Ergod. Th. & Dynam. Sys. (2005), **25**, 793–798
doi:10.1017/S0143385704000914
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Printed in the United Kingdom

A note on horospherical points for flows with hyperbolic structure

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(*Received* 7 *May* 2004 *and accepted in final form* 3 *June* 2004)

Abstract. We define the concept of horospherical points for flows admitting a local product structure, and show that these points have a dense strong stable set. This generalizes a classical result from G. A. Hedlund, in the context of geodesic flows on negatively curved manifolds.

1. *Introduction*

We study the topological dynamics of the stable foliation of hyperbolic flows. We generalize the concept of horospherical points, first defined for geodesic flows on negatively curved manifolds, to flows admitting a local product structure. We then show how it is related to the transitivity and the minimality of the stable foliation.

Let us consider a complete surface of constant negative curvature; this surface can be seen as the quotient of the hyperbolic unit disk $\mathbf{D}$, by the action of a subgroup $\mathbf{\Gamma}$ of the isometry group of the disk. Any geodesic on the surface can be lifted to the disk, and ends on a well-defined point on the boundary of the disk $\partial\mathbf{D}$.

A point x of the boundary $\partial\mathbf{D}$ is said to be a *conical point* if there is a sequence of points in a cone based on x, and in the orbit of the origin under the action of $\mathbf{\Gamma}$, that approaches x. Let us recall that a *horosphere* based on x is a circle in the disk $\mathbf{D}$, tangent to the boundary $\partial\mathbf{D}$ at the point x. A point x of the boundary $\partial\mathbf{D}$ is said to be a *horospherical point* if there is a sequence of points in the orbit of the origin under the action of $\mathbf{\Gamma}$, which converges to x, and enters any horosphere based on x. A conical point is a horospherical point.

Hedlund showed in [**Hed36**] that a point on the unit tangent bundle of the negatively curved surface has a dense horosphere if and only if the lift of the geodesic ray starting from x ends on a horospherical point of the boundary $\partial\mathbf{D}$. This result was generalized to the setting of variable negative curvature by Eberlein [**Eb72**], and by Dal'bo [**Da01**]. If the surface is compact, this result implies that the horospheric flow is minimal: every horosphere on the surface is dense in the surface.

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In order to generalize this result, we must first interpret the concept of conical and horospherical points in terms of the dynamics only. Let X be a locally compact metric space, and ϕ_t a continuous flow on X. With a slight abuse in terminology, we say that a point $x \in X$ is of 'conical type' if its ω-limit set

$$\omega(x) = \{y \in X \mid \exists\, t_i \to \infty \text{ s.t. } \phi_{t_i}(x) \to y\}$$

is non-empty. If X is the unit tangent bundle of a negatively curved surface and ϕ_t is the geodesic flow on X, this is equivalent to the fact that the lift of the geodesic starting from x ends on a conical point of the boundary $\partial\mathbf{D}$ (see, for example, the book of Nicholls [**Ni89**, §8.3], where the points with non-empty ω-limit set are called 'conservative'). On a compact manifold, all points are of conical type.

The strong stable sets of the flow ϕ_t are defined by

$$W^{ss}(x) := \left\{ y \in X \;\middle|\; \lim_{t\to\infty} d(\phi_t(x), \phi_t(y)) = 0 \right\},$$
$$W^{ss}_{\varepsilon}(x) := \{y \in W^{ss}(x) \mid d(\phi_t(x), \phi_t(y)) \le \varepsilon \text{ for all } t \ge 0\}.$$

The strong unstable sets W^{su} and W^{su}_{ϵ} of ϕ_t are the strong stable sets of the inverse of the flow ϕ_{-t}.

On negatively curved manifolds, the set W^{ss} corresponds to the horosphere passing through the point x. We say that a point $x \in X$ is of '*horospherical type*' if the ω-limit set of its stable set

$$\omega(W^{ss}(x)) = \{y \in X \mid \exists\, t_i \to \infty, \exists\, x_i \in W^{ss}(x) \text{ s.t. } \phi_{t_i}(x_i) \to y\}$$

is non-empty. The reader should convince himself that on a negatively curved surface this amounts to saying that the lift of the geodesic starting from x ends on a horospherical point of the boundary $\partial\mathbf{D}$. Note also that a point of conical type is a point of horospherical type: $\omega(x) \subset \omega(W^{ss}(x))$. To summarize, a point is of horospherical type if the images of its strong stable set under the action of the flow have an accumulation point.

Our goal is to prove the following result.

THEOREM 1. *Let X be a metric space, ϕ_t a continuous positively and negatively transitive flow defined on X, which admits a local product structure. We suppose that the relation induced by the stable sets W^{ss} is open. Let us, moreover, assume that the periods of the periodic trajectories of the flow generate a dense subgroup of $\mathbb{R}$: $\overline{\langle l(\gamma)\rangle} = \mathbb{R}$.*

Then a point x of X is of horospherical type if and only if it has a dense strong stable set: $\overline{W^{ss}(x)} = X$.

Remarks.

(i) Note that no compactness assumption on X is needed. On the other hand, if X is compact, the theorem implies that every point has a dense strong stable set.

(ii) The hypothesis on the period of the trajectories is equivalent to the existence of a point with a dense strong stable set. This was proven by Dal'bo for geodesic flows on negatively curved manifolds [**Da01**], and generalized in [**C04**] to include the case of positively transitive flows with local product structure, satisfying the Anosov closing lemma.

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(iii) In the context of flows admitting a local product structure, there exist examples of flows which are not transitive, and still admit points with dense strong stable sets. These can be obtained by suspension of a toral automorphism.

We first recall the definition of a local product structure. Since X is not assumed to be compact, our definition is somewhat weaker than the classical definition by Smale [**Sm67**]. We then prove that periodic points have a dense strong stable set. The last part is dedicated to the proof of the theorem.

2. *Terminology*

First recall that a flow is said to be *transitive* if it admits a dense orbit. It is said to be positively (respectively negatively) transitive if there exists a point with dense forward (respectively backward) orbit: $\overline{\phi_{\mathbb{R}_+}(x)} = X$ (respectively $\overline{\phi_{\mathbb{R}_-}(x)} = X$). If X has the Baire property, positive transitivity is equivalent to negative transitivity; this is explained, for example, in [**Wa82**, §5.6]. Finally, if the flow is transitive, the space X is the closure of an orbit; so it is connected.

An equivalence relation R is said to be *open* if the saturation of any open set is open. Let us denote by $R(x)$ the equivalence class of the point x. An open equivalence relation satisfies the following property: if $x \in \overline{R(y)}$, then $R(x) \subset \overline{R(y)}$.

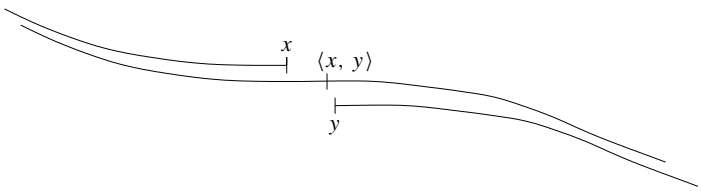
In our case, this means that the set $\bigcup_{x\in U} W^{ss}(x)$ is open if U is an open subset of X, and this implies that for all points x, y, if $x \in \overline{W^{ss}(y)}$, then $W^{ss}(x) \subset \overline{W^{ss}(y)}$.

The latter property is all that we will need to use in the proof, but the openness condition is more natural; it should be seen as a counterpart of the non-singularity assumption in the context of measurable dynamics: saturation of sets of zero measure have zero measure. The hypothesis is satisfied, for example, if the stable sets form a continuous foliation.

Finally, the flow is said to admit a *local product structure* if all points $v \in X$ have a neighborhood V which satisfies: for all $\varepsilon > 0$, there exists a positive constant δ such that, for all $x, y \in V$ with $d(x, y) \leq \delta$, there is a point $\langle x, y\rangle \in X$, and a real number t with $|t| \leq \varepsilon$, so that

$$\langle x, y\rangle \in W^{su}_{\varepsilon}(\phi_t(x)) \cap W^{ss}_{\varepsilon}(y).$$

This property is summarized by the following figure:



Examples of flows admitting a local product structure include Axiom A flows in restriction to their non-wandering sets, suspension of Markov chains, and geodesic flows on CAT(−1) spaces. Other examples are obtained by lifting these flows to a regular cover. That situation was studied in [**C04**], and motivated this work.

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3. *Stable sets of periodic points*

LEMMA 1. *Let X be a metric space, and ϕ_t a continuous positively and negatively transitive flow defined on X, which admits a local product structure. We suppose that the relation induced by the stable sets W^{ss} is open. Let us, moreover, assume that the periods of the periodic trajectories of the flow generate a dense subgroup of $\mathbb{R}$: $\overline{\langle l(\gamma)\rangle} = \mathbb{R}$.*

Then the periodic points of ϕ_t have a dense strong stable set.

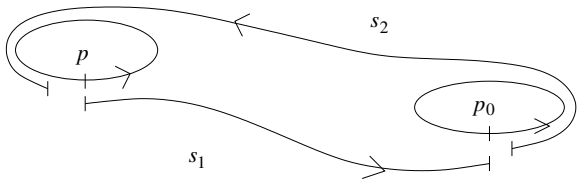
Proof. Let p be a periodic point of ϕ_t; its period is denoted by $l(p)$. We show that, for all $s \in \mathbb{R}$, $\phi_s(p)$ belongs to the closure of $W^{ss}(p)$. Indeed, since W^{ss} defines an open relation, this will imply the inclusion $\phi_s(W^{ss}(p)) \subset \overline{W^{ss}(p)}$. Now, it was shown in [**C04**] that any point $p \in X$ has a dense weak stable set: $\phi_{\mathbb{R}}(W^{ss}(p))$ is dense in X.

Let us recall the argument. First note that the points on the backward orbit of the negatively transitive point form a dense set of points with dense backward orbit. Let y be a point which belongs to $\overline{\phi_{\mathbb{R}}(W^{ss}(p))}$, and let U be a neighborhood of y given by the local product structure. The set U contains a point whose forward orbit is asymptotic to p. It also contains a point with backward dense orbit. This backward orbit can be 'glued' with the point whose forward orbit is asymptotic to p. The resulting orbit is positively asymptotic to the orbit of p, hence included in $\phi_{\mathbb{R}}(W^{ss}(p))$, and accumulates backward on every point.

We assumed that the periods of the periodic orbits generate a dense subgroup of $\mathbb{R}$. So, given some $\varepsilon > 0$, there is another periodic point p_0, and integers $m \in \mathbb{Z}$, $n \in \mathbb{N} - \{0\}$, such that $|ml(p) + nl(p_0)| < \varepsilon$.

Indeed, if there is an $\varepsilon > 0$ with $|ml(p) + nl(p_0)| \geq \varepsilon$ for all periodic points p_0 and all m, n, then the ratios $l(p)/l(p_0)$ are all rational. The number of possible irreducible denominators for these ratios is bounded by $l(p)/\varepsilon$, and hence is finite. This means that all periods belong to $(l(p)/N)\mathbb{Z}$, where N is the product of all the possible irreducible denominators.

The flow ϕ_t is positively transitive, so there exists a piece of orbit starting arbitrarily close to p and getting close to p_0 after time s_1. There is also an orbit starting very close to p_0 and coming close to p after time s_2:

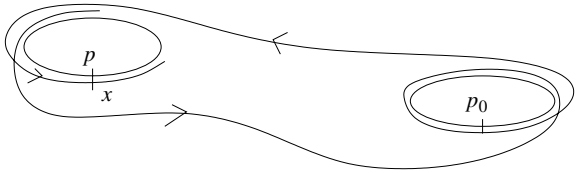


We use the local product structure to 'glue' the backward orbit of p together with the orbit going from p to p_0. We glue the resulting orbit with the periodic orbit defined by p_0. This gives an orbit negatively asymptotic to the orbit of p, which follows the piece of orbit from p to p_0, and then winds around the orbit of p_0. After this orbit has turned k times around the orbit of p_0, we glue it to the orbit going from p_0 to p. The resulting orbit is then glued to the forward orbit of p.

To summarize, we build an orbit negatively asymptotic to the orbit of p, that goes from p to p_0, then winds k times around the periodic orbit defined by p_0, then goes from p_0

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to p, and winds around the periodic orbit defined by p. Note that the neighborhoods used for the gluing of the orbits at p and p_0 do not depend on k:



In other words, for all $\varepsilon_1 > 0$, there exist $s_1,\ s_2$ such that, for all $k \in \mathbb{N}$, there is an $x \in W^{ss}_{\varepsilon_1}(p)$ and a $s_k \in \mathbb{R}$, with $|s_k - s_1 - s_2| < \varepsilon_1$, such that $\phi_{-s_k - kl(p_0)}(x) \in W^{su}_{\varepsilon_1}(p)$. Let $j \in \mathbb{Z}$ be the greatest integer for which $jl(p) < -s_k - kl(p_0)$. Then $\phi_{jl(p)}(x)$ is on $W^{ss}(p)$ and is ε_1-close to $\phi_{s_k + kl(p_0) - jl(p)}(p)$. This last point is equal to $\phi_{s_k + kl(p_0)}(p)$; it also coincides with $\phi_{s_k + kl(p_0) + m'l(p)}(p)$, for all integer m'.

So, taking for k and m' multiples of n and m, there exist points on $W^{ss}(p)$ which are very close to $\phi_{s_1 + s_2 + i(ml(p_0) + nl(p))}(p)$, for all positive integers i. But any point on the orbit of p is ε-close to one of these points. This ends the proof of the lemma. □

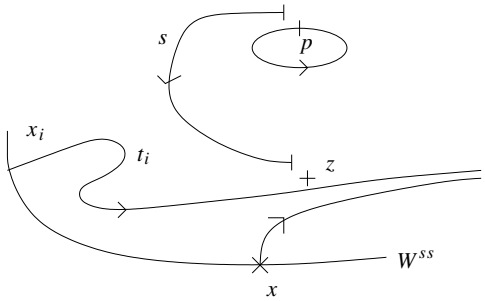
4. *Proof of the theorem*

First, the ω-limit set of the strong stable set of a point $x \in X$ can be rewritten as

$$\omega(W^{ss}(x)) = \bigcap_{t \geq 0} \overline{\phi_{[t,\infty[}(W^{ss}(x))}.$$

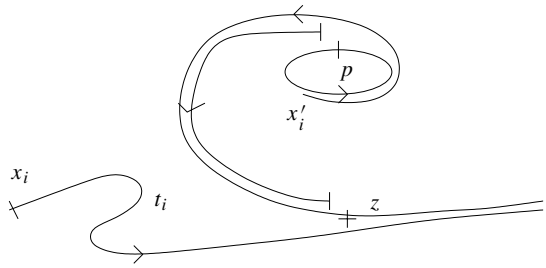
So, if the stable set of x is dense, then $\omega(W^{ss}(x)) = X$.

Let $x \in X$ be a point of horospherical type, and $z \in \omega(W^{ss}(x))$. There exist points $x_i \in W^{ss}(x)$, and a sequence $t_i \to \infty$, such that $\phi_{t_i}(x_i) \to z$ as $i \to \infty$. Let p be some periodic point; the flow is positively transitive, so there exists a piece of orbit starting arbitrarily close to p and getting close to z after some time s. Let us choose $i \in \mathbb{N}$ such that $\phi_{t_i}(x_i)$ is very close to z and t_i is greater than s:



We use the local product structure to 'glue' the backward orbit of p together with the orbit going from p to z. We glue the resulting orbit with the forward orbit of $\phi_{t_i}(x_i)$. The orbit obtained is positively asymptotic to the orbit of x_i, and hence to the orbit of x. It is also negatively asymptotic to the orbit of p:

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So there is a point x_i' on this orbit, which is on the strong stable set of x_i. Since $t_i > s$, it is on the first part of the orbit, that is, close to the orbit of p.

In other words, we have found a subsequence $x_i' \in W^{ss}(x)$ such that $d(x_i', \phi_{\mathbb{R}}(p)) \to 0$. The orbit of p is compact, so upon extracting a subsequence we can assume that the x_i' converge to some point $\phi_{t_0}(p)$ on the orbit of p. This means that $\phi_{t_0}(p)$ belongs to the closure of $W^{ss}(x)$. Since W^{ss} defines an open relation, the following inclusion holds: $W^{ss}(\phi_{t_0}(p)) \subset \overline{W^{ss}(x)}$. The point $\phi_{t_0}(p)$ is periodic, so by the lemma, its strong stable set is dense and the theorem is proven. □

Acknowledgement. We warmly thank F. Dal'bo for the discussions we had concerning horospherical points.

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