Proc. Camb. Phil. Soc. (1971), **69**, 87
PCPS 69–9
Printed in Great Britain

Functions which operate on positive definite functions

BY DANIEL RIDER†
University of Wisconsin

(*Received* 30 *October* 1969)

1. *Introduction.* Let P be a collection of functions mapping some set into the complex plane C and let Δ be a subset of C. A function $F: \Delta \to C$ is said to *operate on* P provided the composition $F \circ \phi$ belongs to P whenever $\phi \in P$ and the range of ϕ is contained in Δ. The basic problem is to characterize the functions which operate.

This paper is concerned with the case where Γ is a locally compact group and $P = P(\Gamma)$ is the set of continuous positive definite functions on Γ. If

$$F(z) = \sum_{m,n=0}^{\infty} c_{m,n} z^m \bar{z}^n, \quad \text{where} \quad c_{m,n} \geqslant 0, \tag{1·1}$$

converges absolutely for $z \in \Delta$ then it is easily seen that F operates on $P(\Gamma)$. That the converse is true has been shown in the following cases:

I. Rudin (3). $\Delta = (-1, 1)$ and Γ is the additive group of integers.

II. Herz (1). $\Delta = (-1, 1)$ and Γ either has no exponent or Γ contains an infinite subgroup with exponent two.

III. Herz (1). Δ is the open unit disc and Γ has no exponent.

Herz has also characterized the functions operating on $P(\Gamma)$ when Δ is the closed unit disc or a closed interval and Γ has no exponent.

Here we will consider the problem when Δ is the open unit disc and Γ has an exponent. For each Γ there is a subset of $\Delta = \{z: |z| < 1\}$ for which the problem is well posed. If Γ has exponent e and $\alpha = \exp(2\pi i/e)$ let

$$\Delta(\Gamma) = \Delta_e = \left\{ \sum_{i=1}^{e} a_i \alpha^i : a_i \geqslant 0, \Sigma a_i < 1 \right\}.$$

If $\phi \in P(\Gamma)$ and range $\phi \subset \Delta$ then range $\phi \subset \Delta(\Gamma)$. Furthermore, if $z \in \Delta(\Gamma)$ there is $\phi \in P(\Gamma)$ such that

$$z \in \text{range}\, \phi \subset \Delta.$$

Hence if F operates on $P(\Gamma)$ we are concerned only with the values of F on $\Delta(\Gamma)$. Our main results are that (1·1) characterizes the functions operating on $P(\Gamma)$ for a certain class of groups with exponent and that this is false for the remaining groups.

DEFINITION. *A locally compact Abelian group is* exceptional *if it is the direct product of a finite group with exponent greater than two and a group of exponent two.*

† This research was supported in part by National Science Foundation Grant 144 A002.

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

THEOREM 1·2. *Let Γ be a non-exceptional locally compact Abelian group. A function F defined on Δ operates on $P(\Gamma)$ if and only if F is represented by an absolutely convergent series, $z \in \Delta(\Gamma)$,*

$$F(z) = \sum_{m,n=0}^{\infty} c_{m,n} z^m \bar{z}^n \quad \text{with} \quad c_{m,n} \geqslant 0.$$

When Γ has no exponent $\Delta(\Gamma) = \Delta$. As mentioned before it is trivial that this condition is sufficient. If Γ has exponent two (in which case $\Delta(\Gamma) = \Delta_2 = (-1, 1)$) or no exponent then the theorem is the result of Herz (II and III above).

That theorem 1·2 is false for exceptional groups follows from the next theorem.

THEOREM 1·3. *If Γ is exceptional there are positive integers p and q such that*

$$p \sum_{i=0}^{q-1} z^i \bar{z}^{q-1} - z^q$$

operates on $P(\Gamma)$.

To prove Theorem 1·2 we first reduce it, in section 2, to the case where Γ is a discrete group (Theorem 2·3). In section 3 it is shown that if F operates on $P(\Gamma)$ then F is continuous and absolutely monotonic on $(0, 1)$. The latter fact is the most important part of the proof and uses the methods of Rudin (3). In section 4 it is shown that F is a sum of functions F_n which operate on $P(\Gamma)$ and satisfy $F_n(sz) = s^n F_n(z)$ for $s > 0$. The F_n are characterized as certain polynomials in section 5. Finally, in sections 6 and 7, Theorems 2·3 and 1·3 are proved.

2. *Preliminaries.* The dual group of the locally compact Abelian group Γ will be denoted by G. $M(G)$ is the space of finite regular Borel measures on G. The Fourier transform of $\mu \in M(G)$ is given by

$$\hat{\mu}(\gamma) = \int_G \overline{(\gamma, x)}\, d\mu(x),$$

where $(\gamma, x) = \gamma(x)$ denotes the character $\gamma \in \Gamma$. The convolution of two measures $\mu, \nu \in \Gamma$ will be written as $\mu * \nu$; $\mu^1 = \mu$, $\mu^{n+1} = \mu^n * \mu$.

A function ϕ on Γ is positive definite if

$$\Sigma c_n \bar{c}_m \phi(\gamma_n - \gamma_m) \geqslant 0$$

for every finite choice of $\gamma_1, \gamma_2, \ldots \in \Gamma$ and complex numbers $c_1, c_2, \ldots$. We will use Bochner's characterization of $P(\Gamma)$.

THEOREM 2·1. ((4); p. 19). *A continuous function ϕ on Γ is positive definite (i.e. $\phi \in P(\Gamma)$) if and only if there is a non-negative measure $\mu \in M(G)$ such that $\hat{\mu} = \phi$.*

Because of Herz's result we will assume from now on that Γ is an infinite group with exponent e. In this case Γ contains a subgroup algebraically isomorphic to the discrete group

$$\Gamma' = Z_e \times Z_p^{\infty},$$

where Z_n is the cylic group of order n, written as the integers mod n, p divides e and Z_p^{∞} is the weak product of countably many copies of Z_p (cf. (2); p. 449). Note that if Γ is not exceptional then either $e = 2$ or p can be chosen so that $p > 2$.

LEMMA 2·2. *If F operates on $P(\Gamma)$ then F operates on $P(\Gamma')$.*

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

Proof. Let $\phi \in P(\Gamma')$ and $\gamma_1, \ldots, \gamma_N \in \Gamma'$. Consider Γ' as a subgroup of Γ and let H be the finite and hence closed subgroup generated by $\gamma_1, \ldots, \gamma_N$. By a lemma of Herz ((1); Lemma 2) F operates on $P(H)$. Since $\phi|H \in P(H)$ it follows that for every choice of complex numbers $c_1, \ldots, c_N$

$$\sum_{n,m=0}^{N} c_n \bar{c}_m F(\phi(\gamma_n - \gamma_m)) \geqslant 0.$$

Since N is arbitrary $F \circ \phi \in P(\Gamma')$.

Since $\Delta(\Gamma) = \Delta(\Gamma') = \Delta_e$ Theorem 1·2 will now follow immediately from

THEOREM 2·3. *Let $\Gamma = Z_e \times Z_p^\infty$ where p divides e and $p > 2$. If F operates on $P(\Gamma)$ then F is represented by an absolutely convergent series for $z \in \Delta_e$,*

$$F(z) = \Sigma c_{m,n} z^m \bar{z}^n \quad \textit{with} \quad c_{m,n} \geqslant 0.$$

3. *Operating functions are absolutely monotonic.* In the following we consider discrete groups Γ which are weak products

$$\Gamma = Z_e \times Z_p^\infty,$$

where p divides e. The dual group G of Γ is the compact group which is the complete product

$$G = Z_e \times \bar{Z}_p^\infty.$$

In particular Γ is a dense subgroup of G.

THEOREM 3·1. *If F operates on $P(\Gamma)$ then F is continuous on Δ_e.*

This can be shown by using the proof of Proposition 1 of (1). However, because of Bochner's theorem it is a corollary of the following stronger result. Let

$$B(\Gamma) = \{\hat{\mu} : \mu \in M(G)\}.$$

A function F defined on Δ is said to map $P(\Gamma)$ into $B(\Gamma)$ if the composition $F \circ \phi \in B(\Gamma)$ whenever $\phi \in P(\Gamma)$ and range $\phi \subset \Delta$. Recall that, for $\phi \in P(\Gamma)$, range $\phi \subset \Delta$ if and only if $\phi(0) < 1$.

THEOREM 3·2. *If F maps $P(\Gamma)$ into $B(\Gamma)$ then F is continuous on Δ_e.*

Proof. Let $\gamma_0 = (1, 0) \in \Gamma, x_i = (i, 0) \in G$ for $0 \leqslant i < e$ and $\alpha = \exp(2\pi i/e) = \gamma_0(x_i)$. Fix $z \in \Delta_e$. In order to prove F is continuous at z we can assume $F(z) = 0$. There are non-negative numbers a_i such that $\Sigma a_i < 1$ and

$$z = \sum_{i=0}^{e-1} a_i \alpha^i.$$

Let

$$\mu = \sum_{i=0}^{e-1} a_i \delta_i \in M(G),$$

where δ_i is the measure with unit mass at $-x_i$. Then μ is a positive measure with norm

$$\|\mu\| = \hat{\mu}(0) = \Sigma a_i < 1.$$

$\hat{\mu}$ is constant on the cosets of Z_p^∞ and

$$\hat{\mu}(\gamma_0) = \Sigma a_i \hat{\delta}_i(\gamma_0) = \Sigma a_i \alpha^i = z.$$

Choose $\epsilon > 0$ such that

$$\|\mu\| + 2\epsilon < 1. \tag{3·3}$$

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

Let z_n be a sequence in Δ_e converging to z so fast that

$$2\sum_1^\infty |z_n - z| < \epsilon. \tag{3·4}$$

For any sequence γ_n of distinct characters in the coset $\gamma_0 + Z_p^\infty$ let

$$\nu = \left[\epsilon + \sum_1^\infty (z_n - z)\,\gamma_n + (z_n - z)^-\overline{\gamma}_n\right] m$$

if $e > 2$ and

$$\nu = \left[\epsilon + \sum_1^\infty (z_n - z)\,\gamma_n\right] m$$

if $e = 2$ where m is the Haar measure of G. In either case by (3·4) ν is a positive measure on G with $\|\nu\| < 2\epsilon$. Thus by (3·3) $\|\mu + \nu\| < 1$ and since $\hat{\mu} = z$ on the coset $\gamma_0 + Z_p^\infty$

$$(\mu+\nu)^\wedge(\gamma) = \begin{cases} z_n & \text{if} \quad \gamma = \gamma_n, \\ z & \text{if} \quad \gamma \in \gamma_0 + Z_p^\infty, \quad \gamma \notin \{\gamma_n\}. \end{cases} \tag{3·5}$$

Since F maps $P(\Gamma)$ into $B(\Gamma)$ there is a measure $\lambda \in M(G)$ such that $\hat{\lambda} = F \circ (\mu+\nu)^\wedge$. If we take the convolution of λ with a measure whose transform is the characteristic function of the coset $\gamma_0 + Z_p^\infty$ it follows from (3·5) and the assumption that $F(z) = 0$ that there is a measure $\theta \in M(G)$ such that

$$\hat{\theta}(\gamma) = \begin{cases} F(z_n) & \text{if} \quad \gamma = \gamma_n, \\ 0 & \text{if} \quad \gamma \notin \{\gamma_n\}. \end{cases}$$

Since the sequence of $\gamma_n \in \gamma_0 + Z_p^\infty$ was completely arbitrary we must have $\lim F(z_n) = 0$ and it follows that F is continuous at z.

The following two theorems use the ideas of the proofs of Theorems II and IV of (3). We use the fact that $\Gamma \subset G$ whereas Rudin considers the integers as a subset of $[0, 2\pi)$ by mapping $n \to n\alpha$ where α/π is irrational.

THEOREM 3·6. *If F operates on $P(\Gamma)$, $\phi \in P(G)$ and $\phi(0) < 1$ then*

$$\int_G F(\phi(x))\,\gamma(x)\,dx \geqslant 0 \quad \textit{for all} \quad \gamma \in \Gamma. \tag{3·7}$$

dx is the Haar measure on G. Since $P(G)$ consists of the continuous functions on G with positive Fourier coefficients this theorem, together with Theorem 3·1, gives that if F operates on $P(\Gamma)$ then F operates on $P(G)$.

Proof of 3·6. Let Z_p^n be the product of n copies of Z_p and $G_n = Z_e \times Z_p^n \times \{0\} \subset G$. Since F and ϕ are continuous

$$\int_G F(\phi(x))\,\gamma(x)\,dx = \lim_{n\to\infty} \int_{G_n} F(\phi(x_n))\,\gamma(x_n)\,dx_n, \tag{3·8}$$

where dx_n is the normalized Haar measure of the finite group G_n. Now $G_n \subset \Gamma$ so that, by ((1); Lemma 1), F operates on $P(G_n)$. Since $\phi|G_n \subset P(G_n)$ there is a positive measure μ_n on G_n (which is its own dual) such that for all $x_n \in G_n$

$$F(\phi(x_n)) = \hat{\mu}_n(x_n) = \int_{G_n} \overline{x}_n(t)\,d\mu_n(t).$$

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

Thus by 3·8 $$\int_G F(\phi(x))\,\gamma(x)\,dx = \lim_{n\to\infty}\int_{G_n}\int_{G_n}\bar{x}_n(t)\,\gamma(x_n)\,d\mu_n(t)\,dx_n. \tag{3·9}$$

Fix γ; then $\gamma\in G_n$ eventually and when it does

$$\int_{G_n}\bar{x}_n(t)\,\gamma(x_n)\,dx_n = \begin{cases} 0 & \text{if} \quad t\neq\gamma, \\ 1 & \text{if} \quad t=\gamma. \end{cases}$$

Thus by 3·9 $$\int_G F(\phi(x))\,\gamma(x)\,dx = \lim_{n\to\infty}\mu_n\{\gamma\}$$

which is non-negative since μ_n is a positive measure.

THEOREM 3·10. *Let F be a continuous function defined on* $(0,1)$ *such that*

$$\int_G F(\phi(x))\,\gamma(x)\,dx \geqslant 0 \quad \textit{for all} \quad \gamma\in\Gamma \tag{3·11}$$

whenever $\phi\in P(G)$ and $0<\phi<1$. Then

$$F(t) = \sum_{n=0}^{\infty} c_n t^n \tag{3·12}$$

on $(0,1)$ *where $c_n\geqslant 0$.*

Proof. A function on an interval (a,b) is absolutely monotonic if it is infinitely differentiable and the derivatives are non-negative. It follows from a theorem of Widder ((5), p. 146) that an absolutely monotonic function on $(0,1)$ has the form of (3·12). Thus we need only show that F is absolutely monotonic.

If π_i is the projection of $\Gamma = Z_e\times Z_p^{\infty}$ onto the ith copy of Z_p let $\gamma_n\in\Gamma$ be such that $\pi_i(\gamma_n)=\delta_{i,n}$ and the projection of Γ onto Z_e takes γ_n to 0. For $0<b<a$ and $a+b<1$ let

$$Q(x) = \sum_1^{\infty}(\gamma_n(x)+\bar{\gamma}_n(x))/2^{n+1} \quad \text{and} \quad \phi(x) = a+bQ(x).$$

Then $\phi\in P(G)$ and for all $x\in G$

$$0<a-b\leqslant\phi(x)\leqslant a+b<1. \tag{3·13}$$

Now choose δ so that $0<\delta<1-a-b$ and let $g\in C^{\infty}(-\infty,\infty)$ be such that $g(t)>0$ on $(0,\delta)$, $g(t)=0$ elsewhere and $\int_0^{\delta} g(t)\,dt = 1$. Define

$$H(y) = \int_0^{\delta} F(y+t)\,g(t)\,dt \quad (0<y<1-\delta).$$

Then $H\in C^{\infty}$ and for $\gamma\in\Gamma$

$$\begin{aligned}\int_G H(\phi(x))\,\gamma(x)\,dx &= \int_G\int_0^{\delta} F(\phi(x)+t)\,\gamma(x)\,g(t)\,dt\,dx \\ &= \int_0^{\delta} g(t)\,dt\left\{\int_G F(\phi(x)+t)\,\gamma(x)\,dx\right\}. \end{aligned}\tag{3·14}$$

Now, for $t\in(0,\delta)$, $\phi+t\in P(G_n)$ and by (3·13)

$$0<\phi(x)+t<a+b+\delta<1.$$

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

Thus by (3·11) the portion of (3·14) in brackets is non-negative, so that

$$\int_G H(\phi(x))\,\gamma(x)\,dx \geqslant 0 \quad (\gamma \in \Gamma). \tag{3·15}$$

Fix a with $0 < a < 1-\delta$ and fix n, an integer. Applying Taylor's theorem to H gives

$$H(\phi(x)) = H(a+bQ(x)) \tag{3·16}$$

$$= \sum_{k=0}^{n} \frac{1}{k!} H_{(a)}^{(k)} b^k Q^k(x) + b^{n+1} R_n(b,x),$$

where $R_n(b,x) = \{1/(n+1)!\}\,Q^{n+1}(x)\,H^{(n+1)}(a+t)$ for some t with $|t| < |bQ(x)|$. Let $\gamma = \gamma_1\gamma_2\ldots\gamma_n$. It is easily seen that

$$\int_G Q^k(x)\,\gamma(x)\,dx = 0 \quad \text{if} \quad k < n \tag{3·17}$$

and

$$\int_G Q^n(x)\,\gamma(x)\,dx > 0. \tag{3·18}$$

Multiply 3·16 by γ and integrate over G. Equations (3·15) and (3·17) then give

$$0 \leqslant \frac{1}{n!} H^{(n)}(a)\, b^n \int_G Q^n(x)\,\gamma(x)\,dx + b^{n+1} \int_G R_n\,\gamma(x)\,dx \tag{3·19}$$

whenever $0 < b < a$ and $a+b+\delta < 1$. Divide 3·19 by b^n and let $b \to 0$. Since R_n is bounded it follows from 3·18 that $H^{(n)}(a) \geqslant 0$ whenever $0 < a < 1-\delta$. Thus H is absolutely monotonic on $(0, 1-\delta)$. Letting $\delta \to 0$ shows that F is the limit of such H and is therefore absolutely monotonic on $(0, 1)$ and the proof is complete.

4. *A series representation for F.* Since $\Gamma = Z_e \times Z_p^\infty$ we can write $\Gamma = \Gamma \times Z_p^\infty$ and $G = G \times \bar{Z}_p^\infty$. If F operates on $P(\Gamma)$ it then operates on $P(\Gamma \times Z_p^\infty)$. Let ϕ and ϕ' belong to $P(G)$ and $P(\bar{Z}_p^\infty)$ respectively; $\phi(0) < 1$ and $\phi'(0) < 1$. For $(x,y) \in G \times \bar{Z}_p^\infty$ let

$$\psi(x,y) = \phi(x)\,\phi'(y).$$

Then $\psi \in P(G \times \bar{Z}_p^\infty)$. Applying Theorem 3·6 to ψ where Γ is replaced by $\Gamma \times Z_p^\infty$ then gives

$$\int_G \int_{\bar{Z}} F(\phi(x)\,\phi'(y))\,\gamma(x)\,\gamma'(y)\,dy\,dx \geqslant 0 \tag{4·1}$$

for all $\gamma \in \Gamma$, $\gamma' \in Z_p^\infty$.

Fix ϕ; $r \in (0,1)$ and $g_r(x) = F(r\phi(x))$. g_r is a continuous function on G and has a Fourier series

$$g_r(x) \sim \sum_{\gamma \in \Gamma} a_r(\gamma)\,\gamma(x) \tag{4·2}$$

where

$$a_r(\gamma) = \int_G F(r\phi(x))\,\bar{\gamma}(x)\,dx. \tag{4·3}$$

For fixed γ, $a_r(\gamma) = a_\gamma(r)$ is a function of r. It is continuous on $(0,1)$ since F, ϕ and γ are all continuous. Let $\phi' \in P(\bar{Z}_p^\infty)$ and $\phi'(0) < 1$. Then by (4·1)

$$\int_{\bar{Z}^\infty} a_\gamma(\phi'(y))\,\gamma'(y)\,dy = \int_{\bar{Z}_p^\infty} \int_G F(\phi(x)\,\phi'(y))\,\bar{\gamma}(x)\,\gamma'(y)\,dx\,dy \geqslant 0 \quad (\gamma' \in Z_p^\infty).$$

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

We can thus apply Theorem (3·10) to the function a_γ where Γ is replaced by Z_p^∞ and obtain

$$a_\gamma(r) = \sum_{n=0}^{\infty} c_n r^n \quad (0 < r < 1), \quad \text{where} \quad c_n \geqslant 0. \tag{4·4}$$

Of course c_n depends on both γ and ϕ and is written $c_n = c_n(\gamma, \phi)$.

In particular it follows from (4·4) that $a_r(\gamma) \geqslant 0$ so that the continuous function g_r on G has non-negative Fourier coefficients and thus an absolutely convergent Fourier series:

$$F(r\phi(x)) = g_r(x) = \sum_\gamma a_r(\gamma)\,\gamma(x). \tag{4·5}$$

Combining (4·4) and (4·5) gives

$$F(r\phi(x)) = \sum_\gamma \sum_n c_n(\gamma, \phi)\, r^n\, \gamma(x), \tag{4·6}$$

where $c_n(\gamma, \phi) \geqslant 0$ and the double series converges absolutely for all $x \in G$ and $r \in (0, 1)$. (4·6) is true whenever $\phi \in P(G)$ and $\phi(0) < 1$. If $t > 0$ is such that $t\phi(0) < 1$ and ϕ is replaced by $t\phi$ in (4·6) then it is easy to see that $c_n(\gamma, t\phi) = t^n c_n(\gamma, \phi)$ and that (4·6) holds whenever $r \in (0, 1/\phi(0))$.

Given $z \in \Delta_e$ there is, just as in the proof of Theorem 3·2, $\phi \in P(G)$ and $x \in G$ such that $\phi(0) < 1$ and $\phi(x) = z$. Define

$$F_n(z) = \sum_\gamma c_n(\gamma, \phi)\,\gamma(x) \tag{4·7}$$

so that by (4·6)

$$F(rz) = \sum_n r^n F_n(z) \quad (0 < r < 1/\phi(0)). \tag{4·8}$$

F_n is well defined; for if ϕ_0 and x_0 is another pair with $\phi_0(x_0) = z$ then by (4·6)

$$F(rz) = F(r\phi_0(x_0)) = \sum_0^\infty r^n \sum_\gamma c_n(\gamma, \phi_0)\,\gamma(x_0). \tag{4·9}$$

Comparing the coefficients of r^n in (4·8) and (4·9) gives that F_n is well defined.

If r and $s \in (0, 1]$ it also follows from (4·8) that

$$F(rsz) = \sum_n r^n s^n F_n(z) = \sum_n r^n F_n(sz)$$

so that

$$F_n(sz) = s^n F_n(z).$$

We state the results of this section as

THEOREM 4·10. *If F is defined on Δ_e and operates on $P(\Gamma)$ then there are functions F_n $(n = 0, 1, 2, \ldots)$ defined on Δ_e such that*

(*a*) *F_n operates on $P(\Gamma)$.*

(*b*) *$F_n(sz) = s^n F_n(z)$ $(z \in \Delta_e; s \in (0, 1])$.*

(*c*) *$F(rz) = \sum_{n=0}^{\infty} r^n F_n(z)$, the series converging absolutely for $z \in \Delta_e, r \in (0, 1]$.*

5. *Characterization of the F_n.* In this section we consider the functions F_n of Theorem 4·10. It is convenient to assume $e > 2$, Theorem 2·3 having been proven by Herz when $e = 2$, so that Δ_e is an open planar set. If F_n operates on $P(\Gamma)$ and

$$F_n(sz) = s^n F_n(z) \tag{5·1}$$

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

for $z \in \Delta_e$ and $s \in (0, 1]$ then F_n can be extended to a function defined on the whole plane such that (5·1) still holds for all z and all $s < 0$. This is done by defining

$$F_n(z) = s^n F_n(s^{-1}z)$$

where s is so large that $s^{-1}z \in \Delta_e$. It is easy to see that the extended function operates on $P(\Gamma)$.

THEOREM 5·2. *If F_n is as above then*

$$F_n(z) = \sum_{i=0}^{\infty} a_i z^{n-i}\bar{z}^i. \tag{5·3}$$

Proof. Define $K(z) = F_n(1+z)$. Then K is defined in the whole plane and operates on $P(\Gamma)$. Applying Theorem 4·10 to K implies there are functions H_m operating on $P(\Gamma)$ such that

$$K(rz) = \sum_{m=0}^{\infty} r^m H_m(z). \tag{5·4}$$

Since K is defined on the whole plane and Theorem 4·10 has an obvious extension to regions of the form $A\Delta_e$, for $A > 0$, it follows that the series in (5·4) converges absolutely for all z and all $r > 0$.

Let $a > 0$. Then by (5·1) $\quad F_n(a+1) = (a+1)^n F_n(1).$ (5·5)

On the other hand, by (5·1) and (5·4)

$$F_n(a+1) = a^n F_n\left(1+\frac{1}{a}\right) = a^n K\left(\frac{1}{a}\right) = a^n \sum_{m=0}^{\infty} a^{-m} H_m(1). \tag{5·6}$$

Since H_m operates $P(\Gamma)$, $H_m(1) \geqslant 0$. Thus by (5·5) and (5·6) for all $a > 0$ and all m

$$(a+1)^n F_n(1) \geqslant a^{n-m} H_m(1). \tag{5·7}$$

Letting $a \to 0$ in (5·7) yields that $H_m(1) = 0$ if $m > n$. Now it follows from (5·4) that $H_m(r) = r^m H_m(1)$ for all $r > 0$. Hence $H_m = 0$ on $(0, \infty)$. Since H_m operates on $P(\Gamma)$ this implies $H_m(z) = 0$ for all z. It follows from this, (5·1) and (5·4) that if $a > 0$ and $r > 0$ then

$$F_n(a+rz) = \sum_{m=0}^{n} a^{n-m} r^m H_m(z). \tag{5·8}$$

This holds also for $a = 0$ or $r = 0$ by continuity. In particular letting $a = 0$ and $r = 1$ gives

$$F_n(z) = H_n(z). \tag{5·9}$$

Let $z = x + iy$ where $x \geqslant 0$ and $y \geqslant 0$. Then by (5·8)

$$\begin{aligned} F_n(z) = F_n(x+iy) &= \sum_{m=0}^{n} x^{n-m} y^m H_m(i) \\ &= \sum_{m=0}^{n} (z+\bar{z})^{n-m} (z-\bar{z})^m \frac{H_m(i)}{2^n i^m}. \end{aligned}$$

Expanding this gives, for some constants a_i,

$$F_n(z) = \sum_{i=0}^{n} a_i z^{n-i}\bar{z}^i \quad (\Re z \geqslant 0, \Im z \geqslant 0). \tag{5·10}$$

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

Now consider z with $\Im z \geqslant 0$. If $a + \Re z \geqslant 0$ then first 5·8 gives

$$F_n(a+z) = \sum_{m=0}^{n} a^{n-m} H_m(z) \tag{5·11}$$

and then (5·10) gives

$$F_n(a+z) = \sum_{i=0}^{n} a_i (a+z)^{n-i} (a+\bar{z})^i. \tag{5·12}$$

Holding z fixed consider $F_n(a+z)$ as a polynomial in a for $a + \Re z \geqslant 0$. Comparing the coefficient of a^0 in (5·11) and (5·12) gives, by (5·9),

$$F_n(z) = H_n(z) = \sum_{i=0}^{n} a_i z^{n-i} \bar{z}^i.$$

Thus (5·10) holds in the upper half plane.

Now since F_n operates on $P(\Gamma)$ we must have $F_n(\bar{z}) = \overline{F}_n(z)$. Thus if z is in the lower half plane

$$F_n(z) = \sum_{i=0}^{n} \bar{a}_i z^{n-i} \bar{z}^i \tag{5·13}$$

so that in order to show that (5·10) holds in the whole plane it is sufficient to show that the a_i are real. This is done as follows. Consider $H(z) = F_n(z^2)$. Then H operates on $P(\Gamma)$ and $H(sz) = s^{2n} H(z)$. Thus the hypotheses of the theorem hold for H and by what has been done

$$H(z) = \sum_{i=0}^{2n} b_i z^{2n-i} \bar{z}^i \tag{5·14}$$

for z in the upper half plane. But if z is in the first quadrant, by (5·10),

$$H(z) = F(z^2) = \Sigma a_i z^{2(n-i)} \bar{z}^{2i} \tag{5·15}$$

and if z is in the second quadrant, by (5·13),

$$H(z) = F(z^2) = \Sigma \bar{a}_i z^{2(n-i)} \bar{z}^{2i}. \tag{5·16}$$

Comparing (5·14) with (5·15) and (5·16) gives $b_{2i} = a_i$ and $b_{2i} = \bar{a}_i$. Hence the a_i are real and the proof is complete.

6. *Proof of theorem* 2·3. Because of Theorems 4·10 and 5·2 in order to show Theorem 2·3 we need only prove that $F(z) = \sum_{i=0}^{n} a_i z^{n-i} \bar{z}^i$ operating on $P(\Gamma)$ involves $a_i \geqslant 0$. Here $\Gamma = Z_e \times Z_p^\infty$ where $p > 2$. F operating on Γ implies it operates on $P(Z_p^\infty)$ as well.

Let π_i be the projection of $\bar{Z}_p^\infty$ onto the ith copy of Z_p and let x_m be such that $\pi_i(x_m) = \delta_{i,m}$ for all i. Then since $p > 2$, $x_m \neq -x_m$. Let $\mu = \sum_1^n \delta_m$ and $\nu = \sum_1^n \delta'_m$ where δ_m and δ'_m are the measures with unit mass at x_m and $-x_m$ respectively. Thus $\hat{\nu} = \hat{\mu}^-$. If $\lambda \in M(\bar{Z}_p^\infty)$ is a positive measure with $\hat{\lambda} = F \circ \hat{\mu}$ then

$$\lambda = \sum_{i=0}^{n} a_i \mu^{n-i} * \nu^i.$$

Now λ is a discrete measure and if

$$g_s = x_i + \ldots + x_{n-s} - x_{n-s-1} - \ldots - x_n$$

then

$$\mu^{n-i} * \nu^i\{g_s\} = \begin{cases} 0 & \text{if } i \neq s, \\ (n-i)!\, i! & \text{if } i = s. \end{cases}$$

Thus $\lambda\{g_i\} = a_i (n-1)!\, i!$. Since λ is a positive measure $a_i \geqslant 0$.

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

7. *Proof of Theorem* 1·3. Γ is exceptional so that $\Gamma = \Gamma_0 \times Q$ where Γ_0 is finite and Q has exponent two. The dual group G is of the form $\Gamma_0 \times R$ where R is the dual group of Q and thus also has exponent two. Let q be the order of Γ_0 and

$$F(z) = q! \sum_{i=0}^{q} z^i \bar{z}^{q-i} - z^q. \tag{7·1}$$

We will show F operates on $P(\Gamma)$. Clearly F maps $P(\Gamma)$ into $B(\Gamma)$.

If μ is a non-negative measure on G let $\nu(E) = \mu(-E)$. Then $\hat{\nu} = \hat{\mu}^-$ so that if $\lambda \in M(G)$ is such that $\hat{\lambda} = F \circ \hat{\mu}$ then

$$\lambda = q! \sum_{i=0}^{q-1} \mu^{i} * \nu^{q-i} - \mu^q. \tag{7·2}$$

If E is a Borel set in G and $\rho \in \pi$, the group of permutations of $1, \ldots, q$, we consider the sets

$$E' = \{(x_1, \ldots, x_q) : x_1 + \ldots + x_q \in E\}$$

$$E^i_\rho = \{(x_1, \ldots, x_q) : x_{\rho(1)} + \ldots + x_{\rho(i)} - x_{\rho(i+1)} - \ldots - x_{\rho(q)} \in E\}.$$

It follows as in ((4); p. 14) that

$$\mu^q(E) = \mu \times \mu \times \ldots \times \mu(E')$$

$$\mu^i * \nu^{q-i}(E) = \mu \times \mu \times \ldots \times \mu(E^i_\rho) \quad (\rho \in \pi) \tag{7·3}$$

where the measure on the right is the product measure on $G \times G \times \ldots \times G$.

Since π has $q!$ elements and $\mu \times \mu \times \ldots \times \mu$ is non-negative it follows from (7·2) and (7·3) that

$$\lambda(E) = \sum_{\rho \in \pi} \sum_{i=0}^{q-1} \mu \times \ldots \times \mu(E^i_\rho) - \mu \times \ldots \times \mu(E') \geqslant \mu \times \ldots \times \mu(A) - \mu \times \ldots \times \mu(E') \tag{7·4}$$

where $A = \bigcup_{\rho} \bigcup_{i=0}^{q-1} E^i_\rho$. λ is a non-negative measure, so that F operates, provided $E' \subset A$.

Let $(x_1, \ldots, x_q) \in E'$ so that $x_1 + \ldots + x_q \in E$. Now each x_i can be written $x_i = y_i + z_i$ where $y_i \in \Gamma_0$ and $z_i \in R$. Thus $z_i = -z_i$. Consider the q elements

$$y_1, y_1 + y_2, \ldots, y_1 + y_2 + \ldots + y_q$$

in Γ_0. Since Γ_0 has order q either two of them are the same or one is 0. In either case there are integers s, t such that $1 \leqslant s \leqslant t \leqslant q$ and

$$y_s + y_{s+1} + \ldots + y_t = 0.$$

Hence

$$x_s + \ldots + x_t = z_s + \ldots + z_t = -z_s - \ldots - z_t$$

$$= -x_s - \ldots - x_t$$

so that

$$x_1 + \ldots + x_{s-1} - x_s - \ldots - x_t + x_{t+1} + \ldots + x_q = x_1 + \ldots + x_q \in E. \tag{7·5}$$

But (7·5) says that $(x_1, \ldots, x_q) \in \bigcup_{\rho \in \pi} E^{q-t+s-1}_\rho \subset A$. Thus $E' \subset A$ so that by (7·4) λ is a non-negative measure.

Remarks. Theorems 4·10 and 5·2 being true for the infinite exceptional groups the problem of classifying the functions operating on $P(\Gamma)$ in this case is that of finding which polynomials

$$\sum_{i=0}^{n} a_i z^{n-i} \bar{z}^i \tag{7·6}$$

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press

operate. The topology of Γ is not important. That is (7·6) operates on the continuous positive definite functions on the infinite exceptional group $\Gamma_0 \times Q$ if and only if it operates on $P(\Gamma_0 \times Z_2^\infty)$.

Finally Theorem 3·2 is easily seen to be true for any infinite group Γ; i.e. if F maps $P(\Gamma)$ into $B(\Gamma)$ it is continuous on $\Delta(\Gamma)$. This leads to the problem of classifying the functions mapping $P(\Gamma)$ into $B(\Gamma)$. I conjecture they are convergent series

$$\Sigma a_{m,n} z^m \bar{z}^n \, (z \in \Delta(\Gamma)).$$

REFERENCES

(1) Herz, C. S. Fonctions opérant sur les fonctions définies-positive. *Ann. Inst. Fourier* (*Grenoble*) **13** (1963), 161–180.
(2) Hewitt, E. and Ross, K. A. *Abstract harmonic analysis.* I (Springer-Verlag; Heidelberg, 1963).
(3) Rudin, W. Positive definite sequences and absolute monotonic functions. *Duke Math. J.* **26** (1959), 617–622.
(4) Rudin, W. *Fourier analysis on groups* (Interscience; New York, 1962).
(5) Widder, D. V. *The Laplace transform* (Princeton, 1941).

https://doi.org/10.1017/S0305004100046454 Published online by Cambridge University Press