$ (in particular, the axiom of determinancy holds) (Solovay, see [**6**], pp. 557–559, from the axiom of determinacy).

(vi) Every κ complete filter can be extended to a κ complete ultrafilter ([**18**], from 2^κ-supercompactness).

(vii) There is a κ^+ saturated ideal on a successor cardinal κ. ([**11**], from hugeness).

Further results will be proved in later papers using stronger versions of the covering lemma. For completeness we list some of these results here:

(viii) Suppose κ is regular in a model M of ZFC, N is a set generic extension of M, and $cf(\kappa) < \kappa$ in N while $cf(\kappa) = \kappa$ in M. Then M has in inner model with $o(\kappa) > 0$. If $cf(\kappa) > \omega$ in N then M has an inner model with $o(\kappa) \geqslant (cf(\kappa))^{(N)}$.

The case $cf(\kappa) = \omega$ is due to Dodd and Jensen [**2**]; Prikry forcing shows that the conclusion $o(\kappa) > 0$ cannot be strengthened. Magidor's extension of Prikry forcing in [**13**] shows that the conclusion $o(\kappa) \geqslant cf(\kappa)^{(N)}$ cannot be strengthened in the case $cf(\kappa) > \omega$.

(ix) If κ is a singular cardinal, $2^\kappa = \kappa$ and $2^\kappa > \kappa^+$ then there is a model with $\{o(\alpha) : \alpha < \kappa\}$ unbounded in κ. If $cf(\kappa) > \omega$ then there is a model of $\exists\kappa(o(\kappa) = \kappa^+)$.

The most likely conjecture seems to be that the case $cf(\kappa) = \omega$ is also strong and the difference in the two cases is an artifact of weaknesses in the proof. However the fact that the first cardinal κ such that $2^\kappa > \kappa^+$ cannot be singular of uncountable cofinality [**17**] but can have cofinality ω [**12**] might suggest caution even though Magidor's result requires much larger cardinals. It follows from (iii) that a $P^2(\kappa)$-measurable cardinal is enough.

(x) If the closed unbounded filter on $\aleph_1$ is an ultrafilter and dependent choice holds then there is a model of $\exists\kappa(o(\kappa) > \kappa^+)$. On the other hand if $M \models o(\kappa) = \kappa^{++}$ then there is a generic extension of M in which the closed unbounded filter on $\aleph_1$ is an ultrafilter and dependent choice holds [**16**].

The second clause is a strengthening of a result of Woodin and Radin [**19**] which required κ to be $P^2(\kappa)$ measurable. The actual conclusion and hypothesis of the two

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clauses are stronger than $o(\kappa) > \kappa^+$ and weaker than $o(\kappa) = \kappa^{++}$, respectively, but a gap does remain.

(xi) Any of the following are equiconsistent with $o(\kappa) = 2$, if $\delta \geqslant \omega$, or with $o(\kappa) = \delta + 1$ if $\omega < \delta < \kappa$:

(*a*) there is a κ^+-saturated, normal ideal I such that $\{\nu < \kappa: cf(\nu) = \delta\} \notin I$.

(*b*) there is a κ-complete ultrafilter U on κ such that $\{\nu < \kappa: cf(\nu) = \delta\} \in U$ and the closed unbounded filter is contained in U.

(*c*) There is an ultrafilter on κ with δ skies (see [**9**]).

These results will appear in [**15**].

This paper is intended to be largely self-contained and for this reason a few arguments from [**7**] and [**14**] have been included. Nevertheless some prior acquaintance with fine structure [**7**] and with [**14**] would be highly desirable, if not actually necessary. A knowledge of iterated ultrapowers (see [**10**]) is absolutely necessary.

The first section is devoted to notation and preliminaries, including the definition of the $J^{\mathscr{F}}_\alpha$ sequence. One interesting fact is also included in section one. Section two is devoted to iterated ultrapowers and to indiscernibles. The most important results are Lemma 2·19, which states that sufficiently large systems of indiscernibles can be thinned out to obtain considerably stronger systems, and Lemma 2·21, which states that sufficiently large system can be obtained by means of iterated ultrapowers.

Section three starts with the definitions of an $\mathscr{F}$ mouse for a sequence $\mathscr{F}$ of filters. A technique taken from [**14**] shows that the class of $\mathscr{F}$ mice is well ordered and hence we can define $K(\mathscr{F})$ to be the class of sets constructable from mice. Evidently $K(\mathscr{F}) \models ZFC$. Most of the rest of the section is devoted to a proof that every member of $K(\mathscr{F})$ is a member of some $\mathscr{F}$-mouse. It follows as a corollary that such elementary properties of L as GCH and $\diamondsuit$ are true in $K(\mathscr{F})$, as are such properties of $L(\mathscr{F})$ as the fact that every normal ultrafilter is a member of the sequence $\mathscr{F}$. Part I concludes with a statement of the fundamental Theorem 3·18 which permits the definition of a sequence $\mathscr{F}$ by recursion: If $\mathscr{F}$ is a sequence such that $\mathscr{F}(\alpha, \beta)$ is a well founded $K(\mathscr{F} \restriction (\alpha, \beta))$-ultrafilter for each (α, β) then $\mathscr{F}$ is a sequence of measures in $K(\mathscr{F})$.

The first section of part II describes the fine structure of $L(\mathscr{F})$. The entire section is devoted to the proof by induction of Theorem 4·1, which asserts that Theorem 3·18 is true if each $\mathscr{F}(\alpha, \beta)$ is countably complete. In addition, theorem 4·1 contains a statement which is later strengthened to assert that Σ^*_n codes exist for each $J^{\mathscr{F}}_\alpha$. Much of the section is devoted to defining Σ^*_n codes. This theory is based on that of Jensen [**7**] in L and Dodd and Jensen [**2**] in $L(U)$, but much of it is considerably more complicated in $L(\mathscr{F})$.

Section five contains two results which would belong in Section 3, except that their proofs require the special case of 3·18 which was proved in Section 4. These results are Theorem 5·1, which states that if $\mathscr{F}$ is a sequence of measures in $K(\mathscr{F})$ then $K(\mathscr{F} \restriction \lambda) \subset K(\mathscr{F})$ for all λ, and Theorem 5·2, which states that if $i: K(\mathscr{F}) \to M$ is elementary and M is well founded then $M = K(i(\mathscr{F}))$.

Finally Section 6 contains the covering lemma and Section 7 contains the main results of the paper which were outlined earlier.

A recent lemma of Dodd and Jensen [**2**, lemma 8·19] could be used to simplify some of the proofs in this paper. This lemma states that

If M is rudimentarily closed and has well founded iterated ultrapowers, $i: M \to N$ is

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an iterated ultrapower from M into a transitive submodel of N, and $\sigma: M \to N$ as any Σ_0-elementary embedding then for all $\nu \in M$ we have $i(\nu) \leqslant \sigma(\nu)$. In particular, if σ is also an iterated ultrapower then $\sigma = i$.

The proof is similar to that of part 2 of Theorem 3·3. Its most dramatic application to this paper is the elimination of ϕ-minimality from the proof of Theorem 3·5 as well as from the proofs in [14] from which Theorem 3·5 was adapted.

The new lemma also yields two simplifications in the fine structure arguments of Section 4. The proof of claim 4·46 in Theorem 4·45, which is somewhat delicate in this paper, becomes almost immediate with the new lemma. In Lemma 4.48, preservation of parameters, the direction $j_m \pi_m^{-1}(p_{m+1}) \leqslant i_m(p_{m+1})$ is immediate with the new lemma. Unfortunately this is the easier direction and only the last paragraph of the proof is eliminated. These simplifications are relatively minor, but appear to be essential in the extension of these results to models with hypermeasures.

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1. *Notation and preliminaries*

Most of the notation in this paper is either standard or is explained where it first appears in the text. This section is devoted primarily to the definitions and basic properties of rudimentary functions, the $J_\alpha^{\mathscr{F}}$ series, and sequences $\mathscr{F}$ of filters.

A parameter is always a finite set of ordinals. We define here the well ordering of parameters which is used in this paper. When the first version of this paper was written I could not recall having seen this ordering used before. Since then several people have pointed out examples of its use; in particular it came up naturally in a first year graduate set theory course since it is the same as the ordering obtained by mapping $\{\alpha_1, \ldots, \alpha_n\}$, with the α's in decreasing order, onto $\omega^{\alpha_1} + \ldots + \omega^{\alpha_n}$. It still seems to me, however, that this ordering does not have the 'standard' status that its naturalness justifies. Improbable as it may seem that such a technical matter could significantly affect the course of the proofs, several of our arguments would have been at least distinctly inconvenient with another ordering.

Definition 1·1. If x and y are finite sets of ordinals then $x < y$ if there is an $\alpha \in y - x$ such that $x - (\alpha + 1) = y - (\alpha + 1)$ or, equivalently, if x is less than y in the lexicographical ordering when both are regarded as descending sequences of ordinals.

Most orderings used for parameters are based on the lexicographical ordering of increasing, rather than decreasing sequences of ordinals, and some modification is needed to make this a well ordering. In our case no such modification is needed:

PROPOSITION 1·2. *The ordering $<$ of parameters is a well ordering.*

Proof. Suppose that X is an infinite set of parameters; we have to show that X has a minimal member. We will define a decreasing sequence of ordinals α_n. Suppose that $\alpha_0, \ldots, \alpha_{n-1}$ have been defined and let X_n be the set of members a of X such that $\alpha_0, \ldots, \alpha_{n-1}$ are the n largest members of a. We assume as an induction hypothesis that $X_n \neq 0$ and that every member of X_n is smaller than any member of $X - X_n$. If $a = \{\alpha_0, \ldots, \alpha_{n-1}\} \in X_n$ then a is the smallest member of X_n and hence of X so we are done. Otherwise let α_n be the least ordinal which is the $(n+1)$st largest member of some

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member b of X_n. Then $b \in X_{n+1}$, so $X_{n+1} \neq 0$, and by the minimality of α_n any member of X_{n+1} precedes any member of $X_n - X_{n+1}$ and hence any member of X.

Now $(\alpha_0, \ldots)$ is a descending sequence of ordinals and hence must be finite. Thus this process must stop, and it can only do so by constructing the minimal member of X. ∎

Definition 1·3. A fuction $f: V^n \to V$ is rudimentary iff it is generated by schemata (i)–(v) below.

(i) $f(x_1, \ldots, x_n) = x_i$ is rudimentary if $1 \leqslant i \leqslant n < \omega$.

(ii) $f(x, y) = \{x, y\}$ is rudimentary.

(iii) $f(x, y) = x - y$ is rudimentary.

(iv) $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$ is rudimentary if h and $g_1, \ldots, g_n$ are.

(v) $f(x, y) = \bigcup_{z \in y} h(x, z)$ is rudimentary if h is.

If D is a class, then f is D-rudimentary if it is generated by (i)–(v) and

(vi) $f(x) = D \cap x$ is D-rudimentary.

More information on D-rudimentary sets can be found in [7] from which the definition and (with modifications) the next two propositions were taken. A set M is said to be D-rudimentarily closed if whenever $x \in M$ and f is D-rudimentary we have $f(x) \in M$. A subset X of M is said to be D-rudimentary if $X = \{x \in M : f(x) = 0\}$ for some D-rudimentary function f. A function f is D-rudimentary in the parameter p if $f(x) = g(x, p)$ for all x and some D-rudimentary function g. If M is D-rudimentarily closed then M^+ is the D-rudimentary closure of $M \cup \{M\}$.

The concept of 'rudimentary function' is very close to that of a 'Δ_0 set'. In particular, if M is transitive and rudimentarily closed then the rudimentary subsets of M are exactly the Δ_0-subsets of M.

Proposition 1·4. *Suppose M is rudimentarily closed and $D \cap x \in M$ for all $x \in M$. Then M is D-rudimentarily closed.*

Proof. We will show that for every D-rudimentary function h and every $w \in M$ there is a rudimentary h' such that $h \restriction w = h' \restriction w$. This is immediate for (i)–(iii) and (vi), so we need only show that the class of D-rudimentary functions with this property is closed under (iv) and (v).

Case (iv). If $f(x) = h(g_1(x), \ldots, g_n(x))$, $w \in M$, and $h, g_1, \ldots, g_n$ have the required property then pick $g_i', \ldots, g_n'$ so that $g_i \restriction w = g_i' \restriction w$. Since M is rudimentarily closed, $w' = \{(g_1'(x), \ldots, g_n'(x)) : x \in w\} \in M$. Pick h' so that $h \restriction w' = h' \restriction w'$ and let

$$f'(x) = h'(g_1'(x), \ldots, g_n'(x)).$$

Then $f \restriction w = f' \restriction w$.

Case (v). Suppose $f(x, y) = U\{h(x, z) : z \in y\}$ and $w, u \in M$. We need f' so that $f \restriction w \times u = f' \restriction w \times u$. Pick h' so that $h' \restriction (w \times Uu) = h \restriction (w \times Uu)$ and set

$$f'(x, y) = U\{h'(x, z) : z \in y\}. \text{ ∎}$$

If the hypothesis of 1·4 holds then (M, D) is said to be *amenable*.

The next proposition says essentially that 'rudimentary in the parameter M' is the same as 'first order definable over M'.

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PROPOSITION 1·5. *Suppose M is transitive and rudimentarily closed and*

$$D \cap M^+ \subset \mathscr{P}(M).$$

Then any subset of M which is D-rudimentary in the parameter M is in the smallest class Γ of subsets of M such that Γ contains every D-rudimentary subset of M, Γ is closed under boolean combinations, and whenever $X \in \Gamma$ then so are: (i) $\{x: \exists y \in M (x, y) \in X\}$ *and* (ii) $\{x: \{y \in M: (x, y) \in X\} \in D\}$.

Proof. The proof of 1·5 requires a somewhat more delicate induction than that of 1·4. We start with a modified characterization of the class of functions D-rudimentary in the parameter M: The functions (i)–(iii) and (vi) are D-rudimentary in M, as is the constant function $f( ) = M$. Also, if $g_1, \ldots, g_n$ are D-rudimentary in M then so are

(iv′) $f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_n(\vec{x}))$ where h is one of (i)–(iii) or (vi).

(v′) $f(\vec{x}) = \bigcup\{g_1(\vec{x}, z) : z \in g_2(\vec{x})\}$.

Claim. This characterizes the set of functions D-rudimentary in M.

Proof. It is clear that every such function is D-rudimentary in M. To see the other direction it is enough to show that these functions are closed under composition, (iv), but a straightforward induction on the length of the derivation of f from this characterization shows that for any $g_1, \ldots, g_n$ so obtained $f(g_1, \ldots, g_n)$ may also be obtained. ∎

Schema (v′) is still too complicated, so we will restrict ourselves to the class R^* of functions obtained by using (v″) below instead of (v′):

(v″) $f(\vec{x}) = \bigcup\{h(\vec{x}, z) : z \in g(\vec{x}) \cap M\}$.

R^* is closed under composition and is contained in the class of functions D-rudimentary in the parameter M.

Now we enumerate the functions $(f_i : i \in \omega)$ of R^* so that f_0 is the identity, $f_0(x) = x$, f_1 is the constant function $f_1( ) = M$, and each f_i is preceded in the list by all of its subformulas. As easy induction shows that whenever $i > 0$ and $x \in f_i(z)$ for some $z \in M$ then there is $i' < i$ and $z' \in M$ such that $x = f_{i'}(z')$.

Now we need to notice that for each $j \in \omega$ the following two sets are both in Γ and have characteristic functions in R^*:

$$\{((i, x), (i', x')) \in (j \times M)^2 : f_i(x) = f_{i'}(x')\} \tag{1}$$

and

$$\{((i, x), (i', x')) \in (j \times M)^2 : f_i(x) \in f_{i'}(x')\}. \tag{2}$$

Assume as an induction hypothesis that it is true for j; we will show that it is true for $j + 1$. First we show that $\{(i, x, x') \in j \times M \times M : f_i(x) \in f_j(x')\}$ is definable by considering five cases, depending on whether f_j has the form (iv) for one of the four functions (i)–(iii) or (vi), or f_j has the form (v″). We'll only discuss the case (v″) here. Then

$$f_j(x) = \bigcup\{h(x, z) : z \in g(x) \cap M\},$$

where h and g are among the $(f_i : i < j)$. Then for $i < j$ we have

$$f_i(x) \in f_j(x') \equiv \exists z \in M (z \in g(x) \quad \text{and} \quad f_i(x) \in h(x, z)). \tag{3}$$

By using the induction hypothesis we can see that $\{(i, x, x') : \exists z \in M (z \in g(x))$ and

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$f_i(x) \in h(x,z)\}$ is in Γ and has characteristic function in R^*, and this proves the induction step for (3). But (1) can be expressed in terms of (3) by using extensionality:

$$f_i(x) = f_{i'}(x') \equiv \forall i'' < j \forall z \in M (f_{i''}(z) \in f_i(x) \equiv f_{i''}(z) \in f_{i'}(x))$$

for $i, i' \leqslant j$. Finally the full strength of (2) can be obtained from (1) and (3).

To complete the proof of Proposition 1·5 it is enough to show that R^* contains every function on M which is D-rudimentary in the parameter M; and to do it is enough to show that R^* is closed under (v'). Suppose $f(x) = \bigcup\{h(x,z) : z \in g(x)\}$ where h and g are in R^*, say $g = f_j$. Now let $h'(x,z) = \bigcup_{i<j} h_i(x,z)$ where

$$h_i(x,z) = \begin{cases} h(x, f_i(z)) & \text{if} \quad f_i(z) \in g(z) \\ 0 & \text{if} \quad f_i(z) \notin g(z). \end{cases}$$

Then h' is in R^* and

$$\begin{aligned} f(x) &= \bigcup\bigcup_{i<j} \{h(x, f_i(z)) : z \in M \text{ and } f_i(z) \in g(z)\} \\ &= \bigcup\{h'(x,z) : z \in M\} \end{aligned}$$

so $f(x)$ is in R^* as well. ∎

In practice we will be working with only two sorts of classes D and will modify our terminology to accommodate them. The first is a set A of ordinals in M; the second is a sequence $\mathscr{F}$ of filters, that is, a function $\mathscr{F}$ with domain of the form $\{(\kappa,\lambda) : \kappa \subset l(\mathscr{F})$ and $\lambda < o^{\mathscr{F}}(\kappa)\}$ where $l(\mathscr{F})$ is an ordinal called the *length* of $\mathscr{F}$ and $o^{\mathscr{F}}$ is a unary function called the *order* of $\mathscr{F}$. We will say that f is $\mathscr{F}$-rudimentary if, in the precise sense of Definition 1·3, f is $\{(x,\kappa,\lambda) : x \in \mathscr{F}(\kappa,\lambda)\}$-rudimentary. We will frequently let $\mathscr{F}$ be understood, writing 'rudimentary' or 'A-rudimentary' rather than '$\mathscr{F}$-rudimentary' or 'A, $\mathscr{F}$-rudimentary'.

The requirement that $\mathscr{F}$ be coherent requires a modification to the obvious definition of the $J^{\mathscr{F}}_\alpha$-sequence. Write $\mathscr{F} \restriction (\kappa,\lambda)$ for

$$\mathscr{F} \restriction \{(\alpha,\beta) : \alpha < \kappa \quad \text{or} \quad \alpha = \kappa \quad \text{and} \quad \beta < \lambda\}$$

and $\mathscr{F} \restriction \kappa$ for $\mathscr{F} \restriction (\kappa, 0)$.

Definition 1·6. A sequence of filters $\mathscr{F}$ is said to be *coherent* in a model M if $\mathscr{F}$ is a sequence of normal ultrafilters in M and for each $(\kappa,\lambda) \in \operatorname{dom} \mathscr{F}$

(1) If $i : M \to M^\kappa/\mathscr{F}(\kappa,\lambda)$ then $i(\mathscr{F}) \restriction (\kappa+1) = \mathscr{F} \restriction (\kappa,\lambda)$, and

(2) If $f \in M$ and $\{\alpha \in \kappa : (f(\alpha) < o^{\mathscr{F}}(\alpha)\} \in \mathscr{F}(\kappa,\lambda)$ then for some $g \in L(\mathscr{F} \restriction (\kappa,\lambda)) \cap M$, $\{\alpha : f(\alpha) = g(\alpha)\} \in \mathscr{F}(\kappa,\lambda)$.

The ultrapowers here and elsewhere in this paper use only functions in M. The purpose of the notion of coherence, which comes from [**14**], will be explained in the next section. It is defined here so that the $J^{\mathscr{F}}_\alpha$ sequence can be defined in such a way that if $\mathscr{F}$ is coherent in $L(\mathscr{F})$ then it is coherent in $J^{\mathscr{F}}_\alpha$ for every ordinal α. We will define $J^{\mathscr{F}}_\alpha$ by recursion over α simultaneously with ordinals $d(\kappa,\lambda)$ for $\lambda < o^{\mathscr{F}}(\kappa)$ and functions $C(\kappa,\lambda',\lambda)$ for $\lambda' < \lambda < o^{\mathscr{F}}(\kappa)$. The function $C(\kappa,\lambda',\lambda)$ will be the least function $g \in L(\mathscr{F} \restriction (\kappa,\lambda))$ such that $[g]_{\mathscr{F}(\kappa,\lambda)} = \lambda'$ as required by 1·6 (ii) and the ordinal $d(\kappa,\lambda)$ will be chosen so that $C(\kappa,\lambda',\lambda)$ is a member of $J^{\mathscr{F}}_{d(\kappa,\lambda)}$ for every $\lambda' < \lambda$. Thus $\mathscr{F}(\kappa,\lambda)$ is coherent in every $J^{\mathscr{F}}_\alpha$ with $\alpha \geqslant d(\kappa,\lambda)$. We will not allow the use of $\mathscr{F}(\kappa,\lambda)$ for constructing sets until after stage $d(\kappa,\lambda)$ so $J^{\mathscr{F}}_{d(\kappa,\lambda)} = J^{\mathscr{F} \restriction (\kappa,\lambda)}_{d(\kappa,\lambda)}$.

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Definition 1·7. Suppose $\mathscr{F}$ is a sequence of filters and $d(\gamma, \nu)$ and $C(\gamma, \nu', \nu)$ have been defined for all (γ, ν', ν) with $(\gamma, \nu) < (\kappa, \lambda)$ and $\nu' < \nu$.

(i) $d(\kappa, \lambda)$ is the least limit ordinal α such that $\alpha > \kappa$, $\alpha > \bigcup\{d(\gamma, \nu) : (\gamma, \nu) < (\kappa, \lambda)\}$ and $[\lambda\nu < \kappa\, o^{\mathscr{F}}(\nu)] \geqslant \lambda$ in ult $(J^{\mathscr{F}}_{\alpha}, \mathscr{F}(\kappa, \lambda))$.

(ii) If $\lambda' < \lambda$ then $C(\kappa, \lambda', \lambda)$ is the least function f in the order of construction of $J^{\mathscr{F}}_{d(\kappa,\lambda)}$ such that $[f] = \lambda'$ in ult $(J^{\mathscr{F}}_{d(\kappa,\lambda)}, \mathscr{F}(\kappa, \lambda))$.

(iii) $J^{\mathscr{F}}_{\alpha+1}$ is the $\mathscr{F} \restriction (\kappa, \lambda)$-rudimentary closure of $J^{\mathscr{F}}_{\alpha} \cup \{J^{\mathscr{F}}_{\alpha}\}$, where (κ, λ) is the least pair such that $d(\kappa, \lambda) \notin \alpha$.

(iv) if α is a limit ordinal then $J^{\mathscr{F}}_{\alpha} = \bigcup_{\alpha' < \alpha} J^{\mathscr{F}}_{\alpha'}$.

Notice that because $J^{\mathscr{F}}_{d(\kappa,\lambda)} = J^{\mathscr{F} \restriction (\kappa,\lambda)}_{d(\kappa,\lambda)}$ the definition of $d(\kappa, \lambda)$ and $C(\kappa, \lambda', \lambda)$ involve only the filter $\mathscr{F}(\kappa, \lambda)$ and the restricted sequence $\mathscr{F} \restriction (\kappa, \lambda)$. Generally when discussing $J^{\mathscr{F}}_{\alpha}$ we will tacitly ignore the part of $\mathscr{F}$ above the least (κ, λ) such that $d(\kappa, \lambda) > \alpha$, making no distinction between $\mathscr{F}$ and $\mathscr{F} \restriction (\kappa, \lambda)$.

In 1·7 (i), $[\lambda\nu < \kappa\, o(\kappa)]$ will actually be equal to λ in ult $(J^{\mathscr{F}}_{\alpha}, \mathscr{F}(\kappa, \lambda))$, since it is equal in M. In 1·7 (iii) we could achieve the same result by setting

$$\mathscr{F}^*(\kappa, \lambda) = \{(\kappa, \lambda, \omega \cdot d(\kappa, \lambda), x) : x \in \mathscr{F}(\kappa, \lambda)\}$$

and then constructing $J^{\mathscr{F}^*}_{\alpha}$ without special rules.

PROPOSITION 1·8. *Suppose $\mathscr{F}$ is coherent in $L(\mathscr{F})$. Then*

(i) *$d(\kappa, \lambda)$ and $C(\kappa, \lambda', \lambda)$ are defined for all $(\kappa, \lambda) \in$ domain $(\mathscr{F})$ and $\lambda' < \lambda$ and hence $L(\mathscr{F}) = \bigcup_{\alpha \in \mathbf{ON}} J^{\mathscr{F}}_{\alpha}$.*

(ii) *$d(\kappa, \lambda) < \kappa^{++}$ in $L(\mathscr{F} \restriction (\kappa, \lambda+1))$ and $d(\kappa, \lambda) \leqslant \kappa^{++} + \omega$ in $L(\mathscr{F} \restriction (\kappa, \lambda))$.*

(iii) *$[C(\kappa, \lambda', \lambda)]_{\mathscr{F}(\kappa,\lambda)} = \lambda'$ in $L(\mathscr{F})$ and hence in any set N such that*

$$J^{\mathscr{F}}_{d(\kappa,\lambda)} \subset N \subset L(\mathscr{F}).$$

(iv) *If $\nu < \delta < \lambda < o^{\mathscr{F}}(\kappa)$ and $f = C(\kappa, \delta, \lambda)$ then*

$$\{\gamma \in \kappa : C(\kappa, \nu, \lambda)(\gamma) = [C(\kappa, \nu, \delta) \restriction \gamma]_{\mathscr{F}(\gamma, f(\gamma))}\} \in \mathscr{F}(\kappa, \lambda).$$

Proof. Let $(\kappa, \lambda, \lambda')$ be the least triple such that $C(\kappa, \lambda', \lambda)$ is not defined, if any such exist. By coherence there is an $f \in L(\mathscr{F} \restriction (\kappa, \lambda))$ such that

$$[f]_{\mathscr{F}(\kappa,\lambda)} = \lambda' \text{ in ult } (L(\mathscr{F} \restriction (\kappa, \lambda)), \mathscr{F}(\kappa, \lambda))$$

and $\{\nu < \kappa : f(\nu) < o^{\mathscr{F}}(\nu)\} \in \mathscr{F}(\kappa, \lambda)$. But then $[f]_{\mathscr{F}(\kappa,\lambda)} = \lambda'$ in ult $(J^{\mathscr{F}}_{\alpha}, \mathscr{F}(\kappa, \lambda))$ for any sufficiently large α. This contradicts the assumption that $C(\kappa, \lambda', \lambda)$ is not defined. Since $C(\kappa, \lambda', \lambda)$ is defined for all $\lambda' < \lambda$, $d(\kappa, \lambda)$ is also defined and clause (i) is true. An easy collapsing argument shows that $\mathscr{P}(\kappa) \cap L(\mathscr{F} \restriction (\kappa, \lambda)) \subset J^{\mathscr{F} \restriction (\kappa,\lambda)}_{\nu}$ for some $\nu < \kappa^{++}$ in $L(\mathscr{F} \restriction (\kappa, \lambda))$ and (ii) follows by a simple computation.

Now suppose that for all $(\gamma, \nu, \nu') < (\kappa, \lambda, \lambda')$ (iii) holds for $C(\gamma, \nu', \nu)$; we claim that (iii) must also hold for $C(\kappa, \lambda', \lambda)$. Set $f = C(\kappa, \lambda', \lambda)$ and $\delta = [C(\kappa, \lambda', \lambda)]_{\mathscr{F}(\kappa,\lambda)}$ in ult $(L(\mathscr{F}), (\mathscr{F}(\kappa, \lambda))$. Then $\lambda' \leqslant \delta < \lambda$ by coherence and the definition of $C(\kappa, \lambda', \lambda)$. For any ordinal $\nu < \delta$ set $g_{\nu}(\gamma) = [C(\kappa, \nu, \delta) \restriction \gamma]_{\mathscr{F}(\lambda, f(\lambda))}$ for all $\gamma < \kappa$. Then in ult $(L(\mathscr{F}), \mathscr{F}(\kappa, \lambda))$ we have

$$[\lambda\gamma \mathscr{F}(\gamma, f(\gamma))]_{\mathscr{F}(\kappa,\lambda)} = \mathscr{F}(\kappa, \delta) \quad \text{and} \quad [\lambda\gamma C(\kappa, \nu, \delta) \restriction \gamma]_{\mathscr{F}(\kappa,\lambda)} = C(\kappa, \nu, \delta)$$

$$\text{so} \qquad [g_{\nu}]_{\mathscr{F}(\kappa,\lambda)} = [\lambda\gamma [C(\kappa, \nu, \delta) \restriction \gamma]_{\mathscr{F}(\gamma, f(\gamma))}]_{\mathscr{F}(\kappa,\lambda)} = [C(\kappa, \nu, \delta)]_{\mathscr{F}(\kappa,\delta)} = \nu \qquad (1)$$

by the induction hypothesis.

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Now $g_\nu \in J^{\mathscr{F}}_{d(\kappa,\delta)} \subset J^{\mathscr{F}}_{d(\kappa,\lambda)}$ for each $\nu < \delta$, so $[f]_{\mathscr{F}(\kappa,\lambda)} \geqslant \delta$ in $\mathrm{ult}(J^{\mathscr{F}}_{d(\kappa,\lambda)}, \mathscr{F}(\kappa,\lambda))$. But by definition $[f]_{\mathscr{F}(\kappa,\lambda)} = \lambda'$ there, so $\lambda' \leqslant \delta \leqslant \lambda'$ and hence $\delta = \lambda'$ and

$$[C(\kappa,\lambda',\lambda)_{\mathscr{F}(\kappa,\lambda)} = \lambda'$$

in $L(\mathscr{F})$. Since this is equivalent to the statement that if $f \in L(\mathscr{F})$ and

$$\{\nu < \kappa : f(\nu) < o(\nu)\} \in \mathscr{F}(\kappa,\lambda)$$

then for some $\delta < \lambda \{\nu : f(\nu) = C(\kappa,\delta,\lambda)(\nu)\} \in \mathscr{F}(\kappa,\lambda)$, the same statement is evidently true for any N with $J^{\mathscr{F}}_{d(\kappa,\lambda)} \subset N \subset L(\mathscr{F})$.

Finally, the right side of (iv) is equal to ν by (1) and the left side is ν by (iii), so (iv) is true. ∎

Note that 1·8 (ii) implies that $d(\kappa,\lambda) < \gamma$ for any measurable cardinal $\gamma > \kappa$ and, in fact, for any indiscernible for any measure greater than κ. Thus we never have to worry about $d(\kappa,\lambda)$ for more than one cardinal κ at a time.

PROPOSITION 1·9. *d and C are uniformly definable; indeed there are first order formulas Ψ and Φ such that*

$$J^{\mathscr{F}}_\alpha \models \Psi(\kappa,\lambda,\mathscr{F}(\kappa,\lambda)) \text{ iff } \alpha = d(\kappa,\lambda) \quad \text{and} \quad J^{\mathscr{F}}_\alpha \models \Phi(\kappa,\lambda,\lambda',f,\mathscr{F}(\kappa,\lambda)) \text{ iff } f = C(\kappa,\lambda',\lambda).$$

Proof. Assume as an induction hypothesis that we have such a first order definition of $C' = C \restriction \{(\gamma,\nu',\nu) : (\gamma,\nu) < (\kappa,\lambda)\}$. Since $g(\delta',\delta,f) = \lambda\gamma < \kappa[C(\kappa,\delta',\delta) \restriction \gamma]_{\mathscr{F}(\gamma,f(\gamma))}$ is definable in $J^{\mathscr{F}}_\alpha$ from C' and $\mathscr{F} \restriction \kappa$ the following are expressible by first order formulas over $J^{\mathscr{F}}_\alpha$:

(1) $$\forall \delta_1 < \delta_2 < \lambda'([g(\delta_1,\lambda',f)]_{\mathscr{F}(\kappa,\lambda)} < [g(\delta_2,\lambda',f)]_{\mathscr{F}(\kappa,\lambda)} < [f]_{\mathscr{F}(\kappa,\lambda)}).$$

(2) $$\forall f' \in J^{\mathscr{F}}_\alpha (\text{if } [f']_{\mathscr{F}(\kappa,\lambda)} < [f]_{\mathscr{F}(\kappa,\lambda)} \text{ then for some } \delta < \lambda'$$
$$([f']_{\mathscr{F}(\kappa,\lambda)} = [g(\delta,\lambda',f)]_{\mathscr{F}(\kappa,\lambda)})).$$

By using Proposition 1·8 we can see that (1) and (2) hold in $J^{\mathscr{F}}_{d(\kappa,\lambda)}$. On the other hand if (1) holds in $J^{\mathscr{F}}_\alpha$ then $[f]_{\mathscr{F}(\kappa,\lambda)} \leqslant \lambda'$ and if (2) holds then $[f]_{\mathscr{F}(\kappa,\lambda)} \leqslant \lambda'$ in

$$\mathrm{ult}(J^{\mathscr{F}}_\alpha, \mathscr{F}(\kappa,\lambda)).$$

Thus $\alpha = d(\kappa,\lambda)$ if α is least such that for all $\lambda' < \lambda$ there is $f \in J^{\mathscr{F}}_\alpha$ such that $\{\nu : f(\nu) < o^{\mathscr{F}}(\nu)\} \in \mathscr{F}(\kappa,\lambda)$ and (1) holds; and $C(\kappa,\lambda',\lambda)$ is the least function f in $J^{\mathscr{F}}_{d(\kappa,\lambda)}$ such that (1) and (2) hold there.

Note that Proposition 1·8 is true in every $J^{\mathscr{F}}_\alpha$ as well as in $L(\mathscr{F})$. Let ϕ be a formula which asserts of a standard model M and a sequence $\mathscr{F}$ that $M = J^{\mathscr{F}}_\alpha$ for some α and Propositions 1·8 and 1·9 are true in M. The formula ϕ is Π_2 and is preserved by cofinal Σ_1-elementary embeddings. In the rest of the paper a '$J^{\mathscr{F}}_\alpha$' will always be a standard model of this formula ϕ.

§ 2. *Iterated ultrapowers and indiscernibles*

This section discusses some aspects of the general theory of iterated ultrapowers and indiscernibles which will be used in the rest of this paper. Throughout this Section we will assume that M is an $\mathscr{F}$-rudimentarily closed set and that every iterated ultrapower of M by $\mathscr{F}$ is well founded. In the applications M will be either some $J^{\mathscr{F}}_\alpha$ or, in Section 4, a modification of $J^{\mathscr{F}}_\alpha$. Except where otherwise stated all ultrapowers are by functions

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in M. In Section 4 we will be using Σ_1-ultrafilters, that is, ultrapowers using all functions which are Σ_1 over M. Much of the discussion in this section still applies to Σ_1 ultrapowers; they will be discussed in more detail when they are used.

Definition 2·1. $\mathscr{F}$ is an ultrafiltler sequence for M if for all $(\kappa, \lambda) \in \text{domain}(\mathscr{F})$ and all $h \in M$,

(i) If $\{\nu < \kappa : h(\nu) < \nu\} \in \mathscr{F}(\kappa, \lambda)$ then for some $\nu_0 \in \kappa$ $\{\nu < \kappa : h(\nu) = \nu_0\} \in \mathscr{F}(\kappa, \lambda)$.

(ii) If $\{\nu < \kappa : h(\nu) < o^{\mathscr{F}}(\nu)\} \in \mathscr{F}(\kappa, \lambda)$ then for some $\lambda' < \lambda$

$$\{\nu < \kappa : h(\nu) = C(\kappa, \lambda', \lambda)(\nu)\} \in \mathscr{F}(\kappa, \lambda).$$

If (i) and (ii) hold for all functions h in a class Γ, then $\mathscr{F}$ is said to be a Γ-ultrafilter sequence.

Notation for iterated ultrapowers 2·2. Our treatment of iterated ultrapowers is based on that of Kunen [**I0**]. We present a detailed system of notation for iterated ultrapowers which will sometimes be convenient but more commonly will be better avoided.

Supports for iterated ultrapowers will be written as sequences of the form

$$\vec{f} = (f_0, g_0)_{a_0} (f_1, g_1)_{a_1} \cdots (f_{k-1}, g_{k-1})_{a_{k-1}}. \tag{1}$$

Here f_i and g_i will be functions in M on the variables $\vec{a} \restriction i = (a_0, \ldots, a_{i-1})$, and the variables $\vec{a}$ will range over the set

$$I(\vec{f}) = \{(\nu_0, \ldots, \nu_{k-1}) : \forall i < k \nu_i \in f_i(\vec{\nu} \restriction i)\}.$$

The subscripts will normally be $a_0, \ldots, a_{k-1}$ and in this case we will omit them, writing $\vec{f} = (f_0, g_0) \cdots (f_{k-1}, g_{k-1})$. The subscripts will be useful when we discuss rearrangements of iterated ultrapowers. They are intended to indicate the 'place' the variables a_i occupy; thus we will write $I(\vec{f}) = \{\vec{a} : \forall i < k\, a_i \in f_i(\vec{a} \restriction i)\}$. If f_0, g_0 and g_1 are the constant functions κ, 1 and 0, and f_1 is the identity on κ then we will write $(f_0, g_0)_{a_0} (f_1, g_1)_{a_1}$ as $(\kappa, 1)_{a_0} (a_0, 0)_{a_1}$ or simply as $(\kappa, 1)(a_0, 0)$.

The sequence $\vec{f}$ is called a *support* if for all $i < k$, domain (f_i) = domain $(g_i) = I(\vec{f} \restriction i)$ and $(f_i(\vec{a}), g_i(\vec{a})) \in \text{domain}(\mathscr{F})$ for all $\vec{a} \in I(\vec{f} \restriction i)$. If $\vec{f}$ is a support then the product ultrapower $\mathscr{F}(\vec{f})$ is defined by induction on k: If $x \subset I(\vec{f})$ then $x \in \mathscr{F}(\vec{f})$ iff

$$\{\vec{a} \in I(\vec{f} \restriction k-1) : \{a_{k-1} : \vec{a}(a_{k-1}) \in x\} \in \mathscr{F}(f_{k-1}(a), g_{k-1}(a))\} \in \mathscr{F}(\vec{f} \restriction k-1).$$

Finally, $\vec{f}$ is a support for a function h if domain $(h) \in \mathscr{F}(\vec{f})$.

In order to permit a subsequence of a support to be a support, we will allow f_i and g_i to depend on a subset of the ordinals $a_0, \ldots, a_{k-1}$. Thus if $\vec{b} = (a_{i_0}, \ldots, a_{i_{n-1}})$ is a subsequence of $\vec{a} = a_0, \ldots, a_{k-1}$ then we will identify a function $f(\vec{b})$ with the function $f^*(\vec{a})$ defined by $f^*(\vec{a}) = f(a_{i_0}, \ldots, a_{i_{n-1}})$. With this convention we can extend the notation to infinite iterated ultrapowers:

If $\vec{f}$ is an infinite sequence then $\vec{f}$ is a support if every $(f_i, g_i)_{a_i}$ is a member of a finite subsequence which is a support; $\mathscr{F}(\vec{f})$ is the union of the ultrapowers

$$\{\vec{a} \in I(\vec{f}) : \vec{a} \restriction x \in \mathscr{F}(\vec{f} \restriction x)\}$$

such that $\vec{f} \restriction x$ is a finite support; and $\vec{f}$ is a support for h if a finite subsequence of $\vec{f}$ is.

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Finally, $\text{ult}(\Gamma, \mathscr{F}(\vec{f}))$ is $\{[h]: h \in \mathbf{\Gamma}\}$ where

$$[h] = \{h' : \{\vec{a} : h(\vec{a}) = h'(\vec{a})\} \in \mathscr{F}(\vec{f})\}.$$

As an example of the use of this notation, consider the iterated ultrapowers

$$(M^\alpha/\mathscr{F}(\alpha,\beta))^{i(\alpha)}/i(\mathscr{F}(\alpha,\beta')), \tag{2}$$

$$(M^\alpha/\mathscr{F}(\alpha,\beta'))^\alpha/\mathscr{F}(\alpha,\beta), \tag{3}$$

where $\beta < \beta' < 0\mathscr{F}(\alpha)$ and $i: M \to M^\alpha/\mathscr{F}(\alpha,\beta)$ is the canonical embedding. Then (2) has support

$$(\alpha,\beta)_{a_0}(\alpha,\beta')_{a_1}$$

since $i(\alpha)$ and $i(\beta)$ are represented in the ultrapower $M^\alpha/\mathscr{F}(\alpha,\beta)$ by the constant functions $\lambda a_0 \alpha$ and $\lambda a_0 \beta$, respectively. Similarly the support of (2) can be written

$$(\alpha,\beta')_{a_0}(a_0, C(\alpha,\beta,\beta')(a_0))_{a_1}$$

since α and β are represented in $M^\alpha/\mathscr{F}(\alpha,\beta)$ by the functions

$$\lambda a_0 a_0 \quad \text{and} \quad \lambda a_0 C(\alpha,\beta,\beta')(a_0),$$

respectively. Now note that (1) and (2) are isomorphic. The isomorphism interchanges the variables:

$$\{a_0 \in \alpha : \{a_1 \in \alpha : (a_0, a_1) \in X\} \in \mathscr{F}(\alpha,\beta')\} \in \mathscr{F}(\alpha,\beta)$$

$$\text{iff } \{a_1 \in \alpha : \{a_0 \in a_1 : (a_0, a_1) \in X\} \in \mathscr{F}(a_1, C(\alpha,\beta,\beta')(a_1)) \in \mathscr{F}(\alpha,\beta').$$

Thus this isomorphism can be emphasized by rewriting the support of (2) so the variables correspond with those of (1):

$$(\alpha,\beta')_{a_1}(a_1, C(\alpha,\beta,\beta')(a_1))_{a_0}.$$

The next proposition asserts formally this commutativity of ultrafilters and hence allows us to change the order of treated ultrapowers. The proof is standard. When Σ_1 ultrafilters are used in Section 4 a modification of 2·3 will be needed.

PROPOSITION 2·3. *Suppose $\mathscr{F}$ is an ultrafilter sequence in the rudimentarily closed set M*

(i) *If $\kappa' < \kappa, \lambda' < o^{\mathscr{F}}(\kappa')$ and $\lambda < o^{\mathscr{F}}(\kappa)$ then the ultrapowers with supports*

$$(\kappa,\lambda)_{a_0}(\kappa',\lambda')_{a_1} \quad \text{and} \quad (\kappa',\lambda')_{a_1}(\kappa,\lambda)_{a_0}$$

are isomorphic.

(ii) *If $\lambda' < \lambda < o^{\mathscr{F}}(\kappa)$ then the ultrapowers with supports $(\kappa,\lambda')_{a_0}(\kappa,\lambda)_{a_1}$ and $(\kappa,\lambda)_{a_1}(a_1, C(\kappa,\lambda',\lambda)(a_1))_{a_0}$ are isomorphic.* |

The rest of this Section is concerned with constructing and using systems of indiscernibles. We begin with the definitions and basic properties, and then we show how a sufficiently large system of indiscernibles can be thinned out to obtain indiscernibles for a larger class of formulas. Finally we show that iterated ultrapowers can be used to generate such large systems of indiscernibles.

We use Γ to denote a class of partial functions on M. Γ will always be assumed to be closed under composition and to contain all $\mathscr{F}$-rudimentary functions and the coherence functions $C(-,-,-)(-)$. Γ may also contain constants, that is, functions of zero variables. If y is a set then we write $y \in \Gamma$ to mean that y is the domain of a function in Γ.

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Definition 2·4. A function h is in Γ with parameters from X if there is $p \in X^{<\omega}$ and $f \in \Gamma$ such that for all $x \in M$, $h(x) = f(x, p)$. The set of such functions is written $\Gamma[X]$, and $\mathbf{\Gamma}$ is used to denote $\Gamma[M]$.

We will normally write $\Gamma''(X)$ for the Skolem hull of X by functions in Γ, that is, for $\{f(p): f \in \Gamma \text{ and } p \in X^{<\omega}\}$. Sometimes this notation is unwieldy and will be modified; thus in Section 4 we will write $\Sigma_1^*(X)$ rather than $\Sigma_1^{*\prime\prime}(X)$.

Definition 2·5. *If* $X \subset M$ then $\mathscr{C}$ is a system of Γ-indiscernibles for $\mathscr{F}$ over X in M if $\mathscr{C}$ is a function satisfying (i) to (v) below:

(i) Domain $(\mathscr{C})$ = domain $(\mathscr{F})$ and $\mathscr{C}(\alpha, \beta) \subset \alpha$ for all $(\alpha, \beta) \in$ domain $(\mathscr{F})$.

(ii) If $\alpha' \in \mathscr{C}(\alpha, \beta)$, $\beta'' \in \Gamma''(X \cup \alpha')$, and $\beta' = C(\alpha, \beta'', \beta)(\alpha')$ then

$$\mathscr{C}(\alpha', \beta') = \mathscr{C}(\alpha, \beta'') \cap \alpha'.$$

(iii) (*a*) If $\beta \neq \beta'$ then $\mathscr{C}(\alpha, \beta) \cap \mathscr{C}(\alpha, \beta') = 0$.

(*b*) Suppose $\nu \in \bigcup_{(\alpha, \beta)} \mathscr{C}(\alpha, \beta)$. Then there are ordinals $(\alpha(\nu), \beta(\nu)) \in \Gamma''(X \cup \nu)$ such that $\nu \in \mathscr{C}(\alpha(\nu), \beta(\nu))$. Furthermore $\alpha(\nu)$ is the largest ordinal α such that $\nu \in \bigcup_\beta C(\alpha, \beta)$. If $\nu \in \mathscr{C}(\alpha', \beta')$ for some $\alpha' < \alpha(\nu)$ then there is a $\beta'' \in \Gamma''(X \cup \alpha')$ such that $\beta'' > \beta(\nu)$, $\alpha' \in \mathscr{C}(\alpha(\nu), \beta'')$, and

$$\beta' = C(\alpha(\nu), \beta(\nu), \beta'')(\alpha').$$

In order to state clauses (iv)–(v) we need the notation of a sequence from $\mathscr{C}$. We say that $\vec{c} = (c_0, \ldots, c_{k-1})$ is a $\Gamma[X]$-sequence from $\mathscr{C}$ with support $\vec{f}$ if $\vec{f}$ is a support from $\Gamma[X]$ of length k, $c_0 < c_1 < \ldots < c_{k-1}$, and $c_i \in \mathscr{C}(f_i(\vec{c} \restriction i), g_i(\vec{c} \restriction i))$ for each $i \in k$.

We will identify increasing sequences of ordinals with finite sets of ordinals; thus we can use such set related notations as $\vec{c} \subset \vec{d}$, $\vec{c} \cap \vec{\nu}$, and $\vec{c} \cup \{\nu\}$ where $\vec{c}$ and $\vec{d}$ are finite sequences of ordinals.

Usually the class $\Gamma[X]$ is clear from the context so $\Gamma[X]$-sequences from $\mathscr{C}$ will usually be called 'sequences from $\mathscr{C}$' or simply 'sequences'. Two sequences are said to be equivalent if they have the same support.

(iv) If $\vec{c}$ is a sequence from $\mathscr{C}$ with support $\vec{f}$ and $y \in \Gamma[X \cup \cap \vec{c}]$ then $c \in y$ iff $y \in \mathscr{F}(\vec{f})$.

(v) If $h(\vec{c}) = \nu$ is an ordinal then there is a sequence $\vec{d}$ and an $h' \in \Gamma[X]$ such that $\vec{d} \subset \nu + 1$ and $\nu = h'(\vec{d})$.

It follows from (iv) that if $\vec{c}$ and $\vec{d}$ are equivalent then for all $y \in \Gamma[X]$ we have $\vec{c} \in y$ iff $\vec{d} \in y$.

Definition 2·6. The Skolem hull of X and $\mathscr{C}$, written $\Gamma''(X; \mathscr{C})$, is $\{h(\vec{c}): h \in \Gamma[X]$ and $\vec{c}$ is a $\Gamma[X]$-sequence from $\mathscr{C}\}$. The system $\mathscr{C}$ is said to Γ-generate M from X if $M = \Gamma''(X; \mathscr{C})$.

Notice that if $\mathscr{C}$ Γ-generates M from X then for all ordinals $\nu \in M$ $\nu \in \Gamma''(X \cup \nu)$ if $\nu \notin U_{(\alpha, \beta)} \mathscr{C}(\alpha, \beta)$, and otherwise $(\alpha(\nu), \beta(\nu)) \in \Gamma''(X \cup \nu)$.

Definition 2·5 assumed that $\mathscr{F}$ was a Γ ultrafilter sequence. In order to permit systems $\mathscr{C}$ which are stronger than $\mathscr{F}$ we will modify Definition 2·5.

Definition 2·7. $\mathscr{C}$ is a system of Γ indiscernible over X if $\mathscr{C}$ satisfies 2·5 (i, ii, iii, and v) and

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(iv′) (*a*) If $\vec{d}$ is a Γ sequence and $\alpha, \beta, x \in \Gamma''\vec{d}$ then $(\mathscr{C}(\alpha,\beta) - \sup(\vec{d} \cap \alpha))$ is either contained in or disjoint from X.

(*b*) If $\vec{c}$ and $\vec{d}$ are equivalent and $h \in \Gamma[X]$ then $h(\vec{c})$ is defined iff $h(\vec{d})$ is defined. (It follows that $h(\vec{c}) = 0$ if $h(\vec{d}) = 0$.)

A system $\mathscr{C}$ of Γ indiscernible over X is *complete* if $\mathscr{C}$ Γ-generates M from X and for all $(\alpha,\beta) \in \text{domain}(\mathscr{C})$ $\mathscr{C}(\alpha,\beta)$ is unbounded in α. If $\mathscr{C}$ is complete then $\mathscr{C}$ generates the $\Gamma[X]$-ultrafilter sequence $\mathscr{F}$ defined by $y \in \mathscr{F}(\alpha,\beta)$ iff $(\mathscr{C}(\alpha,\beta) - y)$ is bounded below α.

We use iterated ultrapowers to generate systems of indiscernibles. Suppose we have an iterated ultrapower $(M_\nu : \nu < \lambda)$, where $M_{\nu+1} = \text{ult}(M_\nu, \mathscr{F}_\nu(\alpha_\nu, \beta_\nu))$. We will assume the iterated ultrapower is normal, that is, $(\alpha_\nu : \nu < \lambda)$ is strictly increasing. Any iterated ultrapower can be reduced to a normal iterated ultrapower by the use of Proposition 2·3, but we shall not need this fact, since our iterated ultrapowers will be constructed to be normal. Then $\mathscr{C}_\nu$ is defined by induction as a system of indiscernibles in M_ν: $\mathscr{C}_{\nu+1}(\alpha,\beta) = \mathscr{C}_\nu(\alpha,\beta)$ for all pairs $(\alpha,\beta) < (\alpha_\nu, \beta_\nu)$. In addition,

$$\mathscr{C}_{\nu+1}(i_{\nu,\nu+1}(\alpha_\nu), i_{\nu,\nu+1}(\beta_\nu)) = \mathscr{C}_\nu(\alpha_\nu,\beta_\nu) \cup \{\alpha_\nu\} \quad \text{and} \quad \mathscr{C}_{\nu+1}(i_{\nu,\nu+1}(\alpha_\nu), i_{\nu,\nu+1}(\beta)) = \mathscr{C}_\nu(\alpha_\nu,\beta)$$

for $\beta \neq \beta_\nu$. This may also be expressed in terms of the notation of 2·2: If the iteration has support $\vec{f} = ((f_\nu, g_\nu)_{a_\nu} : \nu < \lambda)$ then in $M_\lambda = \text{ult}(M, \mathscr{F}(\vec{f}))$ we have $\alpha_\nu = [\lambda \vec{a} a_\nu]$ and $\alpha_\nu \in \mathscr{C}([f_\nu], [g_\nu])$. In fact $\alpha(\alpha_\nu) = [f_\nu]$, $\beta(\alpha_\nu) = [g_\nu]$, and the rest of $\mathscr{C}$ can be defined by definition 2·5 (ii).

PROPOSITION 2·8. *$\mathscr{C}$ is a system of Σ_1 indiscernibles for $i(M)$ over $i''M$.* ∎

The proof is easy, using the fact that $i''M$ is cofinal in $i(M)$. The rest of this section will be dedicated to systems of indiscernibles for stronger classes of functions. We will first show how a sufficiently large system $\mathscr{C}$ can be thinned out to yield a smaller, but stronger, system $\mathscr{C}'$ and then we will show how such large systems can be obtained by iterated ultrapowers.

Suppose $\mathscr{C}$ is a system of Γ-indiscernibles over X. We will define a subsystem $\mathscr{C}'$ of $\mathscr{C}$ which is a system of $\Sigma_1(\Gamma)$ indiscernibles, and we will then show how to iterate the construction. We will assume that $\mathscr{C}$ Γ-generates M from X and to avoid having to write X we will assume $\Gamma = \Gamma[X]$.

Definition 2·9. (i) $\alpha(\nu)$ and $\beta(\nu)$ are the unique pair (α,β) of ordinals in $\Gamma''\nu$ such that either $\nu \in \mathscr{C}(\alpha,\beta)$ or $\alpha = \nu$ and $\beta = o^{\mathscr{F}}(\nu)$.

(ii) If $\beta < o^{\mathscr{F}}(\alpha)$ then $s(\alpha,\beta,\gamma)$ is the least member of $\mathscr{C}(\alpha,\beta)$ larger than γ.

(iii) $\delta(\nu)$ is the least $\delta \in \Gamma''\nu$ such that $\delta = o^{\mathscr{F}}(\alpha(\nu))$ or $\mathscr{C}(\alpha(\nu), \delta)$ is bounded in ν.

(iv) $\mathscr{C}$ is complete if $\mathscr{C}(\alpha,\beta)$ is unbounded in α for all $(\alpha,\beta) \in \text{domain}(\mathscr{C})$.

(v) $\nu \in \mathscr{C}'(\alpha,\beta)$ iff (*a*) $\nu \in \mathscr{C}(\alpha,\beta)$, (*b*) $\delta(\nu) = \delta(\alpha)$, and (*c*) $s(\alpha(\nu), \delta(\nu), \nu) \not< \alpha$ if $\delta(\nu) < o^{\mathscr{F}}(\alpha(\nu))$.

(vi) Γ' is the smallest class containing Γ and s and closed under composition.

We will assume that $\mathscr{C}$ is complete. Notice that if $\Gamma = \Sigma_1$ and X is definable then the relation '$\nu \in \mathscr{C}(\alpha,\beta)$' is also definable and hence $s(\alpha,\beta,\gamma)$ is definable from α, β and γ. In clause (v)*b*, $\delta(\nu) = \delta(\alpha)$ says that for all $\alpha' > \alpha$ and $\beta' \in o^{\mathscr{F}}(\alpha) \cap \Gamma''\nu$, $\mathscr{C}(\alpha',\beta')$ is cofinal in ν iff it is cofinal in α. The assumption that $\mathscr{C}$ is complete could be expressed by saying that $\delta(\alpha) \geqslant \beta(\alpha)$ for all α.

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Suppose ν is an indiscernible in $\mathscr{C}$, so $\nu \in \mathscr{C}(\alpha(\nu), \beta(\nu))$. If $\delta(\nu) = \beta(\nu)$ then $\nu = s(\alpha(\nu), \beta(\nu), \gamma)$ for some $\gamma < \nu$; let $\gamma(\nu)$ be the least such γ. In this case ν is not an indiscernible in $\mathscr{C}'$. If $\beta(\nu) < \delta(\nu)$ then ν is an indiscernible in $\mathscr{C}'$. If $\delta(\nu) = o^{\mathscr{F}}(\alpha(\nu))$ then

$$\nu \in \mathscr{C}'(\alpha(\nu), \beta(\nu))$$

and we can set $\gamma(\nu) = 0$. Otherwise $\nu \in \mathscr{C}'(\alpha, \beta)$ where $\alpha = s(\alpha(\nu), \delta(\nu), \nu)$ and

$$\beta = C(\alpha(\nu), \beta(\nu), \delta(\nu))(\nu).$$

Then $\alpha = s(\alpha(\nu), \delta(\nu), \gamma)$ for some $\gamma < \nu$; let $\gamma(\nu)$ be the least such γ.

Definition 2·10. (i) A sequence $\vec{c}$ from $\mathscr{C}$ is 1-complete if for all $c \in \vec{c}$, the ordinals $\alpha(c)$, $\beta(c)$ are in $\Gamma''(\vec{c} \cap c)$ and $(c - \gamma(c)) \cap \vec{c} \neq 0$ (unless $\gamma(c) = 0$).

(ii) $\vec{c}$ has the extended support $\langle (f_i, g_i, h_i) : i < l(\vec{c}) \rangle$ if for all $i < l(\vec{c})$,

$$\alpha(c_i) = f_i(\vec{c} \restriction i), \quad \beta(c_i) = g_i(\vec{c} \restriction i), \quad \delta(c_i) = h_i(\vec{c} \restriction i),$$

and

$$c_{i-1} \geqslant \gamma(c_i) \quad \text{if} \quad i > 0 \quad \text{and} \quad \gamma(c_0) = 0.$$

(iii) two 1-complete sequences $\vec{c}$ and $\vec{d}$ are 1-equivalent, written $c \sim_1 d$, if they have the same extended support.

The next proposition can be proved using the remarks preceding Definition 2·10 and the fact that if $\nu < \alpha$ then $\Gamma''\nu \cap \alpha \subset s(\alpha, \beta, \nu)$ for all $\beta < o^{\mathscr{F}}(\alpha)$.

PROPOSITION 2·11. (i) *Any sequence from $\mathscr{C}$ can be extended to a 1-complete sequence.*

(ii) *If $\vec{c}$ is a 1-complete sequence and d is the subsequence of members of $\vec{c}$ which are in $\mathscr{C}$ then $\vec{d}$ is a Γ'-sequence from $\mathscr{C}'$.* |

COROLLARY 2·12. *The system $\mathscr{C}'$ Γ'-generates M from X.* |

Proposition 2·11 (ii) defines a map from 1-complete sequences into Γ'-sequences. We will show in Lemma 2·16 that every Γ'-sequence comes from some 1-complete Γ-sequence but first we need two general results.

PROPOSITION 2·13. *Suppose that $\vec{d}$ is 1-complete, $f, g \in \Gamma$, and $w = s(f(\vec{d} \restriction r), g(\vec{d} \restriction r), d_r)$. Let t be largest such that $d_t \leqslant w$. Then the number t, and whether equality holds, depends only on f, g and the 1-equivalence class of $\vec{d}$.*

Proof. Let $(\vec{f}, \vec{g})$ be the extended support of $\vec{d}$. If there is no j such that

$$f_j(\vec{d} \restriction j) = f(\vec{d} \restriction r)$$

then t is largest such that $f_t(\vec{d} \restriction t) < f(\vec{d} \restriction r)$. Otherwise t is the largest number such that $f_t(\vec{d} \restriction t) = f(\vec{d} \restriction r)$, $g_i(\vec{d} \restriction i) < g(\vec{d} \restriction r)$ for all $i < t$ with $f_i(\vec{d} \restriction t) = f(\vec{d} \restriction r)$, and $g_t(d \restriction i) \leqslant g(d \restriction r)$. Equality holds if $g_t(\vec{d} \restriction t) = g(\vec{d} \restriction r)$. All of this depends only on $f, g, \vec{f}$ and $\vec{g}$. |

In the next lemma, something less than the completeness of $\mathscr{C}'$ is actually needed for the proof of the next lemma, but the completeness of $\mathscr{C}$ does not appear to be enough. The necessity for this lemma was pointed out by Solovay, and the proof given here is a simplification of the one he suggested.

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LEMMA 2·14. *Suppose $\mathscr{C}'$ is complete. If $\vec{d} \sim \vec{d}'$, $d_i \leqslant d_i'$ for all $i < n = l(\vec{d})$, and $h \in \Gamma$ then $h(\vec{d}) \leqslant h(\vec{d}')$.*

Proof. We will define a sequence $(\vec{c}^k : k \in \omega)$ of Γ sequences so that for all k the sequence $(c_0^k, \ldots, c_{n-1}^k, c_0^{k+1}, \ldots, c_{n-1}^{k+1})$ is equivalent to $(d_0, \ldots, d_{n-1}, d_0', \ldots, d_{n-1}')$, in the sense that the two sequences are ordered the same and if they are reordered so as to be strictly increasing then they become equivalent by the usual definition. If $h(\vec{d}) > h(\vec{d}')$ then it will follow that $h(\vec{c}^k) > h(\vec{c}^{k+1})$ for all $k \in \omega$, contradicting the well foundedness of M. Thus the construction of $(\vec{c}^k : k \in \omega)$ will prove the lemma.

The indiscernibles c_i^k will be chosen in order of their size. It follows, of course, that for $i' < i$ we will choose $c_{i'}^k$ before c_i^k. If $\vec{d}$ (and hence $\vec{d}'$, and each sequence $\vec{c}^k$) has support $\vec{f} = ((f_i, g_i) : i < n)$ then this means that $\alpha(c_i^k) = f_i(\vec{c}^k \restriction i)$ and $\beta(c_i^k) = g_i(\vec{c}^k \restriction i)$ will always be known before we are required to choose c_i^k.

The basic idea is that if $k < k'$ then we pick c_i^k before $c_{i'}^{k'}$, unless forced to do otherwise. In order to see better when we will be forced to do otherwise, let us fix α for the moment and consider the structure of the set of c_i^k such that $\alpha(c_i^k) = \alpha$. Let $(w_\nu : \nu < \lambda)$ enumerate this set in increasing order, so that $w_\nu = c_{i_\nu}^{k_\nu}$. Then for all $\nu + 1 < \lambda$ $w_{\nu+1}$ can be defined, once the pair $(k_{\nu+1}, i_{\nu+1})$ is known, by $w_{\nu+1} = s(\alpha, \beta(w_{\nu+1}), w_\nu)$. Assume that $\lambda < \omega^2$, so that there are only finitely many limit ordinals ν. Set $\eta_\nu = \sup\{w_{\nu'} : \nu' < \nu\}$ for all $\nu < \lambda$. Clearly $\eta_{\nu+\omega}$ is at most the smallest member of $\mathscr{C}'(\alpha, 0)$ above η_ν, and hence is smaller than α. Thus we can define w_ν (again, assuming that k_ν and i_ν are known) to be $s(\alpha, \beta(w_\nu), \eta_\nu)$. It will turn out that the only limit ordinals ν will come from an $i < n$ such that $d_i = d_i'$. In this case we must have $c_i^k = c_i^{k'}$ for all $k \in \omega$, and the $\nu < \lambda$ such that $c_i^k = W_\nu$ for all $k \in \omega$ may be a limit ordinal. Since this can happen at most n times we have $\lambda \leqslant \omega . n$.

It might seem that indiscernibles c_i^k with $\alpha(c_i^k) \neq \alpha$ might be interspersed with the ordinals w_ν in such a way as to interfere, but this turns out not to be the case. Suppose that $c_i^k < \alpha$, $\alpha(c_i^k) \neq \alpha$, and $c_i^k > s(\alpha, 0, 0)$, the least indiscernible for α. Then $\alpha \in \Gamma''(s(\alpha, 0, 0)) \subset \Gamma''(c_i^k)$, so $\alpha(c_i^k) < \alpha$. Further, $\alpha(c_i^k) = f_i(\vec{c}^k \restriction i) \in \Gamma''(c_{i-1}^k + 1)$ so $c_{i-1}^k < \alpha(c_i^k) < s(\alpha, 0, c_{i-1}^k)$. If $\alpha(c_{i-1}^k) \neq \alpha$ then we may continue by induction to find $i' < i$ so that $\alpha(c_{i'}^k) = \alpha$, so $c_{i'}^k = w_\nu$ for some ν, and $w_\nu = c_{i'}^k < \alpha(c_i^k) < s(\alpha, 0, w_\nu) \leqslant w_{\nu+1}$. Thus the ordinals c_i^k with $\alpha(c_i^k) \neq \alpha$ are automatically put into their proper places with respect to the ordinals c_i^k with $\alpha(c_i^k) = \alpha$.

So far we have seen that we can pick the ordinals c_i^k (so long as we follow the restriction to finitely many limit points of indiscernibles for one α) so that $\vec{c}^k \sim \vec{d}$ for each $k \in \omega$. Furthermore if we can make sure that c_i^k and c_j^{k+1} are in the same order as d_i and d_j whenever $\alpha(c_i^k) = \alpha(c_j^{k+1})$ then c_i^k and c_j^{k+1} will be in their proper order for all $i, j \in n$, so that $(\vec{c}^k, \vec{c}^{k+1})$ is equivalent to $(\vec{d}, \vec{d}')$, as required. Thus our construction below only needs to be verified for pairs c_i^k, c_j^{k+1} with $\alpha(c_i^k) = \alpha(c_j^{k+1})$.

Suppose that the construction has been carried out for some sequence of steps, and for each $k \in \omega$ let i_k be the least (if any) such that $c_{i_k}^k$ has not yet been chosen. Then $\alpha_k = \alpha(c_{i_k}^k) = f_i(\vec{c}^k \restriction i_k)$ is known for all $k \in \omega$; let α be the least member of $\{\alpha_k : k \in \omega\}$ and let k be least such that $\alpha_k = \alpha$. The only obstacle to defining $c_{i_k}^k$ immediately occurs if $\alpha(c_{i_k}^k) = \alpha(c_{i_{k+1}}^{k+1})$ and $d_{i_k} > d_{i_{k+1}}'$. In this case we should pick $c_{i_{k+1}}^{k+1}$ first, so we will

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switch our attention there. This switch to $k+1$ can occur several times, but since $d_{i_k} \leqslant d'_{i_k}$, we must have $i_{k+1} < i_k$. In general, if we switch s times, to $k+s$, then $i_k - i_{k+s} \geqslant s$, so we can only switch finitely many times. After fixing on some $k+s$, we define $c^{k+s}_{i_{k+s}}$ as above.

This will define c^k_i for all k and i provided that the condition that the only points c^k_i which are limit points among the indiscernibles for $\alpha(c^k_i)$ are the c^k_i such that $d_i = d'_i$. Suppose $d_i < d'_i$, we will show that c^k_i is not a limit point. Let $j < i$ be the largest such that $d_j = d'_j$ and $\alpha(c^k_i) = \alpha(c^0_j)$, if any. If j is defined, let $c_j = c^0_j$, so $c_j = c^k_j$ for all k. If j is not defined, set $c_j = 0$. Now if $\alpha(c^{k'}_{i'}) = \alpha(c^k_i)$ and $c_j < c^{k'}_{i'} < c^k_i$ then either $k' < k$ or $k' - k \leqslant i - i'$. There are only finitely many pairs (k', i') of either sort, so c^k_i is not a limit point. ∎

Definition 2·15. A *conservative extension* of a sequence $\vec{c}$ is a sequence $\vec{d}$ such that every member of $\vec{d} - \vec{c}$ is contained in $\mathscr{C} - \mathscr{C}'$.

LEMMA 2·16. *If $\mathscr{C}'$ is complete and $\vec{c}$ is a Γ'-sequence from $\mathscr{C}'$ then there is a conservative extension $\vec{d}$ of $\vec{c}$ which is a 1-complete Γ-sequence from $\mathscr{C}$. If $\vec{c}'$ is another Γ'-sequence equivalent to $\vec{c}$ then there is $\vec{d}'$ extending $\vec{c}'$ such that $\vec{d}' \sim_1 \vec{d}$.*

The proof of Lemma 2·16 will depend on two preliminary results:

PROPOSITION 2·17. (i) *If $\vec{c}$ is a Γ-sequence and $\vec{d}$ is any extension of $\vec{c}$ then there is a conservative extension $\vec{d}'$ of $\vec{c}$ such that $\vec{d}' \sim \vec{d}$.* (ii) *If c and $\vec{c}'$ are 1-equivalent Γ-sequences and $\vec{d}$ is an extension of $\vec{c}$ then there is an extension $\vec{d}'$ of $\vec{c}'$ which is equivalent to $\vec{d}$. If $\vec{d}$ is conservative then $\vec{d}'$ can be taken to be conservative and 1-equivalent to $\vec{d}$.*

Proof. If $\vec{d}$ is an extension of $\vec{c}$ then we can define the sequence $\vec{d}$ required by clause (i) recursively: if $d_i \in \mathscr{C}(f(\vec{d} \restriction u), g(\vec{d} \restriction i))$ and d_i is not in $\vec{c}$ then set

$$d'_i = s(f(\vec{d}' \restriction i), g(\vec{d}' \restriction i), d'_{i-1}).$$

This is possible because the completeness of $\mathscr{C}$ implies that for all $i < l(\vec{c})$, d'_i is defined and (using Lemma 2·14) $d'_i \leqslant d_i$. The construction ensures that $\vec{d}'$ is a conservative extension of $\vec{c}$ and that $\vec{d}' \sim \vec{d}$. Thus (i) is true.

By (i), we can assume that the extension $\vec{d}$ given in clause (ii) is conservative. The requirement that $\vec{d}'$ be 1-equivalent to $\vec{d}$ determines $\vec{d}'$: if

$$d_i = s(f(\vec{d} \restriction i), g(\vec{d} \restriction i), d_{i-1})$$

then d'_i must be $s(f(\vec{d}' \restriction i), g(\vec{d}' \restriction i), d_{i-1})$. By Proposition 2·13 $\vec{d}'$ will be ordered the same as $\vec{d}$ and hence will be 1-equivalent to $\vec{d}$. ∎ 2·17.

PROPOSITION 2·18. *Suppose $\vec{c}$ is a 1-complete sequence and $h' \in \Gamma'$. Then there is a conservative extension $\vec{d} \cup \vec{c}$ of $\vec{c}$ and an $h \in \Gamma$ such that $h(\vec{d} \cup \vec{c}) = h'(\vec{c})$. The function h and the 1-equivalence class of the extension $\vec{d} \cup \vec{c}$ depend only on the function h' and the 1-equivalence class of $\vec{c}$.*

Proof. The proof is by induction over the length of h'. By the definition of Γ', $h'(\vec{c}) = h_1(g_1(\vec{c}), \ldots, g_k(\vec{c}))$ where by the induction hypothesis the proposition holds

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for each g_i and either $h_1 \in \Gamma$ or $h_1 = s$. If $h_1 \in \Gamma$ the proposition clearly holds for h', so we can assume $h'(\vec{c}) = \nu = s(f(\vec{c}), g(\vec{c}), k(\vec{c}))$ where the proposition holds for f, g and k. We may as well assume we have already applied the proposition, so f, g and k are in Γ.

By 2·5 (iii and v) there is a Γ-sequence $\vec{e}' \subset \nu$ and functions f', g' and k' in Γ so that $\alpha(\nu) = f'(\vec{e}')$, $\beta(\nu) = g'(\vec{e}')$, and $k_1(\vec{c}) = k'(\vec{e}' \cap (k_1(\vec{c}) + 1))$. Now by 2·17 there is a conservative extension $\vec{c} \cup \vec{e}$ of $\vec{c}$ which is equivalent to $\vec{c} \cup \vec{e}'$. We will show that

$$(1) \qquad \alpha(\nu) = f'(\vec{e}), \quad \beta(\nu) = g'(\vec{e}), \quad \text{and} \quad k_1(\vec{c}) = k'(\vec{e} \cap (k_1(\vec{c}) + 1)).$$

It will follow that $\nu = s(f'(\vec{e}), g'(\vec{e}), k'(\vec{e} \cap (k_1(\vec{c}) + 1))$. Let $e = \sup(\vec{e} \cap (k_1(\vec{c}) + 1))$. Then $s(f'(\vec{e}), g'(\vec{e}), e) > k'(\vec{e} \cap (e+1)) = k_1(\vec{c})$, of $\nu = s(f'(\vec{e}), g'(\vec{e}), e)$. Thus $\vec{d} = \vec{c} \cup \vec{e} \cup \{\nu\}$ is a conservative extension of c and ν is obtained from $\vec{d}$ by a projection function, which is in Γ.

It can be easily checked that the proof of (1) shows that the same f', g' and k' and the same extended support for the conservative extension $\vec{e}$ will work for any sequence 1-equivalent to $\vec{c}$.

Since $k_1(\vec{c}) = k'(\vec{e}' \cap (k_1(\vec{c}) + 1))$ and $\vec{c} \cup \vec{e}' \sim \vec{c} \cup \vec{e}$, we immediately have

$$k'(\vec{e} \cap (k_1(\vec{c}) + 1)) = k_1(\vec{c}).$$

If $f(\vec{c}) = \alpha(\nu)$ then $g(\vec{c}) = \beta(\nu)$, so $f'(\vec{e}') = f(\vec{c})$ and $g'(\vec{e}') = g(\vec{c})$ and hence $f'(\vec{e}) = f(\vec{c}) = \alpha(\nu)$ and $g'(\vec{e}) = g(\vec{c}) = \beta(\nu)$.

If $f(\vec{c}) \neq \alpha(\nu)$ then for some $\beta < o(\alpha(\nu))$, $f(\vec{c}) \in \mathscr{C}(\alpha(\nu), \beta)$. Since $\vec{c}$ is a Γ sequence it follows that $f(\vec{c}) = c \in \vec{c}$ and there are functions f_c, g_c in Γ so that $\alpha(\nu) = g_c(\vec{c} \cap c)$ and $\beta = g_c(\vec{c} \cap c)$. From $f'(\vec{e}') = \alpha(\nu) = f_c(\vec{c} \cap c)$ we get $f'(\vec{e}) = f_c(\vec{c} \cap c) = \alpha(\nu)$. Finally, we must have $g(\vec{c}) = C(\alpha(\nu), \beta(\nu), \beta)(c) = C(f'(\vec{e}'), g'(\vec{e}'), g_c(\vec{c} \cap c))(c)$ so $g(\vec{c}) = C(f'(\vec{e}), g'(\vec{e}), g_c(\vec{c} \cap c))(c) = C(\alpha(\nu), g'(\vec{e}), \beta)(c)$. By the indiscernibility of c it follows that $g'(\vec{e}) = \beta(\nu)$. ▌2·18.

Proof of 2·16. We assume as an induction hypothesis that Lemma 2·16 holds for sequences shorter than $\vec{c}$. In particular, if ν is the largest member of $\vec{c}$ then there is a conservative extension $\vec{d}$ of $\vec{c} \cap \nu$ which is a 1-complete Γ-sequence from $\mathscr{C}$. Since c is a Γ'-sequence, there are $f', g' \in \Gamma'$ such that $\nu \in \mathscr{C}(f'(\vec{c}), g'(\vec{c}))$. Thus by 2·16 we can assume (taking a further extension of $\vec{d}$ if needed) that $f'(\vec{c}) = f(\vec{d})$ and $g'(\vec{c}) = g(\vec{d})$ for $f, g \in \Gamma$. Let d be the least member of $\vec{d}$ greater than ν. Precisely as in the proof of 2·16, we can assume (taking a conservative extension if necessary) that $\alpha(\nu) \in \Gamma''(\vec{d} \cap \nu)$ and $\beta(\nu) \in \Gamma''(\vec{d} \cap \nu)$. By definition $\delta(\nu) \in \Gamma''\nu$, so there is an extension $\vec{d}'$ of $\vec{d}$ such that $\vec{d}' - \vec{d} \subset \nu$ and $\delta(\nu) = h(\vec{d}' \cap \nu)$ for some $h \in \Gamma$. Since $\nu \in \mathscr{C}'(f(\vec{d}), g(\vec{d}))$, $\delta(\nu) = \delta(f(\vec{d}))$. Either $f(\vec{d}) = \alpha(\nu)$, in which case $\delta(\nu) = o^{\mathscr{F}}(\alpha(\nu)) \in \Gamma''(\vec{d} \cap \nu)$, or $f(\vec{d}) \in \mathscr{C}(\alpha(\nu), \beta)$ for some $\beta > \beta(\nu)$. In this second case $f(\vec{d})$ must be a member of $\vec{d}$, since it is an indiscernible, so $\delta(f(\vec{d})) = \delta(\nu) \in \Gamma''(\vec{d})$, say $\delta(\nu) = h'(\vec{d})$. By Proposition 2·15 we can assume $\vec{d}'$ is a

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conservative extension and still have $h'(\vec{d}) = h(\vec{d}' \cap \nu)$. Thus $\delta(\nu) \in \Gamma''(\vec{d}' \cap \nu)$. We can assume $\vec{d}' = \vec{d}$. We only have to verify that $\Gamma''(\vec{d} \cap \nu) \cap (\nu - \gamma(\nu)) \neq 0$. Now

$$\Gamma''(\vec{d} \cap d) \cap (d - \gamma(d)) \neq 0,$$

but $\vec{d} \cap d = \vec{d} \cap \nu$ and $\Gamma''(\vec{d} \cap \nu) \cap (d - \nu) = 0$ since $d < \alpha(\nu)$, so $\Gamma''(\vec{d} \cap \nu) \cap (\nu - \gamma(d)) \neq 0$ and it is enough to show that $\gamma(d) \geqslant \gamma(\nu)$. We have

$$d = s(\alpha(\nu), \beta(d), \nu) \leqslant f(\vec{d}) \leqslant s(\alpha(\nu), \delta(\nu), \nu),$$

by the definition of $\mathscr{C}'$. It follows that $\beta(d) \leqslant \delta(\nu)$, and hence that $\gamma(d) \geqslant \gamma(\nu)$ as required, since $s(\alpha(\nu), \delta(\nu), \gamma(d)) < \nu$ would imply $s(\alpha)(\nu), \beta(d), \gamma(d)) < \nu$. ▌2·16.

COROLLARY 2·19. *$\mathscr{C}'$ is a system of Γ' indiscernibles.*

Proof. Clause (ii) of the Definition 2·7 has already been proved. The other clauses follow immediately from 2·14 and the fact that $\mathscr{C}$ is a system of Γ indiscernibles. ▌

Corollary 2·19 permits us to iterate the construction of $\mathscr{C}'$:

2·20. *Definition.* (i) $\mathscr{C}^\eta$ and Γ^η are defined by recursion over η: $\mathscr{C}^0 = \mathscr{C}$ and $\Gamma^0 = \Gamma$; $\mathscr{C}^{\eta+1} = (\mathscr{C}^\eta)'$ and $\Gamma^{\eta+1} = (\Gamma^\eta)'$; and if η is a limit ordinal then for each (α, β)

$$\mathscr{C}^\eta(\alpha, \beta) = \bigcap_{\eta' < \eta} \mathscr{C}^{\eta'}(\alpha, \beta) \quad \text{and} \quad \Gamma^\eta = \bigcup_{\eta' < \eta} \Gamma^{\eta'}.$$

(ii) $\mathscr{C}$ is rank λ complete if $\mathscr{C}^\eta$ is complete for each $\eta < \lambda$.

(iii) For each $(\alpha, \beta) \in \text{domain}(\mathscr{F})$ and for each $\eta > 0$

$$\mathscr{F}^\eta(\alpha, \beta) = \{x \subset \alpha : \forall \eta' < \eta\ \mathscr{C}^\eta(\alpha, \beta) \cap x \text{ is unbounded in } \alpha\}.$$

All of the results proved for $\mathscr{C}'$ extend to $\mathscr{C}^\lambda$, providing that $\mathscr{C}$ is rank λ complete. The limit case of the proof is trivial: if η is a limit ordinal then any member of Γ^η is in $\Gamma^{\eta'}$ for some $\eta' < \eta$ and any Γ^η sequence is a $\Gamma^{\eta'}$ sequence for some $\eta' < \eta$. The notation used above can be generalized to $\mathscr{C}^\eta$. We define $\alpha^\eta(\nu)$, $\beta^\eta(\nu)$, $\delta^\eta(\nu)$, $\gamma^\eta(\nu)$, and $s^\eta(\alpha, \beta, \gamma)$ from $\mathscr{C}^\eta$ as α, β, δ, γ and s were defined from $\mathscr{C}$. Notice that $(\alpha^\eta(\nu) : \eta \in \mathbf{ON})$ and $(\gamma^\eta(\nu) : \eta \in \mathbf{ON})$ are non-increasing. Since $\beta^\eta(\nu)$ and $\delta^\eta(\nu)$ only change when $\alpha^\eta(\nu)$ does, it follows that $\alpha^\eta(\nu), \beta^\eta(\nu), \delta^\eta(\nu)$, and $\gamma^\eta(\nu)$ only take finitely many values as η ranges through the ordinals. Thus any sequence has a finite η-completion for any $\eta \in \mathbf{ON}$ and we can define η equivalence of sequences. An extension $\vec{d}$ of $\vec{c}$ is η-conservative if $\vec{d} - \vec{c} \subset \mathscr{C} - \mathscr{C}^{\eta+1}$.

LEMMA 2·21. *Suppose $\mathscr{C}$ is a rank $\omega.\lambda$-complete system of Γ-indiscernibles for $\mathscr{F}$ in $J^{\mathscr{F}}_\alpha$ which Γ-generates $J^{\mathscr{F}}_\alpha$, and that X and Γ are members of $J^{\mathscr{F}}_{\alpha+1}$. Then*

(i) *$\mathscr{F}^{\omega.\lambda}$ is an ultrafilter sequence in $J^{\mathscr{F}^{\omega.\lambda}}_{\alpha+\lambda}$ (provided that all of its iterated ultrapowers are well founded.)*

(ii) *$\mathscr{C}^{\omega.\lambda}$ is a system of Σ_1-indiscernibles in $J^{\mathscr{F}^{\omega.\lambda}}_{\alpha+\lambda}$ over $((\alpha+\lambda)\backslash\alpha) \cup X$ which Σ_1-generates $J^{\mathscr{F}^{\omega.\lambda}}_{\alpha+\lambda}$.*

In our applications the condition that Γ and X are in $J^{\mathscr{F}}_{\alpha+1}$ will be trivially satisfied: Γ will be the class of Σ_1 functions and X will be of the form $\alpha \cup p$ where p is finite. Also, the proviso in (i) that iterated ultrapowers of $J^{\mathscr{F}^{\omega.\lambda}}_{\alpha+\lambda}$ by $\mathscr{F}^{\omega.\lambda}$ are well founded is unnecessary. In Lemma 2·21 we will see that there is an iterated ultrapower $j: J^{\mathscr{F}}_\alpha \to J^{\mathscr{G}}_{\alpha'}$ generating a rank $\omega.\lambda$ complete system of indiscernibles for $\mathscr{G}$ and such that whenever

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$o^{\mathscr{G}}(\kappa) > 0$ then $cf(\kappa) > \omega$. Then $\mathscr{G}(\kappa, \lambda)$ is countably complete, so all iterated ultrapowers by $\mathscr{G}^{\omega \cdot \lambda}$ are well founded. Now j extends to a map $j^*: J^{\mathscr{F}^{\omega \cdot \lambda}}_{\alpha+\lambda} \to J^{\mathscr{G}^{\omega \cdot \lambda}}_{\alpha'+\lambda}$, so $\mathscr{F}^{\omega \cdot \lambda}$ also has all its iterated ultrapowers well founded.

Proof of 2·21. We will prove the lemma by induction on λ. We simultaneously prove that

(iii) Every $\Sigma_1(J^{\mathscr{F}^{\omega \cdot \lambda}}_{\alpha+\lambda})$ sequence $\bar{c}$ is also a $\Gamma^{\omega \cdot \lambda}$-sequence.

(iv) If f is a Σ_1 function in $J^{\mathscr{F}^{\omega \cdot \lambda}}_{\alpha+\lambda}$ and $f(\bar{c}) = \nu$ for some ordinal ν then there is $f' \in \Gamma^{\omega}$ such that $f'(\bar{c}) = \nu$.

The limit case is easy, since any Σ_1 formula true in $J^{\mathscr{F}^{\omega \cdot \lambda}}_{\alpha+\lambda}$ is true in $J^{\mathscr{F}^{\omega \cdot \lambda}}_{\alpha+\lambda'}$ for some $\lambda' < \lambda$. We will prove (i)–(iv) for $\lambda = 1$; this will suffice to prove the successor case.

We also continue to assume $\Gamma = \Gamma[X]$, so as to avoid having to write in the parameter X. The first proposition will be used to show that $\Gamma^{\omega} \subset J^{\mathscr{F}^{\omega}}_{\alpha+1}$, and indeed Γ^{ω} is a Σ_1 definable subset of $J^{\mathscr{F}^{\omega}}_{\alpha+1}$. This proposition also has some independent interest.

PROPOSITION 2·22. *If $\mathscr{C}$ Γ-generates $J^{\mathscr{F}}_{\alpha}$ then $\mathscr{C}$ is determined by Γ. Indeed for every $\nu \in \bigcup_{\kappa, \lambda} \mathscr{C}(\kappa, \lambda)$ there is a unique pair (κ, λ) satisfying* (i)–(iii) *below; and if (κ, λ) satisfies* (i)–(iii) *then $\nu \in \mathscr{C}(\kappa, \lambda)$.*

(i) $\Gamma''\nu \cap \kappa = \nu < \kappa$.

(ii) $(\kappa, \lambda) \in \Gamma''\nu$.

(iii) *For all $x \in P(\kappa) \cap \Gamma''\nu$, $x \in \mathscr{F}(\kappa, \lambda)$ iff $\nu \in x$.*

Proof. Since $\mathscr{C}$ Γ-generates $J^{\mathscr{F}}_{\alpha}$, every $\nu \in J^{\mathscr{F}}_{\alpha}$ can be written in the form $h(\vec{c})$ where $\vec{c}$ is a sequence from $\mathscr{C}$ and $\cup \vec{c} \leqslant \nu$. If $\cup \vec{c} < \nu$ then $\nu \notin \bigcup_{\kappa, \lambda} \mathscr{C}(\kappa, \lambda)$ and ν fails to satisfy (i) for any pair (κ, λ), so 2·22 holds for ν. For the rest of the proof we assume that $\nu = \cup \vec{c}$. Thus there are $f, g \in \Gamma$ such that $\nu \in \mathscr{C}(f(\vec{c} \cap \nu), g(\vec{c} \cap \nu))$. We set $\kappa = f(\vec{c} \cap \nu)$ and $\lambda = g(\vec{c} \cap \nu)$. Clause (ii) holds by definition and (i) holds by standard arguments, and clause (iii) holds by indiscernibility.

We complete the proof by showing that (i)–(iii) determine κ and λ. Clearly (i) and (ii) determine κ, and in order to see that (ii) and (iii) determine λ it is enough to see that if $\lambda, \lambda' \in \Gamma''\nu$ then there is $x \in \Gamma''\nu$ such that $x \in \mathscr{F}(\kappa, \lambda) \not\equiv x \in \mathscr{F}(\kappa, \lambda')$. Assume without loss of generality that $\lambda' > \lambda$. Now the sets

$$B_{\xi, \xi'} = \{\nu: C(\kappa, \xi, \lambda)(\nu) < C(\kappa, \xi', \lambda)(\nu) < o(\nu)\} \in \mathscr{F}(\kappa, \lambda')$$

for all $\xi < \xi' < \lambda$. On the other hand $B_{\xi, \xi'} \notin \mathscr{F}(\kappa, \lambda)$ for some ξ, ξ', since otherwise $o(\kappa)$ would be at least λ' in the ultrapower by $\mathscr{F}(\kappa, \lambda)$. Let $X = B_{\xi, \xi'}$, where (ξ, ξ') is the least pair such that $B_{\xi, \xi'} \notin \mathscr{F}(\kappa, \lambda')$ Then x is rudimentary in λ' and λ, so $x \in \Gamma''\nu$. ∎

Since $\Gamma \in J^{\mathscr{F}^{\omega}}_{\alpha+1}$ and $J^{\mathscr{F}^{\omega}}_{\alpha+1}$ is rudimentarily closed, Proposition 2·19 implies that $\mathscr{C} \in J^{\mathscr{F}^{\omega}}_{\alpha+1}$. An easy induction shows that for each $n \in \omega$ $\mathscr{C}^n$ and Γ^n are in $J^{\mathscr{F}^{\omega}}_{\alpha+1}$, and it follows that $\Gamma^{\omega} \subset J^{\mathscr{F}^{\omega}}_{\alpha+1}$ and is a Σ_1 definable subset of $J^{\mathscr{F}^{\omega}}_{\alpha+1}$.

We are now ready to prove that $\mathscr{C}^{\omega}$ is a system of Σ_1 indiscernibles for $J^{\mathscr{F}^{\omega}}_{\alpha+1}$. The essential step involves the rudimentary sets: We define a rank $n = n(R)$ for every subset R of $J^{\mathscr{F}}_{\alpha}$ which is rudimentary in the parameter $J^{\mathscr{F}}_{\alpha}$ so that if $\vec{c}$ and $\vec{c}'$ are $n(R)$-equivalent Γ sequences and $h \in \Gamma$ then $h(\vec{c}) \in R$ iff $h(\vec{c}') \in R$. The proof uses Proposition 1·5. Any set R which is rudimentary without using the parameter $J^{\mathscr{F}}_{\alpha}$ satisfies the

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property with rank $n(R) = 0$, and any boolean combination of sets, each of which has the property with rank at most n, has the property with the same rank n. Hence we will complete the proof of the claim by showing that if R has rank n then

$$R_1 = \{x: \exists y \in J_\alpha^{\mathscr{F}}((x, y) \in R)\}$$

has rank $n+1$ and $R_2 = \{(x, \alpha, \beta): \{\nu: (x, \nu) \in R\} \in \mathscr{F}(\alpha, \beta)\}$ has rank n.

Suppose $h(\vec{c}) \in R_1$ and $\vec{d}$ is $n+1$-equivalent to $\vec{c}$. Then there is a function $h' \in \Gamma$ and an extension $\vec{c}'$ of $\vec{c}$ such that $(h(\vec{c}), h'(\vec{c}')) \in R$. By Proposition 2·14 (ii) there is an extension $\vec{d}'$ of $\vec{d}$ which is n-equivalent to $\vec{c}'$. Then $(h(\vec{d}), h'(\vec{d}')) \in R$ and hence $h(\vec{d}) \in R_1$. Thus $n(R_1) = n+1$.

Now suppose $(h_1(\vec{c}), h_2(\vec{c}), h_3(\vec{c})) \in R_2$ and $\vec{d}$ is n-equivalent to $\vec{c}$. From the definition of R_2 we have

$$\{\nu: R(h_1(\vec{c}), \nu)\} \in \mathscr{F}^\omega(h_2(\vec{c}), h_3(\vec{c})). \tag{1}$$

By the definition of $\mathscr{F}^\omega$ and the fact that R has rank n, (1) holds iff $(h_1(\vec{c}), \nu) \in R$ holds for any member ν of $\mathscr{C}^u(h_2(\vec{c}), h_3(\vec{c}))$ larger than $\cup(\vec{c} \cap h_2(\vec{c}))$, and in particular (1) holds iff

$$(h_1(\vec{c}), s_n(h_2(\vec{c}), h_3(\vec{c}), \cup(\vec{c} \cap h_2(\vec{c})))) \in R.$$

Since $\vec{c}$ is n-equivalent to $\vec{d}$, $\vec{c} \cup \{s_n(h_2(\vec{c}), h_3(\vec{c}), \cup(\vec{c} \cap h_2(\vec{c})))\}$ is n-equivalent to $\vec{d} \cup \{s_n(h_2(\vec{d}), h_3(\vec{d}), \cup(\vec{d} \cap h_2(\vec{d})))\}$. Thus

$$(h_1(\vec{d}), s_n(h_2(\vec{d}), h_3(\vec{d}), \cup(\vec{d} \cap h_2(\vec{d})))) \in R$$

and hence $(h_1(\vec{d}), h_2(\vec{d}), h_3(\vec{d})) \in R_2$.

This completes the proof of the claim that every subset R of $J_\alpha^{\mathscr{F}}$ which is rudimentary in the parameter $J_\alpha^{\mathscr{F}}$ has a rank $n(R)$. Now suppose the set H is Σ_1 in $J_{\alpha+1}^{\mathscr{F}^\omega}$ with the parameter α, so H can be put in the form

$$H = \{x: \exists y \in J_{\alpha+1}^{\mathscr{F}^\omega} Rxy\}$$

for a rudimentary relation R with parameter α. If $h \in \Gamma$ and $h(\vec{c}) \in H$ then there is a rudimentary function r with parameter $J_\alpha^{\mathscr{F}}$ and an $a \in J_\alpha^{\mathscr{F}}$ such that $R(h(\vec{c}), r(a))$. The set $\{(x, y): R(x, r(y))\}$ is rudimentary; let n be its rank. We claim that if $\vec{d}$ is $n+1$-equivalent to $\vec{c}$ then $h(\vec{d}) \in H$. Since $h(\vec{c}) \in H$ there is $f \in \Gamma$ and a Γ-sequence $\vec{c}$, which we can take to be an extension of $\vec{c}$, such that $R(h(\vec{c}), r(f(\vec{c}')))$. Since $\vec{d}$ is $n+1$-equivalent to $\vec{c}$ there is an extension $\vec{d}'$ of $\vec{d}$ which is n-equivalent to $\vec{c}'$. Then $R(h(\vec{d}), r(f(\vec{d}')))$, so $h(\vec{d}) \in H$.

It follows that subsets of $J_\alpha^{\mathscr{F}}$ which are $\Sigma_1(J_{\alpha+1}^{\mathscr{F}^\omega})$ in the parameter α can be given rank ω: that is, if H is $\Sigma_1(J_{\alpha+1}^{\mathscr{F}^\omega})$, $h \in \Gamma$ and $\vec{c}$ and $\vec{c}'$ are n-equivalent for all $n \in \omega$ then $H(h(\vec{c}))$ iff $H(h(\vec{c}'))$.

Let us summarize what we have so far. First, if $\Gamma^* = \Sigma_1(J_{\alpha+1}^{\mathscr{F}})$ then $\Gamma^\omega \subset \Gamma^*[X \cup \{\alpha\}]$ and the sets rudimentary in the parameter $J_\alpha^{\mathscr{F}}$ are in $\Gamma^*[\{\alpha\}]$. It follows that

$$\Gamma^{*\prime\prime}(X \cup \{\alpha\}; \mathscr{C}^\omega) = J_{\alpha+1}^{\mathscr{F}^\omega}$$

so $\mathscr{C}^\omega \Sigma_1(J_{\alpha+1}^{\mathscr{F}^\omega})$-generates $J_{\alpha+1}^{\mathscr{F}^\omega}$. Second, $\mathscr{C}^\omega$ is a system of indiscernibles for $\Sigma_1(J_{\alpha+1}^{\mathscr{F}^\omega})$ over $X \cup \{\alpha\}$ in at least the weak sense that $\Sigma_1(J_{\alpha+1}^{\mathscr{F}^\omega})$ relations are invariant under equivalent

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Γ^ω sequences. (Recall that $\Gamma[X] = \Gamma$, and hence the constant function $f = c$ is in Γ if $c \in X$.)

We will complete the proof of Lemma 2·21 by proving the following claim:

Claim. If $\vec{c}$ is a Γ^ω sequence, $h \in \Gamma^*[X \cup \{\alpha\}]$ with parameters in $X \cup \{\alpha\}$, and $h(\vec{c}) \in \alpha$ then there is $g \in \Gamma^\omega$ such that $g(\vec{c}) = h(\vec{c})$.

It follows from the claim that every $\Gamma^*[X \cup \{\alpha\}]$ sequence is also a Γ^ω sequence, and hence $\mathscr{C}^\omega$ is a system of $\Gamma^*[X \cup \{\alpha\}]$ indiscernibles.

Proof of claim. There is a Γ sequence $\vec{d}$ extending $\vec{c}$ and an $h' \in \Gamma$ such that $h'(\vec{d}) = h(\vec{c})$. Since the relation $y = h(x)$ is Σ_i, there is an n such that any extension $\vec{d}'$, of $\vec{c}$ which is n-equivalent to $\vec{d}$ also satisfies $h'(\vec{d}') = h(\vec{c})$. Such a $\vec{d}'$, can be chosen to be a n-conservative extension of $\vec{c}$, so $\vec{d}'$ reduces to the Γ^ω sequence $\vec{c}$, and $h'(\vec{d}')$ reduces to a function $g(\vec{c})$ with $g \in \Gamma^\omega$. ⊣

It only remains to show that rank λ complete systems can be generated by iterated ultrapowers:

LEMMA 2·23. *Suppose that $\mathscr{F}$ is an ultrafilter sequence in M and $\lambda \in \mathbf{ON}$. Then there is an iterated ultrapower $i_\phi: M \to M_\phi = \mathrm{ult}(M, \mathscr{F}(\vec{f}))$ such that the system $\mathscr{C}$ of indiscernibles for M_ϕ over $i''_\phi M$ is rank λ-complete.*

Note that here $\Gamma = \Gamma^0 = \{i(f) : f \text{ is a function in } M\}$ and $X = i''M$. This result can be generalized to include ultrapowers by larger classes of functions, but such generalizations will not be needed here.

Proof. We define the iterated ultrapower $i_\nu: M \to M_\nu$ by recursion over ν. For limit ν, i_ν is the direct limit of the embeddings $(i_{\nu',\nu''}: \nu' < \nu'' < \nu)$. For successor ordinals, $i_{\nu,\nu+1}: M_\nu \to \mathrm{ult}(M_\nu, \mathscr{F}_\nu(\alpha_\nu, \beta_\nu))$ where $\mathscr{F}_\nu = i_\nu(\mathscr{F})$ and (α_ν, β_ν) is defined as follows: Let N_ν be the smallest model of ZF' such that M_ν, $i''_\nu M$, and λ are members of N_ν. Then (α_ν, β_ν) is the smallest pair such that

$$\alpha_\nu > \alpha_{\nu'} \quad \text{for all} \quad \nu' < \nu,\ \beta_\nu < o^{\mathscr{F}_\nu}(\alpha_\nu), \tag{1}$$

and

$$\text{either } cf(\alpha_\nu) = \omega,\ \alpha_\nu \text{ is singular in } N_\nu,\ \alpha_\nu \leqslant \lambda, \tag{2}$$

$$\text{or } \mathscr{C}_\nu(\alpha_\nu, \beta_\nu) \text{ is not stationary in the sense of } N_\nu.$$

$\mathscr{C}_\nu$ is the system of indiscernibles in M_ν defined by the iterated ultrapower i_ν. If one of the first three clauses of (2) holds then $\beta_\nu = 0$.

We will first show that this process stops at some $\phi \leqslant |M|^+$, yielding a system $\mathscr{C} = \mathscr{C}_\phi$ of indiscernibles. We will then show that for each $\alpha \in M_\phi$ and $\beta < o^\phi(\alpha_\nu)$ the sets $\mathscr{C}^\eta(\alpha, \beta)$ are definable in N_ν, where $\nu = \phi$ or $\alpha_\nu \geqslant \alpha$, and hence that $\mathscr{C}^\eta(\alpha_\nu, \beta)$ is unbounded in α_ν for all $\eta < \lambda$.

2·24. *Claim.* The process just defined stops at some ordinal $\phi < |M|^+$.

Proof. Set $\delta = |M|^+$ and consider $L_{\delta^+}(M)$. Then the sequence $(M_\nu : \nu < \delta)$ is in $L_{\delta^+}(M)$. Suppose the process still has not stopped after δ steps, so that $(\alpha_\delta, \beta_\delta)$ is defined. There

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is $\bar{\nu} < \delta$ such that $(\alpha_\delta, \beta_\delta) \in i_{\bar{\nu},\delta''} M_{\bar{\delta}}$; say $(\alpha_\delta, \beta_\delta) = i_{\bar{\nu},\delta}(\bar{\alpha}, \bar{\beta})$. Now consider any elementary substructure $\pi: L_{\delta'}(M) \cong H < L_{\delta^+}(M)$ such that $M \cup \{M, \bar{\nu}, \delta\} \subset H$. Then

$$\pi^{-1}(\langle M_\nu : \nu < \delta \rangle) = \langle M_\nu : \nu < \pi^{-1}(\delta) \rangle$$

by absoluteness, and, since $L_{\delta'}(M)$ is a model of ZF^- containing $M_{\pi^{-1}(\delta)}$,

$$\pi^{-1}(\alpha_\delta, \beta_\delta) = \alpha_{\pi^{-1}(\delta)}, \beta_{\pi^{-1}(\delta)}).$$

But also $\pi^{-1}(\alpha_\delta, \beta_\delta) = \pi^{-1}(i_{\bar{\nu},\delta}(\bar{\alpha}, \bar{\beta})) = i_{\bar{\nu},\pi^{-1}(\delta)}(\bar{\alpha}, \bar{\beta})$, so $i_{\pi^{-1}(\delta),\delta}(\alpha_{\pi^{-1}(\delta)}, \beta_{\pi^{-1}(\delta)}) = (\alpha_\delta, \beta_\delta)$. Now let D be a closed unbounded set of ordinals $\bar{\delta} < \delta$ such that there is an H as above with $\bar{\delta} = \pi^{-1}(\delta)$. It can easily be seen that $\alpha_\delta = \sup_{\bar{\delta} \in D} \alpha_{\bar{\delta}}$, so $\alpha_\delta = \delta$ and $\alpha_{\bar{\delta}} = \bar{\delta}$ for all $\bar{\delta} \in D$. Thus $cf(\alpha_\delta) = \delta > \omega$, so α_δ is regular in $\mathbf{V}$, and hence certainly in N_δ. Also $\mathscr{C}_\delta(\alpha_\delta, \beta_\delta) \supset D$, so $\mathscr{C}_\delta(\alpha_\delta, \beta_\delta)$ is stationary in $\mathbf{V}$ and hence certainly in N_δ. Thus $(\alpha_\delta, \beta_\delta)$ does not satisfy (2), contradicting its definition. |

We now show that $\mathscr{C} = \mathscr{C}_\phi$ is rank λ complete in M_ϕ. For any measurable $\alpha \in M_\phi$, let ν be least such that $\alpha_\nu \geqslant \alpha$ and, working inside N_ν, define decreasing sequences of sets $\bar{\mathscr{C}}^\eta(\alpha, \beta)$, for $\beta < o^{\mathscr{F}^{\bar{\nu}}}(\alpha)$, and of ordinals $\bar{\delta}^\eta < o^{\mathscr{F}^{\bar{\nu}}}(\alpha)$ as follows: Set $\bar{\mathscr{C}}^0(\alpha, \beta) = \mathscr{C}_\nu(\alpha, \beta)$ and $\bar{\delta}^0 = o^{\mathscr{F}^{\bar{\nu}}}(\alpha)$. If $\bar{\mathscr{C}}^\eta$ and $\bar{\delta}^\eta$ have been defined then $\bar{\delta}^{\eta+1}$ is the least $\beta < \bar{\delta}^\eta$ such that $\bar{\mathscr{C}}^\eta(\alpha, \beta)$ is bounded in α is there is any, and $\bar{\delta}^{\eta+1} = \bar{\delta}^\eta$ otherwise. Then

$$\bar{\mathscr{C}}^{\eta+1}(\alpha, \beta) = \{\alpha' \in \bar{\mathscr{C}}^\eta(\alpha, \beta) : \beta < \bar{\delta}^{\eta+1} \text{ and } \forall \beta' < \bar{\delta}^{\eta+1}(\bar{\mathscr{C}}^\eta(\alpha, \beta') \text{ is unbounded in } \alpha')\}.$$

For limit η, $\bar{\mathscr{C}}^\eta(\alpha, \beta) = \cap_{\eta' < \eta} \bar{\mathscr{C}}^{\eta'}(\alpha, \beta)$ and $\bar{\delta}^\eta = \cap_{\eta' < \eta} \bar{\delta}^{\eta'}$. Now $\bar{\mathscr{C}}^0(\alpha, \beta) = \mathscr{C}_\nu(\alpha, \beta)$ is stationary in N_ν, for each $\eta < \lambda$ there is a closed unbounded set H_η such that for each β $\bar{\mathscr{C}}^{\eta+1}(\alpha, \beta) = \cap H_\eta$, so each $\bar{\mathscr{C}}^\eta(\alpha, \beta)$ is stationary in N_ν and hence unbounded in α. Thus it will be enough to show that for all $\eta < \lambda$

$$\forall \beta < o^{\mathscr{F}\phi}(\alpha) \bar{\mathscr{C}}^\eta(\alpha, \beta) = \mathscr{C}^\eta(\alpha, \beta). \tag{3}$$

We shall also need some added facts as part of the induction. ($\alpha^\eta(\alpha)$ and $\delta^\eta(\alpha)$ are the α and δ given at stage η by Definition 2·20).

(4) if ν^η is such that $\alpha^\eta(\alpha) = i_{\nu,\nu^\eta}(\alpha)$ then $\delta^\eta(\alpha) = i_{\nu,\nu^\eta}(\bar{\delta}^\eta)$

(5) $\alpha^0(\alpha) = i_{\nu\phi}(\alpha)$, and if $\eta \geqslant 0$ then $\alpha^{\eta+1}(\alpha) = \alpha^\eta(\alpha)$ if $\bar{\delta}^{\eta+1} = \bar{\delta}^\eta$ and otherwise $\alpha^{\eta+1}(\alpha) = s^\eta(\alpha^\eta(\alpha), \delta^\eta(\alpha), \gamma^\eta)$ where $\gamma^\eta = \sup(\bar{\mathscr{C}}^\eta(\alpha, \bar{\delta}^\eta))$.

(6) $$o^{\mathscr{F}\phi}(\alpha^\eta(\alpha)) = i_{\nu,\nu^\eta}(\bar{\delta}^\eta) \quad \text{and} \quad \Gamma^{\eta'}\alpha \cap o^{\mathscr{F}\phi}(\alpha^\eta(\alpha)) = i_{\nu,\nu^\eta}{}''\bar{\delta}^\eta.$$

(7) $$\bar{\mathscr{C}}^\eta(\alpha, \beta) = \mathscr{C}^\eta(\alpha^\eta, i_{\nu,\nu^\eta}(\beta)) \cap \alpha \quad \text{for all} \quad \beta < o^{\mathscr{F}\nu}(\alpha).$$

We will prove (3–7) by induction on η simultaneously for all pairs (α, β). For limit η the proof is immediate, so we only need consider the successor case. Hence assume that (3–7) are true for $\eta - 1$, and we will prove them for η. Note that by (3) we know that $\mathscr{C}$ is rank $\eta - 1$ complete. Statements (3–5) and (7) all follow immediately from the induction hypothesis. To see (6), note that for all μ, Γ^η is obtained by augmenting the functions in $\Gamma^0 = i_\phi'' M$ by adding the function $s^{\eta'}(\alpha, \beta, \gamma)$ for $\eta' < \eta$. We have immediately that $\alpha^\eta \in \mathscr{C}(i_{\nu,\phi}(\alpha), i_{\nu,\phi}(\bar{\delta}^\eta))$ so $o^{\mathscr{F}\phi}(\alpha^\eta(\alpha)) = i_{\nu,\nu^\eta}(\bar{\delta}^\eta)$. An ordinal $\xi < o^{\mathscr{F}\phi}(\alpha^\eta(\alpha))$ is in $\Gamma^{\eta'}\alpha$ iff it is of the form $i_\phi(f)(\vec{a}, \alpha^\eta(\alpha))$ for some $f \in M$ and $\vec{a} \in \alpha^{<\omega}$, since $\alpha^\eta(\alpha)$ is the smallest ordinal above α added by the functions $s^{\eta'}$. If $\alpha^\eta(\alpha) = \alpha^0(\alpha)$ then $\xi = i_{\nu,\phi}(\gamma) = i_{\nu,\nu^0}(\gamma)$ for some $\gamma < \bar{\delta}^0 = o^{\mathscr{F}\nu}(\alpha)$, since $o^{\mathscr{F}\nu}(\alpha) \subset \{i_\nu(f)(\vec{a}) : f \in M \text{ and } \vec{a} \in \alpha^{<\omega}\}$ and in particular $\alpha = i_\nu(f)$ for some $f \in M$ and $\vec{a} \in \alpha^{<\omega}$.

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If $\alpha^\eta(\alpha) < \alpha^0(\alpha)$ then let $h(\vec{u}, \kappa, \lambda) = [\lambda v . f(\vec{u}, v)]_{\mathscr{F}(\mu, \lambda)}$. Since M is rudimentarily closed, $h \in M$. Let $\gamma = i_{\phi}(h)(\vec{a}, \alpha^0(\alpha), i_{\nu, \phi}(\bar{\delta}^\eta))$. Then

$$\xi = i_{\phi}(f)(\vec{a}, \alpha^\eta(\alpha)) = C(\alpha^0(\alpha), \gamma, i_{\nu, \phi}(\bar{\delta}^\eta))(\alpha^\eta(\alpha)),$$

since $\alpha^\eta(\alpha) \in \mathscr{C}(\alpha^0(\alpha), i_{\nu, \phi}(\bar{\delta}^\eta))$. Let $\bar{\gamma} = i_\nu(h)(\vec{a}, \alpha, \bar{\delta}^\eta) < o^\nu(\alpha)$ so $i_{\nu, \phi}(\bar{\lambda}) = \gamma$. Then $\xi = i_{\nu, \nu^\eta}(\bar{\gamma})$. Since ξ was arbitrary, $(\Gamma^{\eta^*}\alpha) \cap o^\phi(\alpha^\eta(\alpha)) = i_{\nu, \nu^\eta *}(\bar{\delta}^\eta)$. ∎

§ 3. *Mice*

Definition 3·1. An $\mathscr{F}$ mouse is a model $M = J^{\mathscr{G}}_\alpha$ such that for some ordinal δ with $l(\mathscr{F}) \leqslant \delta \leqslant \alpha$ and some parameter p in $[\alpha - \delta]^{<\omega}$, (i) $J^{\mathscr{G}}_\alpha = \Sigma_1''(\delta \cup p)$, (ii) $\mathscr{G} \restriction \delta + 1 = \mathscr{F}$, (iii) above δ, $\mathscr{G}$ is an ultrafilter sequence in $J^{\mathscr{G}}_\alpha$ such that every iterated ultrapower is well founded.

If M is an $\mathscr{F}$ mouse then δ_M is the least value of δ for which (i) is satisfied and p_M is the least value of p such that (i) is satisfied with $\delta = \delta_M$.

An $\mathscr{F}$ mouse is designed so that if $\mathscr{F}'$ is any strong end extension of $\mathscr{F}$, so that $\mathscr{F}' \restriction \delta = \mathscr{F}$ and $\mathscr{F}'$ is an ultrafilter sequence above δ, then any subset of δ in $L(\mathscr{F}')$ will be in an $\mathscr{F}$ mouse. Notice that condition (iii) puts no restriction on $\mathscr{F}$, so it need not be an ultrafilter sequence. Later we will use mice to define recursively a sequence $\mathscr{F}$ so that $\mathscr{F}$ is an ultrafilter sequence in $L(\mathscr{F})$.

It might seem that allowing $\mathscr{F}$ not to be an ultrafilter sequence would conflict with the definition of $J^{\mathscr{F}}_\alpha$ in Section 1. This would be true if we were seriously interested in the case when $\mathscr{F}$ was completely arbitrary, but we are only really interested in carrying the construction out far enough to construct a witness to the fact that $\mathscr{F}$ is not an ultrafilter sequence, if in fact it is not. All that is necessary for our definition of the order of construction to work is that $d(\gamma, \beta)$ be defined for $(\gamma, \beta) \in \text{domain}(\mathscr{F})$. This will hold in the cases we are interested in.

The simplest examples of mice are obtained by taking $\delta = \alpha$, giving the 0-mouse J_α and the $\mathscr{F}$ mouse $J^{\mathscr{F}}_\alpha$. (Strictly speaking $J^{\mathscr{F}}_\alpha$ is only an $\mathscr{F}$ mouse if $\alpha \geqslant l(\mathscr{F})$, but in order to reduce the need for special cases we will call $J^{\mathscr{F}}_\alpha$ a $\mathscr{F}$ mouse for any α.)

The smallest nonconstructible 0-mouse is obtained as follows: assume $0^\#$ exists, let $(c_n : n \leqslant \omega)$ be the first $\omega + 1$ indiscernibles for L and let $\mathscr{C}$ be the filter on c_ω generated by the first ω indiscernibles. Then $J^{\mathscr{C}}_{c_\omega + 1} \subset L$, so $\mathscr{C}$ is a $J^{\mathscr{C}}_{c_\omega + 1}$ ultrafilter. Let

$$\pi : M \cong H \prec_1 J^{\mathscr{C}}_{c_\omega + 1},$$

where H is the Σ_1 hull of the empty set and M is transitive. Then $M = J^{\mathscr{C}_0}_{c_0 + 1}$, where $\mathscr{C}_0 = \pi^{-1}(\mathscr{C})$, and M is a 0-mouse with $(\delta_M = p_M = 0)$ which has a single measurable cardinal $c_0 = \pi^{-1}(c_\omega)$.

More generally, suppose x is a subset of δ and $x \in J^{\mathscr{F}}_\alpha$, where $o^{\mathscr{F}}(\delta) = 0$ and $\mathscr{F}$ is an ultrafilter sequence in $J^{\mathscr{F}}_\alpha$ above δ. Let $\pi : M \cong H(\delta + 1 \cup p) \prec_1 J^{\mathscr{F}}_\alpha$, where p is a finite set of ordinals such that x is Σ_1-definable from parameters in $\delta \cup p$ in $J^{\mathscr{F}}_\alpha$. Then M is a $\mathscr{F} \restriction \delta$-mouse with $x \in M$. This is the most common method of constructing mice.

Since $\mathscr{G}$ is a well founded ultrafilter sequence above δ in $M = J^{\mathscr{G}}_\alpha$, we can always take interated ultrapowers $i : M \to \text{ult}(M, \mathscr{G}(\vec{f}))$ of M so long as the support $\vec{f}$ of i does not involve ultrafilters from $\mathscr{G} \restriction \delta$. The ultrapower $M' = \text{ult}(M, \mathscr{G}(f))$ satisfies Definition 3·1 except that in 3·1 (i) the set $\Sigma_1''(\delta \cup p)$ must be replaced by $\Sigma_1''(\delta \cup p; \mathscr{C})$, where $\mathscr{C}$ is

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the system of indiscernibles generated by i. The mouse M can always be recovered from its ultrapower M' since by 3·1 (i) M is isomorphic to the Σ_1 Skolem hull of $\delta \cup i(p)$ in M'.

Definition 3·2. If $M = J^{\mathscr{G}}_{\alpha}$ and $N = J^{\mathscr{H}}_{\beta}$ are $\mathscr{F}$-mice then $M < N$ if there are iterated ultrapowers $i: M \to J^{\mathscr{G}'}_{\alpha'}$ and $j: N \to J^{\mathscr{H}'}_{\beta'}$ with $i \restriction \delta_M$ and $j \restriction \delta_N$ equal to the identity such that $\alpha' \leqslant \beta'$, $\mathscr{H}' \restriction \alpha' = \mathscr{G}'$, and $(\alpha', \delta_M, i(p_M)) < (\beta', \delta_N, j(p_N))$ lexicographically.

THEOREM 3·3. *$<$ is a well ordering of the $\mathscr{F}$ mice such that if $M < N$ then $M \in L(N)$.*

Proof. The proof of Theorem 3·3 divides into three parts:

Part 1. If M and N are $\mathscr{F}$ mice then at least one of $M < N$, $N < M$ or $M = N$ holds.

Proof of Part 1. We will use the technique of the main lemma of [**14**] to compare the sequences $\mathscr{G}$ and $\mathscr{H}$. The proof is based on an idea already used in this paper, in Lemma 2·23. We will be using elaborations of this proof later and will refer back to this proof for notation.

We define by induction on ν a sequence of iterated ultrapowers:

$$i_\nu: M \to M_\nu = J^{\mathscr{G}_\nu}_{\alpha_\nu},$$

$$j_\nu: N \to N_\nu = J^{\mathscr{H}_\nu}_{\beta_\nu}.$$

At limit ordinals ν, i_ν and j_ν are the direct limits of $(i_{\nu'}: \nu' < \nu)$ and $(j_{\nu'}: \nu' < \nu)$. To define $i_{\nu+1}$ and $j_{\nu+1}$ we let (a_ν, b_ν) be the least pair (a, b) of ordinals (if any exist) such that $a < \alpha_\nu \cap \beta_\nu$ and one of the following hold:

(1) $$b = o^{\mathscr{G}_\nu}(a) < o^{\mathscr{H}_\nu}(a).$$

(2) $$b = o^{\mathscr{H}_\nu}(a) < o^{\mathscr{G}_\nu}(a).$$

(3) $$b < o^{\mathscr{H}_\nu}(a) \cap o^{\mathscr{G}_\nu}(a) \quad \text{and for some} \quad x \in M_\nu \cap N_\nu$$
$$x \in \mathscr{G}_\nu(a, b) \not\equiv x \in \mathscr{H}_\nu(a, b).$$

We set $M_{\nu+1} = M^{a_\nu}_\nu / \mathscr{G}_\nu(a_\nu, b_\nu)$ in cases (2) and (3) and $M_{\nu+1} = M_\nu$ in case (1). $N_{\nu+1}$ is defined similarly, taking the ultrapower in cases (1) and (3). Thus a_ν is an increasing sequence, $a_\nu > \delta_M \cap \delta_N$ for all ν, and for each ν, $\mathscr{H}_{\nu+1} \restriction (a_\nu + 1) = \mathscr{G}_{\nu+1} \restriction (a_\nu + 1)$.

Suppose that for some ordinal ν there is no pair (a, b) satisfying the conditions above. Then the process stops at ν and (supposing for example $\alpha_\nu \leqslant \beta_\nu$), $\mathscr{H}_\nu \restriction \alpha_\nu = \mathscr{G}_\nu$ and $M_\nu = J^{\mathscr{F}_\nu}_{\alpha_\nu} \subset N_\nu$. If $(\alpha_\nu, \delta_M, i_\nu(p_M)) < (\beta_\nu, \delta_N, j_\nu(p_N))$ then by the definition $M < N$, and if $(\alpha_\nu, \delta_M, i_\nu(p_M)) > (\beta_\nu, \delta_N, j_\nu(p_N))$ then $N < M$. If $(\alpha_\nu, \delta_M, i_\nu(p_M)) = (\beta_\nu, \delta_N, j_\nu(p_N))$ then both N and M are isomorphic to the Σ_1 hull of $(\delta_M \cup i_\nu(p_M))$ in M_ν, so $N = M$. Hence we will have completed the proof of Part 1 if we show that the process eventually stops.

To see that the process stops we apply the technique of Claim 2·24, letting $\delta = |M \cap N|^+$ and then considering substructures of $L_{\delta^+}(M \cup N)$. If the process has not stopped after δ steps then (a_δ, b_δ) will be defined. The argument of Claim 2·24 gives a closed unbounded set $D \subset \delta$ such that for $\nu \in D$ we have $a_\nu = \nu$ and

$$(a_\delta, b_\delta) = i_{\nu,\delta}(a_\nu, b_\nu) = j_{\nu,\delta}(a_\nu, b_\nu).$$

Since $i_{\nu,\delta}(a_\nu) > a_\nu$ and $j_{\nu,\delta}(a_\nu) > a_\nu$, cases (1) and (2) do not hold for $\nu \in D$ and hence also not for $\nu = \delta$. But D is a set of indiscernibles for both $\mathscr{F}_\delta(a_\delta, b_\delta)$ and $\mathscr{G}_\delta(a_\delta, b_\delta)$, so

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$\mathscr{F}_\delta(a_\delta, b_\delta) = \{x: \exists \lambda \subset \delta\, D \backslash \lambda \subset x\} = \mathscr{G}_\delta(a_\delta, b_\delta)$, and case (3) does not hold at δ either, contradicting the choice of (a_δ, b_δ). |

Part 2. $<$ is well founded.

Proof of Part 2. Suppose on the contrary that $<$ is not well founded. Then there is a sequence $(M_n: n \in \omega)$ of $\mathscr{F}$ mice so that $M_{k+1} < M_k$ for all $k \in \omega$. For each k set $\alpha_k = \alpha_{M_k}$, $\delta_k = \delta_{M_k}$ and $p_k = p_{M_k}$. We will define by induction a sequence

(4)
$$\begin{array}{ccccccccc} N_1 & \xrightarrow{i_{12}} & N_2 & \xrightarrow{i_{23}} & \cdots & \xrightarrow{i_{k-1\,k}} & N_k & \xrightarrow{i_{k\,k+1}} & \cdots N_\omega \\ \| & & \uparrow{\scriptstyle j_2} & & & & \uparrow{\scriptstyle j_k} & & \\ M_1 & & M_2 & & & & M_k & & \end{array}$$

so that each $i_{k\,k+1}$ is an iterated ultrapower, each j_k is a Σ_0 elementary embedding, and if $a_k = \sup\{j_k(\alpha): \alpha < \alpha_k\}$ then $i_{k\,k+1}(a_k, \delta_k, j_k(p_k)) > (a_{k+1}, \delta_{k+1}, j_{k+1}(p_{k+1}))$. This construction will yield a contradiction since the direct limit N_ω of the sequence (4) will be an iterated ultrapower of the mouse M_1 and hence must be well founded, while on the other hand the images of the triples $(a_k, \delta_k, j_k(p_k))$ will form an infinite descending sequence in N_ω. This contradiction shows that no infinite descending sequence $(M_k: k \in \omega)$ exists and hence $<$ is well founded.

Suppose N_k and j_k have been defined and let r and s, $r: M_k \to N$ and $s: M_{k+1} \to \bar{N}$, be iterated ultrapowers witnessing that $M_{k+1} < M_k$. Define the maps $i_{k\,k+1}$ and j so that the square

$$\begin{array}{ccc} N_k & \xrightarrow{i_{k,k+1}} & N_{k+1} \\ \uparrow{\scriptstyle j_k} & & \uparrow{\scriptstyle \bar{j}} \\ M_k & \xrightarrow{r} & N \end{array}$$

commutes. These maps can be defined by induction on the length of the iteration r, using the fact that j_k is Σ_0 elementary. Set $j_{k+1}: M_{k+1} \xrightarrow{s} \bar{N} \subset N \xrightarrow{\bar{j}} N_{k+1}$. Then j_{k+1} is Σ_0 elementary and

$$\begin{array}{ccccccc} N_k & \xrightarrow{i_{k,k+1}} & N_{k+1} & & & & \\ \uparrow{\scriptstyle j_k} & & \uparrow{\scriptstyle \bar{j}} & \searrow^{j_{k+1}} & & & \\ M_k & \xrightarrow{r} & N & \hookleftarrow & \bar{N} & \xleftarrow{s} & M_{k+1} \end{array} \quad \text{commutes.}$$

$$\begin{aligned} i_{k,k+1}(a_k, \delta_k, j_k(p_k)) &\geqslant (U_{\alpha<\alpha_k} i_{k,k+1}\, j_k(\alpha), \delta_k, i_{k,k+1}\, j_k(p_k)) \\ &= (U_{\alpha<\alpha_k} \bar{j} r(\alpha), \delta_k, \bar{j} r(p_k)) \\ &> (U_{\alpha<\alpha_k} \bar{j} s(\alpha), \delta_{k+1}, \bar{j} s(p_{k+1})) \\ &= (a_{k+1}, \delta_{k+1}, j_{k+1}(p_{k+1})), \end{aligned}$$

as required. |

Parts (1) and (2) show that $<$ is a well ordering of the $\mathscr{F}$ mice, and Part 3 below will complete the proof of Theorem 3·3.

Part 3. If $M < N$ then $M \in L(N)$.

Proof of Part 3. Let $i: M \to \bar{M}$ and $j: M \to \bar{N}$ witness that $M < N$, so $\bar{M}$ is an initial

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segment of $\bar{N}$. Then $\bar{M} \in L(\bar{N})$ trivially and $M \in L(\bar{M})$ since M can be obtained by collapsing the Σ_1 hull of $\delta_M \cup i(p_M)$ in $\bar{M}$. It is, however, not clear that $\bar{N} \in L(N)$. We sidestep this problem by factoring j into $N \xrightarrow{r} N^* \xrightarrow{s} \bar{N}$, so that r is an iterated ultrapower in $L(N)$, s is Σ_1 elementary, and $\delta_M \cup i(p_M) \cup \{\cup i''\alpha_M\} \subset \text{range}(s)$. Then the same argument gives $M \in L(N^*)$, so we have $M \in L(N)$ if we can show that $N^* \in L(N)$.

Now $\delta_M \subset \text{range}(j)$ unless $\delta_N < \delta_M$ and there is a ordinal $\eta < \delta_M$ such that $\delta_N \leqslant \eta$ and $o^{\mathscr{F}_N}(\eta) > 0$. If η is the least such ordinal then j starts with i^U_γ where $U = \mathscr{F}_N(\eta, 0)$ and γ is large enough so that $i^U_\gamma(\eta) > \delta_M$. The set $i(p_M) \cup \{\cup i''\alpha_M\}$ is finite and hence has finite support in j. We define r by first taking i^U_γ and following it with the finitely many ultrapowers in the support of $i(p_M) \cup \{\cup(i''\alpha_M)\}$. The map s is the rest of j. Then the maps r and s have the required properties. ∎

Definition 3·4. If $\mathscr{F}$ is a set then $K(\mathscr{F})$, the core model of $\mathscr{F}$, is the class of sets constructible from $\mathscr{F}$-mice. More specifically, if M is a mouse then let $A_M \subset \delta_M$ code the Σ_1 theory of M with parameters in $p_M \cup \delta_M$, so that M is constructible from A_M. Then $K(\mathscr{F}) = L(A^{\mathscr{F}})$, where $A^{\mathscr{F}} = \{(\alpha, \lambda, \delta) : \alpha \in A_M$, where M is the λth mouse (in the $<$-order) with $\delta_M = \delta\}$.

If $\mathscr{F}$ is a class then $K(\mathscr{F}) = L(\{(\xi, \alpha, \lambda, \delta) : (\alpha, \lambda, \delta) \in A^{\mathscr{F}|\xi+1}\})$.

It follows from Theorem 3·3 that $K(\mathscr{F}) \models ZF + AC$. Theorem 3·5 below shows that the *GCH* holds in $K(\mathscr{F})$ above $l(\mathscr{F})$. We will see later, in Theorem 3·9, that the *GCH* and other elementary properties of L hold throughout $K(\mathscr{F})$.

THEOREM 3·5. *If* $l(\mathscr{F}) \neq \mathbf{ON}$ *then* $K(\mathscr{F}) = \cup\{M : M$ *is an* $\mathscr{F}$*-mouse*$\}$.

In fact for each $x \in K(\mathscr{F})$, if $x \subset \xi$ for an ordinal ξ then there is an $\mathscr{F}$-mouse M such that $x \in M$ and κ_0^M, the first measurable cardinal in M above $l(\mathscr{F})$, is larger than ξ.

The Proof of 3·5 will stretch out over several lemmas. Note that if M is an $\mathscr{F}$ mouse with $\delta_M \geqslant l(\mathscr{G}_M)$ then $\mathscr{G}_M$ is $\mathscr{F}$ or an initial segment of $\mathscr{F}$, so $M \in L(\mathscr{F}) \subset \cup\{M : M$ is an $\mathscr{F}$-mouse$\}$. Hence we can assume that all mice M have an ordinal κ larger than δ_M such that $o^{\mathscr{G}_M}(\kappa) > 0$. Let κ_0^M be the least such ordinal and for $0 < n \leqslant \omega$, $\kappa_n^M = i^U_n(\kappa_0^M)$ where $U = \mathscr{G}_M(\kappa_0^M, 0)$.

LEMMA 3·6. *If* $i : M \to M'$ *is an iterated ultrapower of* M *with* $\mathscr{G}_{M'} \restriction \kappa_\omega^M = \mathscr{G}_M \restriction \kappa_0^M$ *then* κ_ω^M *is regular in* M'*; however there is an* $\mathscr{F}$*-mouse* N *with* $\kappa_0^N > \kappa_\omega^M$ *such that* $(\kappa_n^M : n \in \omega) \in N$.

Proof. It is clear that κ_ω^N is regular in any iterated ultrapower M' of M with $\mathscr{G}_{M'} \restriction \kappa_\omega^M = \mathscr{G}_M \restriction \kappa_0^M$. To find N use Lemmas 2·19 and 2·21 to find an ultrapower $i : M \to N = J^{\mathscr{F}}_\beta$ so that the system $\mathscr{C}$ of indiscernibles generated by i gives an extension $\mathscr{H}'$ of $\mathscr{H}$ which is an ultrafilter sequence in $J^{\mathscr{F}'}_{\beta+1}$. But by Proposition 2·20 $\mathscr{C}$, and hence $(\kappa_n^M : n \in \omega)$, is definable in $J^{\mathscr{F}}_\beta$ from $\delta_M \cup i(p_M)$. Thus $(\kappa_n^M : n \in \omega) \in J^{\mathscr{F}'}_{\beta+1}$, and the collapse N of the Σ_1 Skolem hull of $(\kappa_\omega^M + 1) \cup i(p_M)$ is an $\mathscr{F}$ mouse with $\kappa_0^N > \kappa_\omega^M$ and with $(\kappa_n^M : n \in \omega)$ in N. ∎

LEMMA 3·7. *Any set of* $\mathscr{F}$ *mice has an upper bound under* $<$.

Proof. Let X be a set of $\mathscr{F}$ mice. We will find $J^{\mathscr{G}}_\alpha$ so that $\mathscr{G}$ is a well founded ultrafilter sequence in $J^{\mathscr{G}}_\alpha$ above $l(\mathscr{F})$, $\mathscr{C} \restriction \delta = \mathscr{F}$ where $\delta = \cup\{\kappa_\omega^M : M \in X\}$, and $cf(\kappa_\omega^M) = \omega$ in $J^{\mathscr{G}}_\alpha$ for all $M \in x$. Then the transitive collapse of the Σ_1 Skolem hull of $\delta + 1$ in $J^{\mathscr{G}}_\alpha$ is an $\mathscr{F}$ mouse N. For each $M \in X$ $cf(\kappa_\omega^M) = \omega$ in N so $M < N$ by Lemma 3·6. Thus N is the desired upper bound.

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By Lemma 3·6 we can assume that for all $M \in X$ there is $M' \in X$ with $\kappa_0^{M'} > K_\omega^M$ and $cf(\kappa_\omega^M) = \omega$ in M'. We will use the method of Theorem 3·3, Part 1 to compare all the mice in X rather than only two mice. For each $M \in X$ we will define a sequence $i_\nu^M : M \to M_\nu$ of iterated ultrapowers. Clauses (1) and (2) of 3·3 are replaced by

(1′) $$\exists M, M' \in X (a \in M_\nu \cap M'_\nu \quad \text{and} \quad o^{M_\nu}(a) = b < o^{M'_\nu}(a))$$

and (3) is replaced by

(3′) $\mathscr{G}_\nu(a,b) = \cup\{\mathscr{G}^{M_\nu}(a,b) : a \in M_\nu$ and $b < o^{M_\nu}(a)\}$ is not countably complete, that is, there is $s \subset \mathscr{G}_\nu(a,b)$ such that $|s| \leqslant \aleph_0$ and $\cap s = 0$.

Let (a_ν, b_ν) be the least pair such that (1′) or (3′) holds.

Notice that if $a_\nu \notin M_\nu$ then M_ν is an initial segment of M'_ν for all those $M' \in X$ such that $a_\nu \in X$. If $a_\nu \notin M_\nu$ then set $i_{\nu,\mu}^M = id$ for all $\mu > \nu$. Otherwise in Case (3′) set $i_{\nu\nu+1}^M : M_\nu \to \text{ult}(M_\nu, \mathscr{G}^{M_\nu}(a_\nu, b_\nu))$ for all $M \in X$ such that $a_\nu \in M_\nu$; and in Case (1′) take the ultrapower only for those M_ν such that $o^{M_\nu}(a_\nu) > b_\nu$.

If the process stops at some ν then the sequence $\mathscr{G}$ defined by

$$\mathscr{G}(\kappa,\lambda) = \cup\{\mathscr{G}^{M_\nu}(\kappa,\lambda) : M \in X \text{ and } \kappa \in M_\nu\} \quad \text{and} \quad \alpha = \cup\{M_\nu \cap \mathbf{ON} : M \in X\}$$

give the desired $J_\alpha^{\mathscr{G}}$. The proof that it does stop is essentially the same as in Theorem 3·3. ∎

Proof of Theorem 3·5. Suppose $x \in K(\mathscr{F})$. Then x is constructed from an initial segment of the class of $\mathscr{F}$ mice so by Lemma 3·7 there is an $\mathscr{F}$-mouse M and an ordinal ν such that $x \in L_\nu(M)$. Now by Lemma 2·21 there is an iterated ultrapower

$$i : M \to M' = J_\alpha^{\mathscr{G}}$$

such that i induces a rank $\omega \cdot (1+\rho+\nu)$ complete system $\mathscr{C}$ of indiscernibles, where $o = (\mathbf{ON} \cap M) + 1$. By Lemma 2·19 $\mathscr{G}$ can be extended to an ultrafilter sequence for $J_{\alpha'}^{\mathscr{G}}$ where $\alpha' = \alpha + 1 + \rho + 1 + \nu$. We can also assume that i is chosen so that $\mathscr{G} \restriction \nu = \mathscr{F} \restriction \nu$. We will show that $x \in J_\alpha^{\mathscr{G}}$; the usual collapsing argument will then yield a mouse N with $x \in N$.

First, $i''M \in J_{\alpha+1}^{\mathscr{G}}$, since $i''M$ is the set of members of $J_\alpha^{\mathscr{G}}$ which are Σ_1 definable from $\delta_M \cup i(p_M)$. A simple calculation shows that the collapsing map $i^{-1} \restriction (i''M) : i''M \cong M$ is a member of $J_{(\alpha+1+\rho+1)}^{\mathscr{G}}$. But then $x \in L_\nu(M) \subset J_{\alpha'}^{\mathscr{G}}$. ∎

The only fact about the sequence $\mathscr{F}$ which we have used so far is that it is a set small enough to be ignored in our arguments. Now we will want to work with the sequence $\mathscr{F}$ itself and so we will need a further assumption:

Definition 3·8. The sequence $\mathscr{F}$ is *strong* if $\mathscr{F}$ is an ultrafilter sequence in $K(\mathscr{F})$.

THEOREM 3·9. *If $\mathscr{F}$ is strong then*

(i) $K(\mathscr{F}) \models ZFC + GCH +$ (*there is a Δ_3^1 well ordering of the reals*).

(ii) *Every normal ultrafilter in $K(\mathscr{F})$ is a member of the sequence $\mathscr{F}$.*

(iii) *Every countably complete ultrafilter in $K(\mathscr{F})$ is in the smallest class containing the ultrafilters $\mathscr{F}(\kappa,\lambda)$ and the principal ultrafilters and closed under isomorphism and products of the form $\Sigma_{i \in I} U_i / U$.*

Sketch of proof. We first prove the *GCH* in $K(\mathscr{F})$. Assume for the moment that

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$o(\lambda) \leqslant \kappa$ for all $\lambda \leqslant \kappa$. It will be enough to show that every subset x of κ is in some $\mathscr{F} \restriction (\kappa+1)$ mouse in $K(\mathscr{F})$, since the order type of

$$\{M: M \text{ is an } \mathscr{F} \restriction (\kappa+1) \text{ mouse and for some } x \subset \kappa\ M \text{ is the least mouse with } x \in M\}$$

is at most κ^+. Now x is in some $\mathscr{F}$ mouse $J^{\mathscr{G}}_\alpha$. Since $\mathscr{G} \restriction l(\mathscr{F}) = \mathscr{F}$, $\mathscr{G}$ is countably complete below $l(\mathscr{F})$, and we can iterate $J^{\mathscr{G}}_\alpha$ to obtain $i: J^{\mathscr{G}}_\alpha \to J^{\mathscr{G}'}_{\alpha'}$ so that $i \restriction l(\mathscr{F}) = id$ and $\mathscr{G}'$ is countably complete. Now any iterated ultrapower of $J^{\mathscr{G}'}_{\alpha'}$ by ultrafilters in G' on ordinals $\alpha > \kappa$ is well founded, and $x \in J^{\mathscr{G}'}_{\alpha'}$, so we can let $J^{\mathscr{G}''}_{\alpha''} \cong \mathscr{H}(\kappa \cup p) < J^{\mathscr{G}'}_{\alpha'}$ be the collapse of the Skolem hull of $\kappa \cup p$, where p is a finite sequence of ordinals such that x is Σ_1-definable from p in $J^{\mathscr{G}'}_{\alpha'}$. Then $J^{\mathscr{G}''}_{\alpha''}$ is the desired mouse.

Now we can remove the assumption on κ. Suppose κ is least such that $2^\kappa > \kappa^+$. If $o(\kappa) > 0$ then the argument above shows that every $x \subset \kappa$ is in a $\mathscr{F} \restriction \kappa$ mouse in ult $(K(\mathscr{F}), \mathscr{F}(\kappa, 0))$ and hence in $K(\mathscr{F})$, so $2^\kappa = \kappa^+$ in $K(\mathscr{F})$. If $o(\lambda) > \kappa$ for some $\lambda < \kappa$ then $o(\lambda) \leqslant (2^\lambda)^+ \leqslant \kappa^+$, since $2^\lambda = \lambda^+$. Now $\mathscr{G}'' \restriction \kappa$ will be $\mathscr{F} \restriction (\lambda, \delta)$ where δ is κ^+ in $J^{\mathscr{G}''}_{\alpha''}$. Thus every $x \subset \kappa$ is in a $\mathscr{F} \restriction (\lambda, \delta)$-mouse for some $\delta < \kappa^+$. There are only κ^+ such mice for each δ and hence $(\kappa^+)^2 = \kappa^+$ such sets in all.

The rest of (i) can be proved similarly by modifying proofs from L and $L(U)$. Proofs of (ii) and (iii) can be obtained by the same modification to theorems 3·2 and 5·1 of [**14**], but results from section 5 are needed to show that the same notion of θ-minimality works. We will give a different notion of θ-minimality, suggested by Solovay, which does not require fine structure.

Suppose θ is any Σ_1 sentence, for example the sentence asserting that (ii) or (iii) is false. If θ is true in any model of $\mathbf{V} = K(\mathscr{F})$ then θ will be true in a model of the form $L(\mathscr{F}, M)$ such that $\mathscr{F}$ is a set, M is an $\mathscr{F}$-mouse, and $\mathscr{F}$ is strong in $L(\mathscr{F}, M)$. In this case we will say $L(\mathscr{F}, M)$ is a θ-model. In the following discussion $(\mathscr{F}, M)$ will range over such pairs.

If x is any set then $x^\#$ is defined by analogy with $0^\#$:

Definition 3·10. Suppose C is a closed unbounded class of indiscernibles for $L(x)$. Then $x^\#$ is the theory of $L(x)$ in the language with constants for x, for each of the first w members of c, and for each member of the transitive closure of x.

Then $x^\#$ works like $0^\#$: if $x^\# \in M$ then the effect is as if $L(x)$ were a set in M.

Definition 3·11. $L(\mathscr{F}, M)$ is θ-minimal if it is a θ-model and there is no θ-model $L(\mathscr{F}', M')$ with $(\mathscr{F}', M')^\# \in L(\mathscr{F}, M)$.

The ordering implied by this definition is well founded, since if $(\mathscr{F}', M')^\# \in L(\mathscr{F}, M)$ then for any $\kappa > \sup(\text{ord}(M), \text{ord}(M'))$ we have $\kappa^{+L(\mathscr{F}', M')} < \kappa^{+L(\mathscr{F}, M)}$. Thus there is a θ-minimal model. Also, the property of being θ-minimal is first order and hence is preserved by elementary embeddings. All we need now in order to use this notion of θ-minimality is the following lemma:

LEMMA 3·12. *Suppose $L(\mathscr{F}, M)$ and $L(\mathscr{F}', M')$ are both θ-minimal and $\mathscr{F}' = \mathscr{F} \restriction \delta$ for some ordinal δ. Then $\mathscr{F} = \mathscr{F}'$ and $L(\mathscr{F}, M) = L(\mathscr{F}', M')$.*

We will use two subsidiary results, which are of interest in their own right.

PROPOSITION 3·13. $L(\mathscr{F}, M) \models \mathbf{V} = K(\mathscr{F})$.

Proof. Let N be $K(\mathscr{F})$ as defined in $L(\mathscr{F}, M)$. Clearly $M \in N$ and $\mathscr{F} \cap N \in N$, so $L(\mathscr{F}, M) \subset N$. Thus $N = L(\mathscr{F}, M)$, but $N \models \mathbf{V} = K(\mathscr{F})$. ∎

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LEMMA 3·14. *Suppose M is an $\mathscr{F}$-mouse and $M \notin L(\mathscr{F}, N)$. Then $(\mathscr{F}, N)^{\#} \in L(M)$.*

Proof. Assume without loss of generality that M is the least $\mathscr{F}$-mouse not in $L(\mathscr{F}, N)$. If M had no measurable cardinals larger than $l(\mathscr{F})$ then M would be $J_\alpha^{\mathscr{F}}$ for some ordinal α, so $M \in L(\mathscr{F}) \subset L(\mathscr{F}, N)$. Hence we can assume $M = J_\alpha^{\mathscr{G}}$ where $o^{\mathscr{G}}(\kappa) > 0$ for some $\kappa > l(\mathscr{F})$. Let κ_0 be the least such κ and let U be $\mathscr{G}(\kappa_0, 0)$.

We claim that $L(\mathscr{F}, N) \subset \mathrm{ult}_{\mathbf{ON}}(M, U) \cap \mathbf{V} = \{x : \exists \delta\, x \in R_\delta \cap \mathrm{ult}_\delta(M, U)\}$.

If $x \in L(\mathscr{F}, N) \cap R_\delta$ then by Theorem 3·5 and Proposition 3·13 there is an $\mathscr{F}$-mouse N' in $L(\mathscr{F}, N)$ with $\kappa_0^{N'} > \delta$ and $x \in N'$. But $N' < M$, since $M \leqslant N'$ would imply $M \in L(N') \subset L(\mathscr{F}, N)$, contrary to assumption. Thus $x \in \mathrm{ult}_\delta(M, U)$.

Now pick δ large enough that $\kappa_\delta = i_\delta^U(\kappa_0) > \mathrm{rank}(N)$ and let U_δ be $i_\delta^U(U)$. Then U_δ is a normal ultrafilter on $P(\kappa_0) \cap \mathrm{ult}_{\mathbf{ON}}(M, U) \cap \mathbf{ON}$ such that the ultrapower by U_δ is well founded. Hence the same is true of $L(\mathscr{F}, N)$, so $(\mathscr{F}, N)^{\#}$ is constructible from $\mathscr{F}, N$ and U_δ. But all three are in $L(M)$, so $(\mathscr{F}, N)^{\#} \in L(M)$. ∎

We will need a finer analysis than that given by Lemma 3·14.

LEMMA 3·15. *Let M, N and $M_\delta = \mathrm{ult}_\delta(M, U_0)$ be as in Lemma 3·14. Then $(\mathscr{F}, N)^{\#}$ is Σ_1-definable in M_δ.*

Proof. Consider the common iterates of M and N:

$$i : M \to M' = J_\alpha^{\mathscr{G}}, \quad i' : N \to N' = J_\beta^{\mathscr{G}'},$$

where $\mathscr{G}' = \mathscr{G} \restriction \beta$. We can carry out the iteration so that $\mathscr{G}$ is rank-κ'-complete in N', where $\kappa' = i(\kappa_0^M)$ is the first measurable cardinal in M' and N', and so that $\kappa' > \delta$ and $U' = i(U_0)$ is countably complete. We consider two cases, depending on whether $\alpha > \beta + \kappa'$.

Case 1. $\alpha \leqslant \beta + \kappa'$. We show in this case that $M \in L(\mathscr{F}, N)$, contrary to assumption. Since the indiscernibles $\mathscr{C}$ generated by i' are rank κ' complete, we can use them to extend $\mathscr{G}'$ to $\mathscr{G}''$ so that $J_{\beta+\kappa'}^{\mathscr{G}''} \supset J_\alpha^{\mathscr{G}} = M'$. Thus $M' \in L(N', \mathscr{C})$ and, since M' can be collapsed to M, $M \in L(N', \mathscr{C})$. Now i', and hence N' and $\mathscr{C}$, might not be in $L(N, \mathscr{F})$. However we can construct in $L(N, \mathscr{F})$ a minimal iteration $\bar{i} : N \to \bar{N}$ such that the system $\bar{\mathscr{C}}$ of indiscernibles defined by $\bar{i}$ is rank κ' complete. There is a canonical embedding $\pi : \bar{N} \to N'$ which takes $\bar{\mathscr{C}}$ into $\mathscr{C}$ so that π can be extended to $\pi^* : J_{\bar{\beta}+\kappa'}^{\mathscr{G}'} \to J_{\beta+\kappa'}^{\mathscr{G}''}$ where $\bar{N} = J_{\bar{\beta}}^{\mathscr{G}'}$. But then $\bar{M} = \pi^{*-1}(M')$ is in $L(\mathscr{F}, N)$ and can be collapsed to M. Thus $M \in L(\mathscr{F}, N)$. ∎

Case 2. $\alpha > \beta + \kappa'$. The collapse of N' to N can be carried out in $J_{\alpha+\mathrm{ord}(N)}^{\mathscr{G}}$. Since $\mathrm{ord}(N) < \kappa'$, $L_{\kappa'}(\mathscr{F}, N)$ is definable in $J_{\alpha+\mathrm{ord}(N)+\kappa'}^{\mathscr{G}} = J_{\alpha+\kappa'}^{\mathscr{G}}$. Now use the measure $U' = i(U)$ to find a homogeneous set for formulas in $L_{\kappa'}(\mathscr{F}, N)$. We can't quite carry that out in $J_{\alpha+\kappa'+1}^{\mathscr{G}}$, but for each n and m we will have a set $C_{n,m}$ such that whenever ϕ is a Σ_n formula and $\vec{c}, \vec{d} \in [C_{n,m}]^m$ then $L_{\kappa'}(\mathscr{F}, N) \models \phi(\vec{c}) \equiv \phi(\vec{d})$. Now $C = \cap_{n,m} C_{n,m}$ is a set of indiscernibles for $L_{\kappa'}(\mathscr{F}, N)$, and they yield $(\mathscr{F}, N)^{\#}$:

$$\phi(\vec{v}) \in (\mathscr{F}, N)^{\#} \equiv \forall \vec{c} \in [C_{n,m}]^m \quad L_{\kappa'}(\mathscr{F}, N) \models \phi(\vec{c})$$

for the appropriate pair m, n. Thus, $(\mathscr{F}, N)^{\#}$ is Σ_1 definable in M'. Now M_δ is a collapse of M', so the same Σ_1 definition will work in M_δ. ∎

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Proof of lemma 3·12. We have θ-minimal models $L(\mathscr{F}, M)$ and $L(\mathscr{F}', M')$ with $\mathscr{F}' = \mathscr{F} \restriction \delta$. If $\mathscr{F} = \mathscr{F}'$ then M' is an $\mathscr{F}$ mouse and since by θ-minimality

$$(\mathscr{F}, M')^{\#} \notin L(\mathscr{F}, M)$$

we must have from the lemma that $M \in L(\mathscr{F}, M')$. Similarly $M' \in L(\mathscr{F}, M)$, so $L(\mathscr{F}, M) = L(\mathscr{F}', M')$.

Now suppose that $\mathscr{F}' \neq \mathscr{F}$, so $\mathscr{F}' = \mathscr{F} \restriction \delta$ where $\delta < l(\mathscr{F})$. We will show that one of $L(\mathscr{F}, M)$ and $L(\mathscr{F}', M')$ is not θ-minimal. We can assume (by taking an iterated ultrapower of $L(\mathscr{F}, M)$ if necessary) that the first cardinal $\kappa > \delta$ with $o^{\mathscr{F}}(\kappa) > 0$ is larger than ord (M'). If $M' \in L(\mathscr{F}, M)$ then $L(\mathscr{F}', M') \subset L(\mathscr{F}, M)$ and $\mathscr{F}(\kappa, 0)$ is a normal, well founded measure on $L(\mathscr{F}', M')$, so $(\mathscr{F}', M')^{\#} \in L(\mathscr{F}, M)$ and $L(\mathscr{F}, M)$ is not θ-minimal. Thus we can assume $M' \notin L(\mathscr{F}, M)$.

Let $M' = J^{\mathscr{G}}_{\alpha}$ and consider ultrapowers

$$i: L(\mathscr{F}, M) \to L(i(\mathscr{F}), i(M)), \quad j: J^{\mathscr{G}}_{\alpha} \to J^{j(\mathscr{G})}_{j(\alpha)} = j(M')$$

so that one of $i(\mathscr{F})$ and $j(\mathscr{G})$ is an initial segment of the other. Now $j(\mathscr{G})$ cannot be an initial segment of $i(\mathscr{F})$: otherwise $j(M')$ would be in $L(i(\mathscr{F}), i(M))$ so M would be in $L(\mathscr{F}, M)$. Then $j(M')$ is an $i(\mathscr{F})$ mouse and $j(M') \notin L(i(\mathscr{F}), i(M))$, so by Lemma 3·15 $(i(\mathscr{F}), i(M))^{\#}$ is Σ_1 definable in $j(M')$. Now of course we cannot conclude that $(i(\mathscr{F}), i(M))^{\#} \in L(M')$, but consider $L(\mathscr{F}'', M'') \cong \mathscr{H}(C) \prec L(\mathscr{F}, M)$ where $\mathscr{H}(C)$ is the Skolem hull of the class C of indiscernibles for $L(\mathscr{F}, M)$. Then $\mathscr{F}''$ and M'', and hence $(\mathscr{F}'', M'')^{\#}$, will be countable and Σ_1-definable in $j(M')$. The collapse of $j(M')$ to M' will preserve this Σ_1-definition, so $(\mathscr{F}'', M'')^{\#}$ is Σ_1-definable in M' and hence is in $L(\mathscr{F}', M')$. But $L(\mathscr{F}'', M'') \models \theta$, so this contradicts the θ-minimality of $L(\mathscr{F}', M')$. |

We now come to the problem of constructing a strong sequence $\mathscr{F}$. The basic procedure is to define $\mathscr{F}(\kappa, \lambda)$ by recursion over the pairs (κ, λ). The major problem is to ensure when $\mathscr{F}(\kappa, \lambda)$ is chosen that it will still be an ultrafilter in $K(\mathscr{F})$ when all of $\mathscr{F}$ has been defined. One solution, which was used in theorem 4·1 of [**14**], is simply to require that $\mathscr{F}(\kappa, \lambda)$ be an ultrafilter in the real world. In theorem 4·1 of [**14**] we used the hypothesis that κ is 2^{κ}-supercompact. In order to find sequences $\mathscr{F}$ under weaker hypotheses which do not imply the existence of even measurable cardinals in the real world this procedure must be modified. The rest of this section will describe the modification and the second part will show, among other things, that the modification works.

Definition 3·16. Suppose $\kappa \geqslant l(\mathscr{F})$ and U is a ultrafilter on $\mathscr{P}(\kappa) \cap K(\mathscr{F})$. We say U is a $K(\mathscr{F})$-ultrafilter if

(i) (*normality*) if $f \in K(\mathscr{F})$ and $\{\nu: f(\nu) < \nu\} \in U$ then for some $\nu_0 < \kappa$

$$\{\nu: f(\nu) = \nu_0\} \in U.$$

(ii) (*coherence*) $\mathscr{F} = i^U(\mathscr{F}) \restriction \kappa + 1$; and if $f \in K(\mathscr{F})$ and $\{\nu: f(\nu) < o^{\mathscr{F}}(\nu)\} \in U$ then for some $f' \in L(\mathscr{F})$ $\{\nu: f(\nu) = f'(\nu)\} \in U$.

(iii) ($\mathscr{F}$ *is absolutely well founded*) if $\mathscr{F}'$ is any sequence such that $\mathscr{F}' \restriction \kappa + 1 = \mathscr{F}$ and $\mathscr{F}'$ is strong above κ then $K(\mathscr{F}')^{\kappa}/U$ is well founded.

In clause (ii), $i^U(\mathscr{F}) \restriction \kappa + 1 = \mathscr{F}$ means only that the domain of $(i^U(\mathscr{F})) \restriction \kappa + 1$ is the same as that of $\mathscr{F}$, and that $x \in \mathscr{F}(\alpha, \beta)$ iff $x \in i^U(\mathscr{F})(\alpha, \beta)$ for all $(\alpha, \beta) \in$ domain $(\mathscr{F})$ and all $x \in K(\mathscr{F})$. In general there may be subsets of κ in ult $(K(\mathscr{F}), U) \backslash K(\mathscr{F})$, so $i^U(\mathscr{F})(\kappa, \beta)$ may strictly contain $\mathscr{F}(\kappa, \beta)$. We show that if $\mathscr{F}$ is strong then this cannot

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happen. We had originally proved this as part of the fine structure argument in part II; Solovay pointed out that it could be proved directly here.

THEOREM 3·17. *If $\mathscr{F}$ is strong, U is a $K(\mathscr{F})$-ultrafilter on κ, $x \subset \kappa^2$, and $x \in K(\mathscr{F})$ then $z = \{\nu : \{\xi : (\nu, \xi) \in x\} \in U\} \in K(\mathscr{F})$. Thus U is a $K(\mathscr{F})$-ultrafilter in Kunen's sense.*

Proof. Let $W = \text{ult}(K(\mathscr{F}), U)$. Then $z = \{\nu : (\nu, \kappa) \in i(x)\} \in W$. Since $W \models V = K(i(\mathscr{F}))$, z is in an $i(\mathscr{F}) \restriction (\kappa+1)$ mouse in W. It will therefore be enough to show that any subset of κ in an $i(\mathscr{F}) \restriction (\kappa+1)$ mouse in W is in an $\mathscr{F}$ mouse. Suppose not, and let $M = J^{\mathscr{G}}_{\alpha+1}$ be the least $i(\mathscr{F}) \restriction (\kappa+1)$ mouse in W which contains a set $x \subset \kappa$ not in $K(\mathscr{F})$. There is an $\mathscr{F}$ mouse N in $K(\mathscr{F})$ with $P(\kappa) \cap K(\mathscr{F}) \subset N$; let us compare N and M, say

$$i : M \to M' = J^{\mathscr{G}'}_{\alpha'+1}, \quad j : N \to N' = J^{\mathscr{F}'}_{\gamma'},$$

so that if $\delta = \inf(\alpha'+1, \gamma')$ then $\mathscr{F}' \restriction \delta = \mathscr{G}' \restriction \delta$, at least in the sense that their domains are equal and the ultrafilters agree on sets in $M' \cap N'$. Since this is already true of $\mathscr{F}$ and $i(\mathscr{F}) \restriction \kappa+1$, $i \restriction (\kappa+1) = j \restriction (\kappa+1) = \text{id}$. Now if $\gamma' \geqslant \alpha'+1$ then since $\mathscr{F}'$ is an ultrafilter on $J^{\mathscr{F}'}_{\gamma'}$ we must have $G' = \mathscr{F}'$ in $J^{\mathscr{F}'}_{\alpha'+1}$, so $J^{\mathscr{F}'}_{\alpha'+1} = J^{\mathscr{G}'}_{\alpha'+1}$ and $x \in N \subset K(\mathscr{G})$, contrary to assumption. Hence $\gamma' < \alpha'+1$ and

$$P(\kappa) \cap K(\mathscr{F}) = P(\kappa) \cap J^{\mathscr{G}'}_{\alpha'} \in N' \subset M'.$$

Now define $\mathscr{F}^*$ by

$$\mathscr{F}^*(\nu, \beta) = \begin{cases} \mathscr{F}(\nu, \beta) = \mathscr{G}'(\nu, \beta) \cap K(\mathscr{F}) & \text{if } \nu \leqslant \kappa \\ \mathscr{G}'(\nu, \beta) & \text{if } \nu > \kappa. \end{cases}$$

Then $J^{\mathscr{G}'}_{\alpha'} = J^{\mathscr{F}^*}_{\alpha'} \subset J^{\mathscr{F}^*}_{\alpha'+1} \subset J^{\mathscr{G}'}_{\alpha'+1}$. Now $\mathscr{F}^*$ is an ultrafilter sequence above κ. The coherence is inherited from $J^{\mathscr{G}'}_{\alpha'+1}$, since the required coherence function $C(\alpha, \gamma, \beta)$ for any $\mathscr{G}(\alpha, \beta)$ used at level $\alpha'+1$ is already in $J^{\mathscr{G}'}_{\alpha'}$. The fact that all iterated ultrapowers are well founded is also inherited from $J^{\mathscr{G}'}_{\alpha'+1}$. Thus $P(\kappa) \cap J^{\mathscr{F}^*}_{\alpha'+1} \subset K(\mathscr{F})$, so

$$P(\kappa) \cap J^{\mathscr{F}^*}_{\alpha'+1} \subset J^{\mathscr{F}^*}_{\alpha'}.$$

Since $\mathscr{F}^*(\nu, \beta) = \mathscr{G}'(\nu, \beta)$ for $\nu > \kappa$ and $\mathscr{F}^*(\nu, \beta) \cap J^{\mathscr{F}^*}_{\alpha'} = \mathscr{G}'(\nu, \beta) \cap J^{\mathscr{F}^*}_{\alpha'}$ for $\nu \leqslant \kappa$, this implies that $J^{\mathscr{F}^*}_{\alpha'+1} = J^{\mathscr{G}'}_{\alpha'+1}$. But this is absurd, since $x \in J^{\mathscr{G}'}_{\alpha'+1}$ but $x \notin K(\mathscr{F})$ and hence $x \notin J^{\mathscr{F}^*}_{\alpha'+1}$. ∎

THEOREM 3·18. *If $\mathscr{F}$ is a sequence such that each $\mathscr{F}(\alpha, \beta)$ is a $K(\mathscr{F}) \restriction (\alpha, \beta)$-ultrafilter then $\mathscr{F}$ is strong.*

Theorem 3·18 is the basic result which permits us to construct strong sequences by recursion over pairs (α, β). It will be proved in part II in two steps: In Section 4 we will prove the special case in which well foundedness is ensured by requiring each $\mathscr{F}(\alpha, \beta)$ to be countably complete. This proof is an essential part of the proof of the uniformization lemma. Section 5 contains a basic lemma on $K(\mathscr{F})$ which requires this special case of Theorem 3·18, and Section 6 contains the covering lemma. The general proof of Theorem 3·18 will then be given in Section 7 as one of the applications of the covering lemma to the basic theory of $K(\mathscr{F})$.

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