*Ergod. Th. & Dynam. Sys.* (2006), **26**, 1491–1509
doi:10.1017/S0143385706000411
© *2006 Cambridge University Press*
*Printed in the United Kingdom*

# Hyperbolic sets that are not locally maximal

TODD FISHER
*University of Maryland, College Park, MD 20742, USA*
*(e-mail: tfisher@math.umd.edu)*

(*Received* 7 *July* 2004 *and accepted in revised form* 6 *December* 2004)

*Abstract.* This paper addresses the following topics relating to the structure of hyperbolic sets: first, hyperbolic sets that are not contained in locally maximal hyperbolic sets; second, the existence of a Markov partition for a hyperbolic set. We construct new examples of hyperbolic sets which are not contained in locally maximal hyperbolic sets. The first example is robust under perturbations and can be constructed on any compact manifold of dimension greater than one. The second example is robust, topologically transitive, and constructed on a four-dimensional manifold. The third example is volume-preserving and constructed on $\mathbb{R}^4$. We show that every hyperbolic set is included in a hyperbolic set with a Markov partition. In addition, we describe a condition that ensures a hyperbolic set is included in a locally maximal hyperbolic set.

## 1. *Introduction*

A general field of study in dynamical systems is examining complicated, or 'chaotic', dynamics. A recipe for complicated dynamics is expansion plus recurrence. In the 1960s, Anosov [**2**] and Smale [**12**] began studying compact invariant sets called hyperbolic sets, whose tangent space splits into invariant uniformly contracting and uniformly expanding directions. On a compact manifold these sets often possess a great deal of recurrence and complicated dynamics. The pioneering article by Smale [**12**] states many of the standard results for hyperbolic sets.

There are many classically studied examples of hyperbolic sets, including hyperbolic toral automorphisms, solenoids, hyperbolic attractors, and horseshoes. These examples show that the structure of hyperbolic sets can be very rich. All of these classical examples possess a useful property called local maximality.

We recall the definition of locally maximal hyperbolic sets. Let $f : M \to M$ be a diffeomorphism of a compact smooth manifold $M$. A hyperbolic set $\Lambda$ is said to be *locally maximal* (or *isolated*) if there exists a neighborhood $V$ of $\Lambda$ in $M$ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$.

Locally maximal hyperbolic sets possess many useful properties; a few are as follows:

(1) locally maximal hyperbolic sets have a local product structure;

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

(2) locally maximal hyperbolic sets have a Markov partition;
(3) locally maximal hyperbolic sets with positive entropy have only finitely many invariant probability measures maximizing entropy.

The structure of locally maximal hyperbolic sets is well understood at this point.

In [**5**, p. 272], Hasselblatt and Katok pose the following question.

*Question 1.1.* Let $\Lambda$ be a hyperbolic set, and let $V$ be an open neighborhood of $\Lambda$. Does there exist a locally maximal hyperbolic set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset V$?

A related question is as follows.

*Question 1.2.* Given a hyperbolic set $\Lambda$, does there exist a neighborhood $V$ of $\Lambda$ such that $\tilde{\Lambda} = \bigcap_{n\in\mathbb{Z}} f^n(V)$ is a locally maximal hyperbolic set containing $\Lambda$?

It appears that Question 1.1 was originally asked by Alexseyev in the 1960s. These questions remained open until 2001, when Crovisier [**4**] showed there is a diffeomorphism $f$ of the 4-torus with a hyperbolic set $\Lambda$, such that $\Lambda$ is not contained in any locally maximal hyperbolic set. Crovisier's construction, based on the elegant example of Hirsch *et al* in [**6**], is delicate, and leaves open the question of whether such examples are somehow unusual. Immediate questions raised by this construction are:

(1) Does there exist an open set $\mathcal{U}$ (in the $C^1$ topology) of diffeomorphisms such that every $f \in \mathcal{U}$ possesses an invariant hyperbolic set that is not contained in a locally maximal one?
(2) Does there exist an example answering Question 1.2 in the negative on other manifolds, in a lower dimension, on all manifolds?
(3) Does there exist a topologically transitive hyperbolic set that is not contained in a locally maximal one?
(4) Does there exist a volume-preserving diffeomorphism with a hyperbolic set that is not contained in a locally maximal one?

In §3 we answer the first two of these questions affirmatively, using techniques different from Crovisier's.

THEOREM 1.3. *Let $M$ be a compact smooth manifold and $\mathcal{U}$ be the set of all diffeomorphisms $f$ of $M$ such that there exists a hyperbolic set $\Lambda$ of $f$ which is not contained in a locally maximal hyperbolic set. If* $\dim(M) \geq 2$, *then* $\mathrm{int}(\mathcal{U}) \neq \emptyset$.

The hyperbolic sets in Crovisier's example and those constructed in §3 are all contained in hyperbolic sets with a Markov partition. Sinai originated the use of Markov partitions in dynamical systems. He constructed Markov partitions for Anosov diffeomorphisms in [**11**] and [**10**]. Bowen later extended these results to the existence of a Markov partition for a locally maximal hyperbolic set [**3**]. A Markov partition allows a symbolic representation of dynamics. This raises the following question, which is a weakened version of Question 1.2.

*Question 1.4.* Let $\Lambda$ be a hyperbolic set for a diffeomorphism $f : M \to M$ and $V$ an open neighborhood of $\Lambda$. Does there exist a hyperbolic set $\tilde{\Lambda}$ with a Markov partition such that $\Lambda \subset \tilde{\Lambda} \subset V$?

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

Using a shadowing theorem due to Alekseyev *et al* [**1**] and adapting Bowen's construction [**3**], we are able to show the following.

THEOREM 1.5. *If $\Lambda$ is a hyperbolic set and $V$ is a neighborhood of $\Lambda$, then there exists a hyperbolic set $\tilde{\Lambda}$ with a Markov partition such that $\Lambda \subset \tilde{\Lambda} \subset V$.*

In §5 we use Theorems 1.3 and 1.5 to extend the results for hyperbolic sets not contained in any locally maximal hyperbolic sets. The example is four dimensional, robust, and topologically transitive. For each $y \in M$ let $\mathcal{O}^+(y) = \{f^n(y) \mid n \in \mathbb{N}\}$ and $\mathcal{O}^-(y) = \{f^{-n}(y) \mid n \in \mathbb{N}\}$ be, respectively, the *forward orbit* and *backward orbit* of $y$. A point $y \in \Lambda$ is *transitive* in $\Lambda$ if $\mathrm{cl}(\mathcal{O}^+(y)) = \Lambda$ and $\mathrm{cl}(\mathcal{O}^-(y)) = \Lambda$, and the set $\Lambda$ is *transitive* if $\Lambda$ contains a transitive point.

THEOREM 1.6. *Let $M$ be a compact smooth surface and $\mathbb{T}^2$ the flat 2-torus. On the compact manifold $M \times \mathbb{T}^2$ there exists a $C^1$ open set of diffeomorphisms, $\mathcal{U}$, such that for any $F \in \mathcal{U}$ there exists a transitive hyperbolic set $\Lambda$ for $F$ that is not contained in a locally maximal hyperbolic set.*

The proofs of Theorems 1.3 and 1.6 use diffeomorphisms that do not preserve any smooth volume form. The arguments can be modified to show the following.

THEOREM 1.7. *There exists a symplectic diffeomorphism of $\mathbb{R}^4$ that contains a hyperbolic set that is not contained in a locally maximal hyperbolic set.*

Note that in the above theorem we obtain a symplectic diffeomorphism, but with the loss of compactness for $M$ and robustness of the construction. The proof of Theorem 1.7 uses classical techniques from Hamiltonian dynamics. The idea is to embed a diffeomorphism as a subsystem of a twist map.

## 2. *Background*

We now give some useful definitions and results concerning hyperbolic sets. Let $M$ be a smooth manifold, $U \subset M$ an open set, and $f : U \to M$ a $C^1$ diffeomorphism onto its image.

*Definition.* A compact $f$-invariant set $\Lambda \subset M$ is called a *hyperbolic set* for $f$ if there is a $Df$-invariant splitting $T_\Lambda M = E^u \oplus E^s$ and positive constants $C$ and $\lambda < 1$ such that, for any point $x \in \Lambda$ and any $n \in \mathbb{N}$,

$$\|Df_x^n v\| \leq C\lambda^n \|v\| \quad \text{for } v \in E_x^s$$

and

$$\|Df_x^{-n} v\| \leq C\lambda^n \|v\| \quad \text{for } v \in E_x^u.$$

Note that it is always possible to make a smooth change of the metric near the hyperbolic set so that $C = 1$. Such a metric is called an *adapted metric*.

For $\epsilon > 0$ sufficiently small and $x \in \Lambda$, the *local stable and unstable manifolds* are respectively

$$W_\epsilon^s(x, f) = \{y \in M \mid \text{ for all } n \in \mathbb{N}, d(f^n(x), f^n(y)) \leq \epsilon\}$$

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

and

$$W^u_\epsilon(x, f) = \{y \in M \mid \text{ for all } n \in \mathbb{N}, d(f^{-n}(x), f^{-n}(y)) \leq \epsilon\}.$$

The *stable and unstable manifolds* are respectively

$$W^s(x, f) = \bigcup_{n\geq 0} f^{-n}(W^s_\epsilon(f^n(x), f))$$

and

$$W^u(x, f) = \bigcup_{n\geq 0} f^{n}(W^u_\epsilon(f^{-n}(x), f)).$$

For $\Lambda$ a hyperbolic set for a $C^r$ diffeomorphism $f$ the stable and unstable manifolds are $C^r$ injectively immersed submanifolds characterized by uniform contraction and uniform expansion, respectively.

For two points $p$ and $q$ in a hyperbolic set $\Lambda$, denote the set of points in the transverse intersection of $W^s(p)$ and $W^u(q)$ as $W^s(p) \pitchfork W^u(q)$. Two hyperbolic periodic points $p$ and $q$ contain a *transverse heteroclinic point* if $W^s(p) \pitchfork W^u(q) \neq \emptyset$.

We will often extend a hyperbolic set, $\Lambda$, by adding the orbit of a transverse heteroclinic point between two periodic points contained in $\Lambda$. Denote the orbit of a point $z$ under a map $f$ by $\mathcal{O}_f(z)$.

LEMMA 2.1. *Let $\Lambda$ be a hyperbolic set for a diffeomorphism $f$ containing periodic points $p$ and $q$ of the same index. If there exists a point $z \in W^s(p) \pitchfork W^u(q)$, then $\Lambda' = \Lambda \cup \mathcal{O}_f(z)$ is a hyperbolic set.*

*Proof.* For $x \in \Lambda$ keep the splitting of $T_xM$ to be the same for $\Lambda'$. Let $\mathbb{E}^s_{f^n(z)} = T_{f^n(z)}(W^s(p))$ and $\mathbb{E}^u_{f^n(z)} = T_{f^n(z)}(W^u(q))$. The lambda lemma [**9**, p. 203] shows the above will be a continuous splitting on $\Lambda'$. To see that the vectors in $\mathbb{E}^s_{x'}$ are uniformly contracted for any $x' \in \Lambda'$, we first start with an adapted metric for $\Lambda$. By continuity of $Df$ there is a neighborhood of $U$ of $p$ and $V$ of $q$ such that $\|Df_{f^n(z)}|\mathbb{E}^s_{f^n(z)}\| < \lambda$ and $\|Df^{-1}_{f^n(z)}|\mathbb{E}^u_{f^n(z)}\| < \lambda$ for $f^n(z) \in U \cup V$. Since there are only a finite number of $f^n(z)$ outside of $U \cup V$, it follows that there exists a $C \geq 1$ such that $\|Df_{f^n(z)}|\mathbb{E}^s_{f^n(z)}\| < C\lambda$ and $\|Df^{-1}_{f^n(z)}|\mathbb{E}^u_{f^n(z)}\| < C\lambda$ for all $n \in \mathbb{Z}$. □

A consequence of hyperbolicity is that the orbit of a point depends sensitively on the initial position. A remarkable result is that approximate periodic orbits are close to real orbits.

Let $(X, d)$ be a metric space, $U \subset M$ open, and $f : U \to M$. For $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z} \cup \{\infty\}$, a sequence $\{x_n\}_{a<n<b} \subset U$ is an $\epsilon$-pseudo orbit if $d(f(x_{n-1}), x_n) < \epsilon$ for all $a < n < b$. An $\epsilon$-pseudo orbit for $f$ is *$\delta$-shadowed* by the orbit $\mathcal{O}(x)$ of $x \in U$ if $d(x_n, f^n(x)) < \delta$ for all $a < n < b$. The following is a more general version of the shadowing theorem for hyperbolic sets.

THEOREM 2.2. (Shadowing theorem) [**1**] *Let $M$ be a Riemannian manifold, $d$ the natural distance function, $U \subset M$ open, $f : U \to M$ a diffeomorphism, and $\Lambda \subset U$ a compact hyperbolic set for $f$. Then there exist a neighborhood $U(\Lambda) \supset \Lambda$ and $\epsilon_0, \delta_0 > 0$ such that for all $\delta > 0$ there is an $\epsilon > 0$ with the following property: if $f' : U(\Lambda) \to M$ is a diffeomorphism $\epsilon_0$-close to $f$ in the $C^1$ topology, $Y$ a topological space, $g : Y \to Y$*

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

*a homeomorphism, $\alpha \in C^0(Y, U(\Lambda))$, and $d_{C^0}(\alpha g, f'\alpha) := \sup_{y\in Y} d(\alpha g(y), f'\alpha(y)) < \epsilon$ then there is a $\beta \in C^0(Y, U(\Lambda))$ such that $\beta g = f'\beta$ and $d_{C^0}(\alpha, \beta) < \delta$. Furthermore, $\beta$ is locally unique: if $\bar{\beta} g = f'\bar{\beta}$ and $d_{C^0}(\alpha, \bar{\beta}) < \delta_0$, then $\bar{\beta} = \beta$.*

To get the more standard version of the shadowing theorem take the space $Y = \mathbb{Z}$, the diffeomorphism $f' = f$, the constant $\epsilon_0 = 0$, and the function $g(n) = n + 1$. In addition, replace $\alpha \in C^0(Y, U(\Lambda))$ by $\{x_n\}_{n\in\mathbb{Z}} \subset U(\Lambda)$ and $\beta \in C^0(Y, U(\Lambda))$ such that $\beta g = f'\beta$ by $\{f^n(x)\}_{n\in\mathbb{Z}} \subset U(\Lambda)$. Then $d(x_n, f^n(x)) < \delta$ for all $n \in \mathbb{Z}$.

A consequence of the shadowing theorem is the strong structural stability of hyperbolic sets [**5**, p. 571]. This theorem states that, given a hyperbolic set $\Lambda \subset M$ of the diffeomorphism $f : U \to M$ and any open neighborhood $V \subset U$ of $\Lambda$, there exists a $C^1$ neighborhood $\mathcal{U}$ of $f$ such that each $f' \in \mathcal{U}$ contains a hyperbolic set $\Lambda' \subset V$ topologically conjugate to $\Lambda$. Moreover, the conjugacy is unique when $\mathcal{U}$ is small enough. The shadowing theorem and strong structural stability of hyperbolic sets do not require local maximality of the hyperbolic set.

A corollary of the strong structural stability of hyperbolic sets is that a hyperbolic set being locally maximal is a robust property and motivates whether a hyperbolic set may robustly not be contained in a locally maximal one. For a hyperbolic set $\Lambda$ of a diffeomorphism $f$ we will denote the continuation of $\Lambda$ for a diffeomorphism $f'$ near $f$ by $\Lambda(f')$. Similarly, for a point $p \in \Lambda$ the continuation will be denoted by $p(f')$ and the continuation of the stable and unstable manifolds of $p$, respectively, as $W^s(p, f')$ and $W^u(p, f')$.

Throughout we will use the fact that locally maximal hyperbolic sets possess a local product structure. A hyperbolic set possesses a *local product structure* provided there exist constants $\delta, \epsilon > 0$ such that if $x, x' \in \Lambda$ and $d(x, x') < \delta$, then $W^s_\epsilon(x, f)$ and $W^u_\epsilon(x', f)$ intersect in exactly one point which is contained in $\Lambda$.

PROPOSITION 2.3. [**5**, p. 581] *For a hyperbolic set, locally maximal and possessing a local product structure are equivalent conditions.*

Many of the constructions will involve hyperbolic sets with a heteroclinic tangency. A hyperbolic set $\Lambda$ for a $C^1$ diffeomorphism has a *heteroclinic tangency* if there exist points $x, y \in \Lambda$ such that $W^s(x) \cap W^u(y)$ contains a point of tangency. A point of *quadratic tangency* for a $C^2$ diffeomorphism is defined as a point of heteroclinic tangency where the curvature of the stable and unstable manifolds differs at the point of tangency.

To obtain robust properties for a manifold $M$, where $\dim(M) > 2$, we will use results from normal hyperbolicity (see [**6**] for more details). Let $f$ be a smooth diffeomorphism of a smooth compact manifold $M$ leaving a $C^1$ closed submanifold $V \subset M$ invariant.

*Definition.* A diffeomorphism $f$ is *normally hyperbolic* at $V$ if there is a continuous $Df$-invariant splitting of the tangent bundle $T_V M = N^u \oplus TV \oplus N^s$ such that

$$\inf_{p\in V} m(Df_p|_{N^u_p}) > 1, \quad \sup_{p\in V} \|DF|_{N^s_p}\| < 1,$$

$$\inf_{p\in V} m(Df_p|_{N^u_p})/\|Df_p|_{TV}\| > 1, \quad \sup_{p\in V} \|Df_p|_{N^s_p}\|/m(Df_p|_{TV}) < 1,$$

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

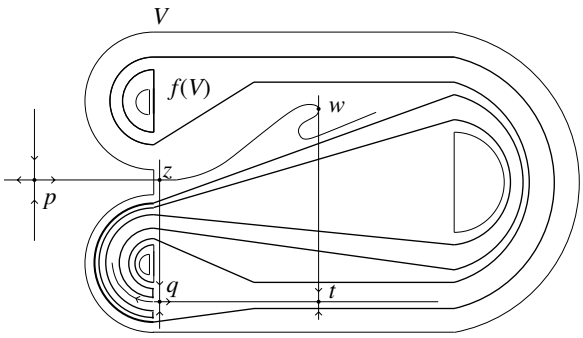



FIGURE 1. Robust example.

where $m(A) = \|A^{-1}\|^{-1}$ is the *co-norm* of an operator $A$. For a diffeomorphism the co-norm $m(Df_p) = \inf_{\|v\|=1, v\in T_pM} \|Df(v)\|$.

The above implies the normal behavior to $V$ of $Df$ is hyperbolic and dominates the behavior tangent to $V$. The diffeomorphism $f$ is *normally contracting* if $T_V M = N^s \oplus TV$. The main theorem for normal hyperbolicity states that if $f'$ is another diffeomorphism of $M$ and $f'$ is $C^1$ near $f$, then $f'$ is normally hyperbolic at some unique submanifold $V'$ which is $C^1$ near $V$.

## 3. *Hyperbolic sets that are not locally maximal*

This section concerns hyperbolic sets that are not contained in any locally maximal hyperbolic sets. Specifically, we prove Theorem 1.3.

The idea is to construct a hyperbolic set $\Lambda$ containing two points $p$ and $q$ with a quadratic tangency. We construct $\Lambda$ so that if there were a locally maximal hyperbolic set $\Lambda' \supset \Lambda$, then $\Lambda'$ would contain the point of tangency, a contradiction. We will use the following construction to prove Theorem 1.3.

Let $f : M \to M$ be a diffeomorphism of a smooth compact manifold $M$ containing a normally contracting two-dimensional compact submanifold $U$ ($U = M$ if $\dim(M) = 2$) so that $f|_U$ satisfies:

(1) an attractor $\Lambda_a$ with trapping region $V$;
(2) a fixed point $q \in \Lambda_a$;
(3) a saddle fixed point $p \notin \Lambda_a$;
(4) a point $z$ such that $z \in W^u(p) \pitchfork W^s(q)$;
(5) and there exist closed intervals $J$ of $q$ in $W^u(q)$ and $I$ of $z$ in $W^u(p)$ with the following properties:
   - each $x \in I$ is connected by a local stable manifold to a point $y \in J$,
   - the closed interval $I$ contains a point $w$ of quadratic tangency between $W^u(p)$ and $W^s(t)$, for some $t \in J$.

The above properties are illustrated in Figures 1 and 2.

PROPOSITION 3.1. *For a smooth compact manifold $M$, where $\dim(M) \geq 2$, there exists a diffeomorphism $f$ with the above properties.*

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

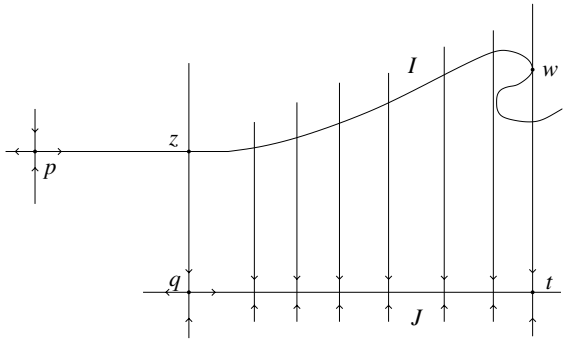



FIGURE 2. Intervals $I$ and $J$.

The proof of Proposition 3.1 is reserved for the appendix.

PROPOSITION 3.2. *There exists a neighborhood $\mathcal{U}$ of $f$ in* $\mathrm{Diff}^1(M)$ *such that every $g$ in $\mathcal{U}$ has an extension of $U$ satisfying properties (1)–(5) as above.*

*Proof.* First, for $\mathcal{U}$ sufficiently small, normal hyperbolicity implies the existence of a $C^1$ submanifold, denoted $U(g)$, that is $C^1$ close to $U$. For $\mathcal{U}$ perhaps smaller, the structural stability of hyperbolic sets implies that there exists a continuation $\Lambda_a(g)$ of the Plykin attractor and $p(g)$ of the hyperbolic saddle fixed point.

Normal hyperbolicity implies that there is a diffeomorphism $h : U(g) \to U$ with derivatives of $h$ and $h^{-1}$ uniformly bounded near the identity. Then $\hat{f} = h \circ (g|_{U(g)}) \circ h^{-1}$ is a diffeomorphism of $U$ which is $C^1$ near $f|_U$. Hence, $\hat{f}$ and $g|_{U(g)}$ are smoothly conjugate. Therefore, it is sufficient to show that properties (1)–(5) are robust under $C^1$ perturbations for perturbations of $f|_U$ in $\mathrm{Diff}^1(U)$.

The strong structural stability of hyperbolic sets implies there exists a neighborhood $\mathcal{V}$ of $f_U$ in $\mathrm{Diff}^1(U)$ such that any $\hat{f} \in \mathcal{V}$ contains a continuation $\Lambda_a(\hat{f})$ of $\Lambda_a$ and $p(\hat{f})$ of $p$.

We now show there exist continuations of $z$, $w$, $I$, and $J$ finishing the proof of the proposition. The continuity of the stable and unstable manifolds implies for $\mathcal{V}$ perhaps smaller that the points $p(\hat{f})$ and $q(\hat{f})$ will have a point $z(\hat{f}) \in W^u(p, \hat{f}) \pitchfork W^s(q, \hat{f})$ a continuation of $z$. The existence of the continuation of $w$ will follow since $w$ is a point of quadratic tangency for $p$ and $t$ under the action of $f|_U$.

To see this, we choose $C^2$ local coordinates $(x_1, x_2)$ near $w$ such that the submanifolds $W^s(t)$ and $W^u(p)$ can be expressed near $w$ by

$$W^s(t) = \{(x_1, x_2) \mid x_2 = 0\}$$

and

$$W^u(p) = \{(x_1, x_2) \mid x_2 = ax_1^2\},$$

where $a$ is a non-zero constant. Let $D$ be a small neighborhood of $t$ in $W^u(q)$ such that for each $y \in D$ the submanifolds of $W^s(y)$ in the local coordinate system can be expressed by

$$W^s(y) = \{(x_1, x_2) \mid x_2 = \psi_y(x_1)\}$$

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

where $\psi_y$ is $C^1$ close to $x_2 = 0$. Let $\hat{f}$ be a sufficiently small $C^1$ perturbation of $f|_U$. Then the continuation of the submanifolds $W^u(p, \hat{f})$ and $W^s(y, \hat{f})$ near $w$ can be expressed by

$$W^u(p, \hat{f}) = \{(x_1, x_2) \mid x_2 = \mu(x_1)\}$$

where $\mu$ is $C^1$ close to $ax_1^2$ and

$$W^s(y, \hat{f}) = \{(x_1, x_2) \mid x_2 = \psi_{y,\hat{f}}(x_1)\}$$

where $\psi_{y,\hat{f}}$ is $C^1$ near $x_2 = 0$, and $y$ is in the continuation of $D$.

Then for $x_2 > 0$ sufficiently large and some $y$ in the continuation of $D$, the stable manifold $W^s(y, \hat{f})$ crosses $W^u(p, \hat{f})$ transversally twice in the local coordinate system. Furthermore, for $x_2 < 0$ and $|x_2|$ sufficiently large, $W^s(y, \hat{f})$ does not intersect $W^u(p, \hat{f})$ in the local coordinate system for any $y$ in the continuation of $D$. It then follows that there is some $y \in U$ such that $W^s(y, \hat{f})$ and $W^u(p, \hat{f})$ are tangent, and the tangency is near $w$.

Since the stable and unstable manifolds vary continuously in $\mathcal{V}$, it follows that there exist intervals $I(\hat{f})$ and $J(\hat{f})$ in $W^u(p, \hat{f})$ and $W^u(q, \hat{f})$, respectively. □

The proof of Theorem 1.3 is a direct consequence of the following proposition.

PROPOSITION 3.3. *Let $g \in \mathcal{U}$ be as defined in Proposition 3.2. Then the set $\Lambda(g) = \Lambda_a(g) \cup \{p(g)\} \cup \mathcal{O}(z(g))$ is a hyperbolic set not contained in any locally maximal hyperbolic set.*

*Proof.* From Lemma 2.1, it is clear that $\Lambda(g)$ is a hyperbolic set. Suppose for the sake of contradiction that there exists a locally maximal hyperbolic set $\Lambda' \supset \Lambda(g)$. We will show that $w(g) \in \Lambda'$, where $w(g)$ is as in Proposition 3.1. The point $w(g)$ is a point of tangency between $W^u(p, g)$ and $W^s(t, g)$, and both $p(g)$ and $t(g)$ are contained in the hyperbolic set $\Lambda(g)$. Hence, if $w(g) \in \Lambda'$, the hyperbolic splitting fails to extend continuously to $w(g)$, a contradiction.

We now prove that $w(g) \in \Lambda'$. Since $\Lambda'$ carries a local product structure, there exist constants $\delta, \epsilon > 0$ such that for all $x, y \in \Lambda'$, if $d(x, y) < \delta$, then $W^u_\epsilon(x) \cap W^s_\epsilon(y)$ contains a single point $[x, y] \in \Lambda'$. Fix such $\delta$ and $\epsilon$.

Let $I' \subset I(g)$ be the closed subinterval whose endpoints are $z(g)$ and $w(g)$. We next show that $I' \subset \Lambda'$. From property (3) of Proposition 3.2 and compactness of $I(g)$ we have that there exists $n > 0$ such that, for all $x \in I(g)$, there is a point $y \in J(g) \cap W^s(x)$ such that $d(f^n(x), f^n(y)) < \delta/2$.

The set $f^n(I') \cap \Lambda'$ is closed since $I'$ and $\Lambda'$ are closed. On the other hand, the local product structure on $\Lambda'$ implies that $f^n(I') \cap \Lambda'$ is also open and non-empty in $f^n(I')$. Specifically, if $x \in f^n(I') \cap \Lambda'$, then there is a $y \in f^n(J(g)) \subset \Lambda'$ so that $d(x, y) < \delta/2$. Note that the unstable manifold $W^u(y)$ restricted to $U(g)$ is contained in $\Lambda_a(g) \subset \Lambda'$, since, restricted to $U(g)$, the set $\Lambda_a(g)$ is a hyperbolic attractor under the action of $g$. Now, by local product structure, if $y' \in W^u(y) \subset \Lambda'$ is sufficiently close to $y$ (within $\delta/2$, say), then the point $[x, y']$ is contained in $\Lambda'$. By varying $y'$ in a neighborhood of $y$, we thus obtain a neighborhood of $x$ in $f^n(I')$ contained in $\Lambda'$.

Since $f^n(I') \cap \Lambda'$ is non-empty, open and closed in $f^n(I')$, and $f^n(I')$ is connected, it follows from the invariance of $\Lambda'$ that $I' \subset \Lambda'$. □

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

*Remark 3.4.* The property that a hyperbolic set cannot be included in a closed locally maximal hyperbolic set is not necessarily stable under $C^1$ perturbations.

If the original tangency between $W^u(p)$ and $W^s(t)$ is topologically transverse but not of quadratic type, then under an arbitrarily small $C^1$ perturbation of $f$ there may be no point of heteroclinic tangency between a point of $W^u(q)$ and $p$. Therefore, under the perturbed function it may be possible to include $\Lambda$ in a locally maximal hyperbolic set.

For example, in the construction of $f$, suppose $M$ is the flat 2-torus and suppose instead there existed $C^3$ local coordinates $(x, y)$ near $w$ such that locally $W^s(t)$ and $W^u(p)$ can be expressed by

$$W^s(t) = \{(x, y) \mid y = 0\}$$

and

$$W^u(p) = \{(x, y) \mid y = ax^3\}$$

for some $a \neq 0$. In addition, the only point of tangency between $W^u(p)$ and stable manifolds of points in $W^u(q)$ occurs for the set $\mathcal{O}(w)$. Then $\Lambda = \Lambda_a \cup \{p\} \cup \mathcal{O}(z)$ cannot be included in a locally maximal hyperbolic set, but there exists an arbitrarily small $C^1$ perturbation of $F$ such that there is no heteroclinic tangency between a point of $W^u(q)$ and $p$. Therefore, the new perturbed hyperbolic set can be included in a locally maximal hyperbolic set, namely the separatrix of $W^u(p)$ containing $z$ and $\Lambda_a$.

## 4. *Markov partitions for hyperbolic sets*

The proof of Theorem 1.5 relies heavily on the shadowing theorem and an adaptation of Bowen's construction for Markov partitions in [**3**].

Next, we recall a few definitions that are useful in this and the following section. We use the following results from symbolic dynamics. Specifically, we will be looking at subshifts of finite type, which were introduced by Parry [**8**]. Let $A = [a_{ij}]$ be an $n \times n$ matrix with non-negative integer entries. The *graph* of $A$ is the directed graph $G_A$ with vertices $\mathcal{V}(G_A) = \{1, \ldots, n\}$ and $a_{ij}$ distinct edges with initial state $i$ and terminal state $j$. For the graph $G_A$ with edge set $\mathcal{E}$ and adjacency matrix $A$, let $\Sigma_A$ be the space over $\mathcal{E}$ specified by

$$\Sigma_A = \{\omega = (\omega_j)_{j\in\mathbb{Z}} \mid t(\omega_j) = i(\omega_{j+1}) \text{ for all } j \in \mathbb{Z}\},$$

where $t(\omega_j)$ is the terminal state of edge $\omega_j$ and $i(\omega_{j+1})$ is the initial state of edge $\omega_{j+1}$. The map on $\Sigma_A$ defined by $\sigma_A(\omega) = \omega'$, where $\omega'_j = \omega_{j+1}$, is called the *edge shift map*. The *subshift of finite type* is the space $(\Sigma_A, \sigma_A)$.

A matrix $A$ is irreducible if for each pair $1 \leq i, j \leq n$ there is a $k \in \mathbb{N}$ such that $(A^k)_{ij} > 0$. A standard result states that if $A$ is irreducible, then $\Sigma_A$ under the action of $\sigma_A$ contains a transitive point (see [**9**, p. 277]).

Define the distance between two points $\omega, \omega' \in (\Sigma_A, \sigma_A)$ by

$$d(\omega, \omega') = \sum_{k\in\mathbb{Z}} \frac{\delta(\omega_k, \omega'_k)}{4^{|k|}},$$

where $\delta(i, j)$ is the delta function. Then the local stable and unstable manifolds of a point $\omega$ are, respectively,

$$W^s_{\text{loc}}(\omega) = \{\omega' \in (\Sigma_A, \sigma_A) \mid \omega'_i = \omega_i \text{ for } i \geq 0\}$$

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

and

$$W^u_{\mathrm{loc}}(\omega) = \{\omega' \in (\Sigma_A, \sigma_A) \mid \omega'_i = \omega_i \text{ for } i \leq 0\}.$$

Roughly speaking, a Markov partition of a hyperbolic set is a 'nice' dynamical decomposition of the set and a finite-to-one semiconjugacy to a subshift of finite type. The partitions of a hyperbolic set $\Lambda$ are called rectangles.

We now make these notions precise. For a hyperbolic set $\Lambda$, the continuity of the stable and unstable distributions implies there exist constants $\epsilon \geq \alpha > 0$ such that

$$f^{-1}(W^s_\epsilon(f(p))) \cap f(W^u_\epsilon(f^{-1}(q))) = W^s_\epsilon(p) \cap W^u_\epsilon(q)$$

is a single point whenever $p, q \in \Lambda$ satisfy $d(p, q) \leq \alpha$. This point is denoted $[p, q]$.

A set $R \subset \Lambda$ is a *rectangle* provided $R$ has diameter less than $\alpha$ and $p, q \in R$ implies that $W^s_\epsilon(p) \cap W^u_\epsilon(q) \in R$ where $\epsilon$ and $\alpha$ are as above. A rectangle is *proper* if $R = \mathrm{cl}(\mathrm{int}(R))$. If $R$ is a rectangle, then there exists a natural homeomorphism

$$R \simeq W^s_\epsilon(x, R) \times W^u_\epsilon(x, R),$$

where

$$W^s_\epsilon(x, R) = R \cap W^s_\epsilon(x) \quad \text{and} \quad W^u_\epsilon(x, R) = R \cap W^u_\epsilon(x).$$

*Definition.* A *Markov partition* of $\Lambda$ is a finite collection of closed rectangles $\{R_i\}_{i=1}^M$ that satisfy the following properties:

(1) $\Lambda = \bigcup_{j=1}^m R_j$;

(2) $R_i = \mathrm{cl}(\mathrm{int}(R_i))$ for each $i$;

(3) if $i \neq j$, then $\mathrm{int}(R_j) \cap \mathrm{int}(R_i) = \emptyset$;

(4) if $z \in \mathrm{int}(R_i) \cap f^{-1}(\mathrm{int}(R_j))$, then

$$f(W^u(z, R_i)) \supset W^u(f(z), R_j)$$

and

$$f(W^s(z, R_i)) \subset W^s(f(z), R_j);$$

and

(5) if $z \in \mathrm{int}(R_i) \cap f^{-1}(\mathrm{int}(R_j))$, then

$$\mathrm{int}(R_j) \cap f(W^u(z, R_i) \cap \mathrm{int}(R_i)) = W^u(f(z), R_j) \cap \mathrm{int}(R_j)$$

and

$$\mathrm{int}(R_i) \cap f^{-1}(W^s(z, R_j) \cap \mathrm{int}(R_j)) = W^s(z, R_i) \cap \mathrm{int}(R_i).$$

The definition of a Markov partition implies that if $x, y \in R_i$, then $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ and $W^u_\epsilon(x) \cap W^s_\epsilon(y)$ are contained in $R_i$. Note that this is different than a local product structure for $\Lambda$. Specifically, if $x, y \in \Lambda$, the points $x$ and $y$ are in different rectangles, and $d(x, y) < \delta$, then $W^s_\epsilon(x) \cap W^u_\epsilon(y)$ or $W^u_\epsilon(x) \cap W^s_\epsilon(y)$ may not be contained in $\Lambda$.

*Proof of Theorem 1.5.* Let $U$ be a neighborhood of $\Lambda$ satisfying the hypothesis of the shadowing theorem. For $U$ perhaps smaller, $\bar{U} \subset V$ and $\Lambda_U = \bigcap_{n\in\mathbb{Z}} f^n(\bar{U})$ is hyperbolic. Let $d(\cdot, \cdot)$ be an adapted metric on $\Lambda_U$ and extend the metric continuously to a neighborhood $U'$ of $\Lambda_U$.

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

Fix $\eta > 0$ and $\delta \leq \eta$ such that, for any two points $x, y \in \Lambda_U$, if $d(x, y) < \delta$ then

$$f^{-1}(W^s_\eta(f(x))) \cap f(W^u_\eta(f^{-1}(y))) = W^s_\eta(x) \cap W^u_\eta(y)$$

consists of one point, and the set $\bigcup_{x\in\Lambda} B_{2\eta}(x)$ is contained in $U \cap U'$. Fix $0 < \epsilon \leq \delta/2$ as in the conclusion of the shadowing theorem so that every $\epsilon$-orbit is $\delta/2$-shadowed.

Let $\nu < \epsilon/2$ such that $d(f(x), f(y)) < \epsilon/2$ and $d(f^{-1}(x), f^{-1}(y)) < \epsilon/2$ when $d(x, y) < \nu$ for any $x, y \in \Lambda_U$. Take a $\nu$-dense set $\{p_i\}_{i=1}^N$ in $\Lambda$, and let $A$ be the transition matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } d(f(p_i), p_j) < \epsilon, \\ 0 & \text{if } d(f(p_i), p_j) \geq \epsilon. \end{cases}$$

Let $(\Sigma_A, \sigma_A)$ be the subshift of finite type determined by $A$. We see by the way we have chosen $\nu$ that every $\omega$ corresponds to a unique $\epsilon$-orbit.

For $\omega \in \Sigma_A$, let $\alpha(\omega) = p_{\omega_0}$. For $\omega$ and $\omega'$ close, then $\omega_0 = \omega'_0$, so $\alpha(\omega) = \alpha(\omega')$ and $\alpha$ is continuous. The sequence $\alpha(\sigma^n_A(\omega)) = p_{\omega_n}$ is an $\epsilon$-pseudo orbit. We now verify that, since $d(f(p_{\omega_n}), p_{\omega_{n+1}}) < \epsilon$, we have a well-defined map $\beta$ from $\Sigma_A$ into $\Lambda_U$.

The shadowing theorem shows that the $\epsilon$-orbits $\{\alpha(\sigma^n_A(\omega))\}_{n\in\mathbb{Z}} = \{p_{\omega_n}\}_{n\in\mathbb{Z}}$ are uniquely $\delta$-shadowed by $\beta(\omega) \in U$. In addition, the shadowing theorem implies that $\beta\sigma_A = f\beta$. Therefore, $f^n(\beta(\omega)) \in U$ for all $n \in \mathbb{Z}$ and it follows that $\beta(\omega) \in \Lambda_U$.

Corresponding to the product structure $[\cdot, \cdot]$ on $\Lambda_U$ as defined in the preliminary section on symbolic dynamics, there is a product structure on $\Sigma_A$ denoted also by $[\cdot, \cdot]$ and defined by

$$[\omega, \omega'] = \begin{cases} \omega_i & \text{for } i \geq 0, \\ \omega'_i & \text{for } i \leq 0, \end{cases}$$

for any $\omega, \omega' \in \Sigma_A$ with $\omega_0 = \omega'_0$. Our choice of constants $\eta$, $\delta$, $\epsilon$, and $\nu$ along with the continuity of $\beta$ implies that $[\cdot, \cdot]$ commutes with $\beta$, so $\beta([\omega, \omega']) = [\beta(\omega), \beta(\omega')]$.

Let $R_i = \{\beta(\omega) \mid \omega_0 = i\}$ be the image of the $i$ cylinder set in $\Sigma_A$. If $x = \beta(\omega)$ and $y = \beta(\omega')$ are both in $R_i$, then $\omega_0 = \omega'_0 = i$, so that

$$[x, y] = [\beta(\omega), \beta(\omega')] = \beta([\omega, \omega']) \in R_i.$$

Hence, $R_i$ is a rectangle. The continuity of $\beta$ implies that each $R_i$ is closed.

Let $\tilde{\Lambda} = \bigcup_{i=1}^N R_i = \beta(\Sigma_A)$. Since $\beta\sigma_A = f\sigma_A$ the set $\tilde{\Lambda}$ is invariant. The set $\tilde{\Lambda}$ is also hyperbolic since $\Lambda \subset \tilde{\Lambda} \subset \Lambda_U$. Finally, we refine the rectangles $R_1, \ldots, R_N$ to construct a Markov partition for $\tilde{\Lambda}$.

Define $W^s(x, R_j) = W^s_\eta(x) \cap R_j$ and $W^u(x, R_j) = W^u_\eta(x) \cap R_j$. The sets $R_i$ are each rectangles and by definition the set $\{R_i\}$ covers $\tilde{\Lambda}$. As a reminder, the local stable manifold and unstable manifold of a point $\omega \in \Sigma_A$ is defined in the preliminary section on symbolic dynamics.

CLAIM 4.1. *For any $\omega \in \Sigma_A$ the following hold:*

$$\beta(W^s_{\text{loc}}(\omega)) \subset W^s_\eta(\beta(\omega), R_{\omega_0})$$

*and*

$$\beta(W^u_{\text{loc}}(\omega)) \subset W^u_\eta(\beta(\omega), R_{\omega_0}).$$

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

*Proof.* Let $\omega' \in W^s_{\mathrm{loc}}(\omega)$. Then $\omega'_i = \omega_i$ for $i \geq 0$. Since $\omega_0 = \omega'_0$, we have that $\beta(\omega') \in R_{\omega_0}$. Also, we have that

$$d(f^n(\beta(\omega')), f^n(\beta(\omega))) \leq \epsilon \quad \text{for all } n \geq 0,$$

so that $\beta(\omega') \in W^s_\eta(\beta(\omega))$. Hence,

$$\beta(W^s_{\mathrm{loc}}(\omega)) \subset W^s_\eta(\beta(\omega)) \cap R_{\omega_0} = W^s_\eta(\beta(\omega), R_{\omega_0}).$$

A similar argument shows that $\beta(W^u_{\mathrm{loc}}(\omega)) \subset W^u_\eta(\beta(\omega), R_{\omega_0})$. □

CLAIM 4.2. *For any $x \in \tilde{\Lambda}$ the following hold:*

$$f(W^s(x, R_{\omega_0})) \subset W^s(f(x), R_{\omega_1})$$

*and*

$$f(W^u(x, R_{\omega_1})) \supset W^u(f(x), R_{\omega_0}),$$

*where $x = \beta(\omega)$ for some $\omega \in (\Sigma_A, \sigma_A)$.*

*Proof.* Let $\beta(\omega) = x \in R_i$, and let $y \in W^s(x, R_i)$. This implies that $y = \beta(\omega')$ for some $\omega' \in (\Sigma_A, \sigma_A)$ such that $w_0 = w'_0 = i$. Let $\omega^* = [\omega, \omega'] = W^s_{\mathrm{loc}}(\omega) \cap W^u_{\mathrm{loc}}(\omega')$. Hence, $\beta(\omega^*) = [\beta(\omega), \beta(\omega')] = [x, y] = y$, and $\sigma^n_A(\omega^*)_i = \omega_i$. Since $d$ is an adapted metric and $y \in W^s_\epsilon(x)$, we have that $d(f(x), f(y)) < \epsilon$. This implies that $\beta(\sigma_A(\omega^*)) = f(y) \in W^s(f(x), R_{\omega_1})$, so $f(W^s(x, R_{\omega_0})) \subset W^s(f(x), R_{\omega_1})$. A similar argument shows that $f(W^u(x, R_{\omega_1})) \supset W^u(f(x), R_{\omega_0})$. □

We now modify the rectangles $R_i$ to obtain proper rectangles satisfying the Markov conditions. Subdivide each rectangle $R_j$ for each $R_k$ that intersects $R_j$ as follows:

$$\begin{aligned}
R^1_{j,k} &= R_j \cap R_k, \\
R^2_{j,k} &= \{x \in R_j \mid W^u(x, R_j) \cap R_k \neq \emptyset \text{ and } W^s(x, R_j) \cap R_k = \emptyset\}, \\
R^3_{j,k} &= \{x \in R_j \mid W^u(x, R_j) \cap R_k = \emptyset \text{ and } W^s(x, R_j) \cap R_k \neq \emptyset\},
\end{aligned}$$

and

$$R^4_{j,k} = \{x \in R_j \mid W^u(x, R_j) \cap R_k = \emptyset \text{ and } W^s(x, R_j) \cap R_k = \emptyset\}.$$

The boundary of each rectangle can be written as $\partial(R_i) = \partial_s(R_i) \cup \partial_u(R_i)$, where

$$\partial_s(R_i) = \{x \in R_i \mid x \notin \mathrm{int}(W^u(x, R_i))\}$$

and

$$\partial_u(R_i) = \{x \in R_i \mid x \notin \mathrm{int}(W^s(x, R_i))\}.$$

Define $X = \{x \in \tilde{\Lambda} \mid W^s_\eta(x) \cap \partial_s(R_i) = \emptyset \text{ and } W^u_\eta(x) \cap \partial_u(R_i) = \emptyset\}$. It follows that $X$ is open and dense in $\tilde{\Lambda}$. For $x \in X$, define

$$R(x) = \bigcap \{\mathrm{int}(R^q_{j,k}) \mid x \in R^q_{j,k} \text{ and } R_j \cap R_k \neq \emptyset\}.$$

It then follows that for $x, y \in X$ either $R(x) = R(y)$ or $R(x) \cap R(y) = \emptyset$.

The following claims now follow from Bowen's construction (see [**9**, p. 432] for proofs). Since there are only a finite number of $R^q_{j,k}$, each $R(x)$ is non-empty and open in $\tilde{\Lambda}$.

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

Since each $R^q_{j,k}$ is a rectangle, it follows that $\mathrm{cl}(R(x))$ is a rectangle. By construction, each $\mathrm{cl}(R(x))$ is proper. The collection $\{\mathrm{cl}(R(x)) \mid x \in X\}$ forms a new partition $\{R'_i\}$ of $\tilde{\Lambda}$. Since there are only a finite number of $R^q_{j,k}$, there are only a finite number of sets $\mathrm{cl}(R(x))$. We now show these form a Markov partition of $\tilde{\Lambda}$.

The same argument as in Claims 4.1 and 4.2 shows that, for the subshift of finite type $(\Sigma_{A'}, \sigma_{A'})$ defined by $\{R'_i\}$ and $\omega \in (\Sigma_{A'}, \sigma_{A'})$, the following hold:

$$f(W^s(x, R_{\omega_0})) \subset W^s(f(x), R_{\omega_1})$$

and

$$f(W^u(x, R_{\omega_1})) \supset W^u(f(x), R_{\omega_0}).$$

Lastly, it follows from an argument as in Claims 4.1 and 4.2 that if $x \in \mathrm{int}(R'_i) \cap f^{-1}(\mathrm{int}(R'_j))$, then

$$\begin{aligned} W^s(x, R'_i) &\subset R'_i \cap f^{-1}(W^s(f(x), R'_j)) \\ &\subset R'_i \cap f^{-1}(W^s_\eta(f(x))) = W^s(x, R'_i). \end{aligned}$$

The above is a stronger version of property (5) for a Markov partition. Hence, $\tilde{\Lambda}$ has a Markov partition. □

4.1. *Condition to guarantee local maximality.* In this subsection a condition is given guaranteeing that a hyperbolic set $\Lambda$ is contained in a locally maximal hyperbolic set.

COROLLARY 4.3. *If $\Lambda$ is a hyperbolic set and in the shadowing theorem it is possible, for $\delta$ sufficiently small, to chose $\delta = \epsilon$ such that the set $\bigcap_{n\in\mathbb{Z}} f^n\big(\overline{\bigcup_{x\in\Lambda} B_\epsilon(x)}\big)$ is hyperbolic, then $\Lambda$ is contained in a locally maximal hyperbolic set.*

*Proof.* Take $\{p_i\}_{i=1}^N$ to be a sufficiently dense set as in the construction used to prove Theorem 1.5. Let $\tilde{\Lambda}$ be the hyperbolic set with a Markov partition constructed as in the proof of Theorem 1.5. Each $x \in \tilde{\Lambda}$ is a $\delta$-shadowing of an $\epsilon$-pseudo orbit given by elements of $\{p_i\}_{i=1}^N$. Hence, we have

$$x \in \bigcap_{n\in\mathbb{Z}} f^n\Big(\bigcup_{i=1}^N B_\delta(p_i)\Big) = \bigcap_{n\in\mathbb{Z}} f^n\Big(\bigcup_{i=1}^N B_\epsilon(p_i)\Big),$$

since $\delta = \epsilon$. This implies that

$$\tilde{\Lambda} \in \bigcap_{n\in\mathbb{Z}} f^n\Big(\bigcup_{i=1}^N B_\epsilon(p_i)\Big).$$

Next, fix

$$x \in \bigcap_{n\in\mathbb{Z}} f^n\Big(\bigcup_{i=1}^N B_\epsilon(p_i)\Big).$$

Then $x$ has an $\epsilon$-pseudo orbit given by the $p_i$, so $x$ is in the set $\tilde{\Lambda}$. Therefore,

$$\tilde{\Lambda} = \bigcap_{n\in\mathbb{Z}} f^n\Big(\bigcup_{i=1}^N B_\epsilon(p_i)\Big),$$

so $\tilde{\Lambda}$ is locally maximal. □

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

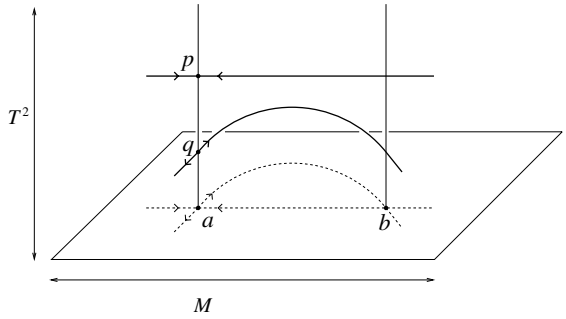



FIGURE 3. Definition of $F_0$.

5. *Further results of hyperbolic sets that are not locally maximal*

In this section we prove Theorems 1.6 and 1.7. In each proof the idea is to extend the diffeomorphism $f_0$ from §3 to higher dimensions. The proof of Theorem 1.6 will use consequences from normal hyperbolicity and Theorem 1.5.

5.1. *Proof of Theorem 1.6.* Let $M$ be a smooth compact surface, and let $\mathbb{T}^2$ denote the flat 2-torus. Fix $F_0 \in \mathrm{Diff}^1(M \times \mathbb{T}^2)$ of the form $F_0(x, y) = (g(x), f(y))$ and satisfying the following properties (see Figure 3):

- $g$ has a hyperbolic saddle fixed point, $a$, with a transverse homoclinic point $b$;
- the submanifold $\{a\} \times \mathbb{T}^2$ is normally hyperbolic under $F_0$;
- $f$ contains a hyperbolic set $\Lambda_0$, a hyperbolic attractor $\Lambda_a$ with fixed point $q$, a saddle fixed point $p \notin \Lambda_a$, and a point $z \in W^s(q) \pitchfork W^u(p)$. Furthermore, $f$ satisfies the conclusions of Theorem 1.3 as in §3.

The stable and unstable manifolds are two dimensional and have components inside and outside of the fiber $\{a\} \times \mathbb{T}^2$. Figure 3 depicts their components transverse to the fiber. To construct $F$ satisfying the conclusions of Theorem 1.6, we will deform $F_0$ such that $W^s(a, p)$ and $W^u(a, q)$ intersect transversally in the fiber $\{b\} \times \mathbb{T}^2$. Specifically, the deformation will be supported in a small neighborhood of $g(b)$. Near $g(b)$ the diffeomorphisms $F$ and $F_0$ will differ by a translation in the fiber direction.

Denote the orbit of a point $x \in M$ under $g$ by $\mathcal{O}_g(x)$. Choose a small neighborhood $U$ of $g(b)$ such that

$$\mathcal{O}_g(b) - g(b) \subset M - U.$$

Fix $\hat{U}$, a small neighborhood of $g(b)$ contained in $U$, and $\gamma : \mathbb{T}^2 \to \mathbb{T}^2$ a translation sending $q$ to $p$ . Let $G$ be a diffeomorphism of $M \times \mathbb{T}^2$ given by $G = (\mathrm{Id}, \gamma)$. By possibly perturbing $f$ near $p$ we obtain that $W^u(a, q)$ and $G(W^s(a, p))$ are transverse at $(b, q)$.

Let $F \in \mathrm{Diff}^1(M \times \mathbb{T}^2)$ be any diffeomorphism satisfying

$$F(x, y) = F_0(x, y) \quad \text{for all } x \in M - U$$

and

$$F(x, y) = (g, \gamma \circ f) \quad \text{for all } x \in \hat{U},$$

see Figure 4.

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

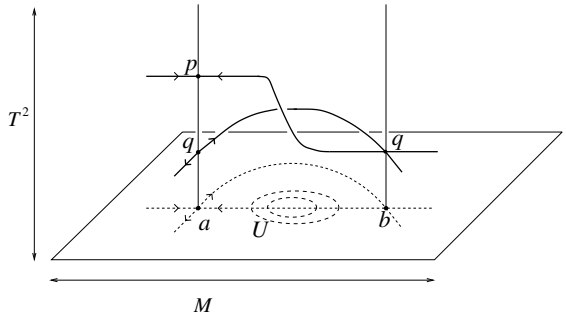



FIGURE 4. Definition of $F$.

By construction of $F$ the fixed points $(a, p)$ and $(a, q)$ are heteroclinically related with

$$(a, z) \in W^u((a, p), F) \pitchfork W^s((a, q), F) \quad \text{and} \quad (b, q) \in W^s((a, p), F) \pitchfork W^u((a, q), F).$$

By Lemma 2.1, the set

$$\Lambda_1 = (a, p) \cup (a, \Lambda_a) \cup \mathcal{O}(a, z) \cup \mathcal{O}(b, q)$$

is a hyperbolic set for $F$.

LEMMA 5.1. *The hyperbolic set $\Lambda_1$ is included in a transitive hyperbolic set $\Lambda$.*

*Proof.* It suffices to show that $\Lambda_1$ is included in a hyperbolic set with a Markov partition as constructed in Theorem 1.5 such that the matrix $A$ is irreducible, so that $(\Sigma_A, \sigma_A)$ is transitive.

The beginning of the proof of Theorem 1.5 gives the following claim.

CLAIM 5.2. *If $\Lambda$ is a hyperbolic set, then there exist a neighborhood $U$ of $\Lambda$ and positive constants $\delta$, $\epsilon$, and $\nu$ such that, corresponding to any $\nu$ dense set of points $\{p_i\} \subset \Lambda$, there is a shift space $(\Sigma_A, \sigma_A)$ and maps $\alpha$ and $\beta$ satisfying the following:*

(1) *for any $\omega \in \Sigma_A$ the set $\{\alpha(\sigma_A^n(\omega))\}$ is an $\epsilon$-pseudo orbit;*
(2) *the set $\beta(\Sigma_A)$ is a hyperbolic set with a Markov partition;*
(3) *the set $\beta(\Sigma_A)$ contains $\Lambda$; and*
(4) *any $x \in \beta(\Sigma_A)$ is a $\delta$-shadowing of an $\epsilon$-pseudo orbit given by some $\omega$.*

We now fix a set $\{p_i\}_{i=1}^n$ that is $\nu$ dense such that there exist constants $n_1 < n_2$ satisfying the following:

- $p_1, \ldots, p_{n_1} \in \{a\} \times \Lambda_a$;
- $p_{n_1+1}, \ldots, p_{n_2} \in \mathcal{O}(a, z)$;
- $p_{n_2+1} = (a, p)$;
- and $p_{n_2+2}, \ldots, p_n \in \mathcal{O}(b, q)$.

Since $\Lambda_a$ is mixing, it follows from the above construction that for any $p_i$ and $p_j$ in $\Lambda_a$ it is possible to construct an $\epsilon$-pseudo orbit from $p_i$ to $p_j$. If $p_i \in \mathcal{O}(b, q)$ and $p_j \in \Lambda_a$, it is possible to construct an $\epsilon$-pseudo orbit from $p_i$ to $p_j$ as follows. First, follow $p_i$ to $p_n$. Since the set $\{p_i\}$ is $\nu$ dense, there is a point $p_k \in \Lambda_a$ such that $d(f(p_n), p_k) < \epsilon$.

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

Then, there is an $\epsilon$-pseudo orbit from $p_k$ to $p_j$. Similarly, for $p_i \in \Lambda_a$ and $p_j \in \mathcal{O}(a, z)$ we can construct an $\epsilon$-pseudo orbit from $p_i$ to $p_j$. Lastly, for $p_i \in \mathcal{O}(a, z)$ and $p_j \in \mathcal{O}(b, q)$ it is possible to construct an $\epsilon$-pseudo orbit by simply following $p_i$ to $p_j$. Therefore, the transition matrix $A$ is irreducible.

For the map $\beta$ as given by the above claim, the following diagram commutes:

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma_A} & \Sigma_A \\ {\scriptstyle\beta}\downarrow & & {\scriptstyle\beta}\downarrow \\ \Lambda & \xrightarrow[F]{} & \Lambda \end{array}$$

We now find a transitive point in $\Lambda$. It will follow from surjectivity and continuity of $\beta$. We will show that there exists a $y \in \Lambda$ such that $\mathrm{cl}(\mathcal{O}^+(y)) = \mathrm{cl}(\mathcal{O}^-(y)) = \Lambda$. The shadowing theorem shows that, for any $\omega \in \Sigma_A$, we have $\beta(\mathcal{O}^+_{\sigma_A}(\omega)) = \mathcal{O}^+_F(\beta(\omega))$ and $\beta(\overline{\mathcal{O}^+_{\sigma_A}(\omega)}) \subseteq \overline{\mathcal{O}^+_F(\beta(\omega))}$. Let $y = \beta(\omega')$ where $\omega'$ is transitive in $\Sigma_A$ under $\sigma_A$. Then,

$$\Lambda = \beta(\Sigma_A) = \beta(\overline{\mathcal{O}^+_{\sigma_A}(\omega')}) \subset \overline{\mathcal{O}^+_F(\beta(\omega'))} = \overline{\mathcal{O}^+_F(y)} \subseteq \Lambda.$$

Hence, $\overline{\mathcal{O}^+_F(y)} = \Lambda$ and similarly $\overline{\mathcal{O}^-_F(y)} = \Lambda$. Therefore, $y$ is transitive in $\Lambda$. □

*Proof of Theorem 1.6.* Using normal hyperbolicity we show that the above construction is robust. Structural stability of hyperbolic sets implies that there exists a neighborhood $\mathcal{U}$ of $F$ in $\mathrm{Diff}^1(M)$ such that, for any $G \in \mathcal{U}$, there exists a hyperbolic set $\Lambda(G)$ conjugate to $\Lambda$. Furthermore, for $\mathcal{U}$ perhaps smaller the conjugacy is unique and near the identity.

Normal hyperbolicity implies that for $\mathcal{U}$ perhaps smaller there is a unique invariant manifold $V(G)$ which is $C^1$ near $V$ and a diffeomorphism $h : V(G) \to V$ such that the derivatives of $h$ and of $h^{-1}$ are uniformly bounded and near the identity. Then $\hat{f} = h \circ (G|_{V(G)}) \circ h^{-1}$ is a diffeomorphism of $V$ which is $C^1$ near $F|_V$. Let $\mathcal{V}$ be a neighborhood of $F|_V$ satisfying Theorem 1.3. For $\mathcal{U}$ perhaps smaller, $\hat{f}$ will be in $\mathcal{V}$. Hence, $G|_{V(G)}$ is smoothly conjugate to $\hat{f}$ and contains a hyperbolic set as constructed in the proof of Theorem 1.3.

Suppose $\Lambda$ is contained in a locally maximal hyperbolic set $\tilde{\Lambda}$. Then there exists a neighborhood $U$ of $\Lambda$ such that $\bigcap_{n\in\mathbb{Z}} G^n(U) = \tilde{\Lambda}$, and $V(G) \cap U$ is a neighborhood of $\Lambda \cap V(G)$ in $V(G)$. By the invariance of the normally hyperbolic fiber it follows that $\bigcap_{n\in\mathbb{Z}} G^n(U \cap V(G)) = \tilde{\Lambda} \cap V(G)$. Hence, under the action of $G|_{V(G)}$ the hyperbolic set $\Lambda \cap V(G)$ is contained in a locally maximal hyperbolic set, a contradiction. □

5.2. *Proof of Theorem 1.7.* In this subsection we show that we can strengthen the previous result to a symplectic diffeomorphism, but with the loss of compactness for $M$ and robustness of the construction.

The proof of Theorem 1.7 uses classical techniques from Hamiltonian dynamics. The idea is to embed a diffeomorphism as a subsystem of a twist map.

As the proof uses classical results for symplectic diffeomorphisms we review some basic concepts (see [**7**, pp. 97–98] for more details). Equip $\mathbb{R}^{2n}$ with the standard symplectic form, $\omega = dx \wedge dy = \sum_{i=1}^n dx_i \wedge dy_i$. Let $F$ be a symplectic diffeomorphism

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

of $\mathbb{R}^{2n}$ and denote $F(x, y) = (X, Y)$. Since $F$ is symplectic and $\mathbb{R}^{2n}$ is simply connected, the closed 1-form $\theta = Y\,dX - y\,dx$ is exact. Hence, there is a function $L(x, X)$ such that $dL = Y\,dX - y\,dx$. Since $dL = (\partial L/\partial X)\,dX + (\partial L/\partial x)\,dx$, it follows that

$$Y = \frac{\partial L}{\partial X} \quad \text{and} \quad -y = \frac{\partial L}{\partial x}. \tag{5.1}$$

The symplectic diffeomorphism $F$ is called a twist map if equation (5.1) implicitly defines a unique solution for all $(x, X)$. For $F$ symplectic, the Hessian of $L$ must be non-singular. Hence, $X$ and $Y$ can be solved in terms of $x$ and $y$.

The following technical lemma is the heart of proving Theorem 1.7. The lemma extends a diffeomorphism $f$ of $\mathbb{R}^n$ to a diffeomorphism $F$ of $\mathbb{R}^{2n}$ so that $F$ is volume preserving, the set $\mathbb{R}^n \times \{0\}$ is invariant, and a hyperbolic set $\Lambda$ of $f$ is a hyperbolic set $\Lambda \times \{0\}$ of $F$.

LEMMA 5.3. *Let $f$ be a diffeomorphism of $\mathbb{R}^n$ that equals the identity outside of a ball of finite radius containing a hyperbolic set $\Lambda$. Then $F$ as defined above is a symplectic diffeomorphism of $\mathbb{R}^{2n}$ such that $F(x, 0) = (f(x), 0)$ and the set $\Lambda \times \{0\}$ is a hyperbolic set for $F$.*

*Proof.* Define the function $L : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$L(x, X) = \tfrac{1}{2}\|X - f(x)\|^2.$$

Furthermore, define

$$Y = \frac{\partial L}{\partial X} = (X - f(x))^{\mathrm{T}}$$

and

$$-y = \frac{\partial L}{\partial x} = -(X - f(x))^{\mathrm{T}} Df_x,$$

where $x$ and $X$ are thought of as column vectors and $y$ and $Y$ are thought of as row vectors. Then $y \cdot Df_x^{-1} = X^{\mathrm{T}} - f(x)^{\mathrm{T}}$. Solving the above for $X$ and $Y$ yields

$$X = f(x) + (y \cdot Df_x^{-1})^{\mathrm{T}} \quad \text{and} \quad Y = y \cdot Df_x^{-1}. \tag{5.2}$$

Define $F(x, y) = (X(x, y), Y(x, y))$; then equation (5.2) implies that $F$ is a symplectic diffeomorphism preserving the standard volume form. Furthermore, $F(x, 0) = (f(x), 0)$ and

$$DF(x, 0) = \begin{bmatrix} Df_x & * \\ 0 & Df_x^{-1} \end{bmatrix}.$$

Hence, $\Lambda \times \{0\}$ is a hyperbolic invariant set for $F$. □

*Proof of Theorem 1.7.* First let $f_0$ be a diffeomorphism of the two-dimensional disk $D$ such that $f_0$ contains a hyperbolic set $\Lambda$ as in §3. Let $f$ be a diffeomorphism of $\mathbb{R}^2$ such that $f|_D = f_0$ and, for some $R > 0$ sufficiently large, $f|_{\mathbb{R}^2 - B(R,0)} = \mathrm{Id}$.

Lemma 5.3 then implies that $f$ can be extended to a symplectic diffeomorphism $F$, such that $\Lambda \times \{0\}$ is a hyperbolic set for $F$. Furthermore, we have $F|_{\mathbb{R}^2 \times \{0\}} = f$. It follows from the same argument used in Proposition 3.3 that $\Lambda \times \{0\}$ cannot be contained in a locally maximal hyperbolic set. □

*Remark 5.4.* This example may not be robust, since the submanifold is not normally hyperbolic.

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

A. *Appendix. Proof of Proposition 3.1*

In this appendix we construct a diffeomorphism satisfying the conclusions of Proposition 3.1.

We start with a Plykin attractor $\Lambda_a$ for a diffeomorphism $f$ of $S^2$ as constructed in [**5**, p. 540]. The diffeomorphism contains a hyperbolic repelling fixed point $p_0$. The set $W^s(\Lambda_a)$ consists of $S^2$ minus four hyperbolic repelling periodic points. Fix a periodic point $q$ contained in $\Lambda_a$. Since $\Lambda_a$ is a locally maximal topologically transitive hyperbolic set with periodic points dense and the closure of a heteroclinic class, we have that $\overline{\bigcup W^s(\mathcal{O}(q))} = W^s(\Lambda_a)$. Hence, there exist points $q_0 \in \mathcal{O}(q)$ and $z \in W^s(q_0)$ such that $p_0 \in \alpha(z, f)$.

Now puncture the sphere at $p_0$ and replace $p_0$ with a closed circle obtaining a closed disk $D_0$. The induced homeomorphism from $f$ on $D$ is not a diffeomorphism, but can be deformed near $\partial D$ to obtain a diffeomorphism $\tilde{f}$ of $D_0$ such that the following hold:

(1) $\tilde{f}|_{\partial D} = \mathrm{Id}$;
(2) $W^s(\Lambda_a) = D \backslash \partial D$; and
(3) there exists a periodic point $q_0 \in \Lambda_a$, a fixed point $p \in \partial D_0$, and a point $z \in W^s(q_0)$ such that $p \in \alpha(z, \tilde{f})$.

By construction, the derivative of $\tilde{f}$ at $p$ in the direction tangent to $\partial D_0$ is 1. Also, the derivative of $p$ in the direction tangent to $W^s(y)$ is greater then or equal to 1. Hence, we can perturb $\tilde{f}$ in a neighborhood of $p$, such that $p$ is a hyperbolic saddle fixed point with $\mathbb{E}^s$ tangent to $\partial D_0$ and $\mathbb{E}^u$ perpendicular to $\partial D$. Fix $n \in \mathbb{N}$ such that $q_0$ is fixed under $\tilde{f}^n$.

Next embed $D_0$ as a disk in the interior of a closed disk $D$ and extend $\tilde{f}^n$ to a diffeomorphism $f_0$ of $D$ such that, within a neighborhood of the boundary of $D$, the diffeomorphism $f_0$ is the identity. By construction, $z \in W^u(p) \cap W^s(q_0)$. Note that a perturbation near the heteroclinic point $f_0^{-1}(z)$ does not change $W^s(q_0)$ near $z$. Thus, by possibly perturbing $f_0$ in a sufficiently small neighborhood of $f_0^{-1}(z)$, we may determine that $z \in W^u(p) \pitchfork W^s(q_0)$.

The transversality of $W^u(p)$ and $W^s(q_0)$ at $z$ now imply the existence of a neighborhood $J_0$ of $q_0$ in $W^u_{\mathrm{loc}}(q_0)$ and a neighborhood $I_0$ of $z$ in $W^u(p)$ such that each $x \in I_0$ is connected by a local stable manifold to a point $y \in J_0$. Let $z' \in I_0 - z$. Then by deforming $f_0$ in a sufficiently small neighborhood of $f_0^{-1}(z')$, we create a point $w \in I_0$ of quadratic tangency between $W^u(p)$ and $W^s(t)$, for some $t \in J_0$. Let $I$ be the segment of $W^u(p)$ from $z$ to $w$, and let $J$ be the segment of $W^u(q_0)$ from $q_0$ to $t$.

It is now simple to construct a diffeomorphism $f$ satisfying the conclusions of Proposition 3.1, when $M$ is a compact boundaryless surface. We just take a sufficiently small coordinate chart $\varphi : D \to M$ such that the diffeomorphism $\varphi \circ f_0 \circ \varphi^{-1}$ satisfies the conclusions of Proposition 3.1, and extend $\varphi \circ f_0 \circ \varphi^{-1}$ trivially to a diffeomorphism $f$ of $M$.

Suppose that $\dim(M) = n \geq 3$. Let $f_1 : S^2 \to S^2$ be a diffeomorphism satisfying the conclusions of Proposition 3.1. We extend $f_1$ to a diffeomorphism $f_2$ of an $n$-dimensional open disk, $D_n$, such that the following hold:

(1) in a neighborhood of the boundary $\partial D_n$ the diffeomorphism $f_2$ is the identity;
(2) $S^2$ is a normally contracting invariant submanifold under $f_2$; and
(3) $f_2|_{S^2} = f_1$.

https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press

Now take a sufficiently small coordinate chart $\varphi : D_n \to M$ such that $\varphi \circ f_1 \circ \varphi^{-1}$ satisfies the conclusions of Proposition 3.1. Lastly, we extend $\varphi \circ f_1 \circ \varphi^{-1}$ trivially to all of $M$, completing the proof. □

## REFERENCES


**[1]** V. M. Alekseyev, A. Katok and A. G. Kusnirenko. *Three Papers in Dynamical Systems (Trans. Amer. Math. Soc., Series 2, 116)*. American Mathematical Society, Providence, RI, 1981.

**[2]** D. V. Anosov. *Geodesic Flows on Riemannian Manifolds with Negative Curvature (Proc. Steklov Inst. Math., 90)*. American Mathematical Society, Providence, RI, 1967.

**[3]** R. Bowen. *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms (Lecture Notes in Mathematics, 470)*. Springer, Berlin, 1975.

**[4]** S. Crovisier. Une remarque sur les ensembles hyperboliques localement maximaux. *C. R. Math. Acad. Sci. Paris* **334**(5) (2001), 401–404.

**[5]** B. Hasselblatt and A. Katok. *Introduction to the Modern Theory of Dynamical Systems*. Cambridge University Press, 1995.

**[6]** M. Hirsch, C. Pugh and M. Shub. *Invariant Manifolds (Lecture Notes in Mathematics, 583)*. Springer, Berlin, 1977.

**[7]** K. R. Meyer and G. R. Hall. *Introduction to Hamiltonian Dynamical Systems and the N-body Problem*. Springer, New York, 1992.

**[8]** W. Parry. Intrinsic Markov chains. *Trans. Amer. Math. Soc.* **112** (1964), 55–56.

**[9]** C. Robinson. *Dynamical Systems Stability, Symbolic Dynamics, and Chaos*. CRC Press, 1999.

**[10]** Ya. G. Sinai. Construction of Markov partitions. *Funct. Anal. Appl.* **2** (1968), 245–253.

**[11]** Ya. G. Sinai. Markov partitions and c-diffeomorphisms. *Funct. Anal. Appl.* **2** (1968), 61–82.

**[12]** S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.* **73** (1967), 747–817.


https://doi.org/10.1017/S0143385706000411 Published online by Cambridge University Press