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# Zeta functions for certain non-hyperbolic systems and topological Markov approximations

MICHIKO YURI

*Department of Business Administration, Sapporo University, Nishioka, Toyohira-ku, Sapporo 062, Japan*

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*Abstract.* We study dynamical (Artin–Mazur–Ruelle) zeta functions for piecewise invertible multi-dimensional maps. In particular, we direct our attention to non-hyperbolic systems admitting countable generating definite partitions which are not necessarily Markov but satisfy the *finite range structure* (FRS) condition. We define a version of Gibbs measure (*weak Gibbs measure*) and by using it we establish an analogy with thermodynamic formalism for specific cases, i.e. a characterization of the radius of convergence in terms of pressure. The FRS condition leads us to nice countable state symbolic dynamics and allows us to realize it as towers over Markov systems. The Markov approximation method then gives a product formula of zeta functions for certain weighted functions.

## 1. *Introduction*

The main purpose of this paper is to study Artin–Mazur–Ruelle zeta functions for piecewise invertible multi-dimensional maps in non-Fredholm situations. More specifically, the dynamical systems to which we direct our attention typically admit non-hyperbolic periodic orbits and their countable definite partitions are not necessarily Markov. The standard method based on a (generalized) Perron–Frobenius operator severely rely on Renyi's condition (see the definition in §5) [**1, 9–11, 14–16**]. On the other hand, it is easy to find examples for which the condition fails so that the operator is not compact. Such situations naturally happen in a number theoretical context and we cannot expect good spectral properties of the Perron–Frobenius operators for such systems. In previous papers [**5, 23–28**], we succeeded in establishing several ergodic properties for certain non-hyperbolic systems. The non-standard method we used in the previous papers requires neither Renyi's condition nor the Markov condition, but severely relies on the *local Renyi condition* and the *finite range structure* (FRS) condition (see §§5 and 2). These conditions allow us to have absolutely continuous $\sigma$-finite invariant measures with unbounded densities [**5, 23**]. The singularities of the invariant densities

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typically happen on non-hyperbolic periodic orbits (see [**24**] and [**25**]). We also assumed several conditions for establishing a variational principle and convergence to equilibrium states in [**25**] and [**26**]. Some of the conditions also contribute to a characterization of the radius of convergence of zeta functions in terms of pressures and an analogy with the thermodynamic formalism, for specific cases. These problems are discussed in §5. Further we ask:

*Question.* How can we obtain analyticity properties of dynamical zeta functions in non-Fredholm situations?

Our approach, which makes a small contribution to answering the question uses the symbolic dynamics. More precisely, the FRS condition allows us to have countable state sofic shifts as their symbolic dynamics, i.e. labelling graphs with finitely many vertices and countably many edges describe the motion of the dynamical systems [**25**]. Then we have a nice collection of cylinders so that the sofic systems can be described by towers over Markov systems. A product formula for the zeta functions will be obtained in the following way.

(1) Construct the jump transformations over the set of Markov cylinders (see the definition in §2), which are just infinite Markov systems.
(2) Study a relation between the zeta functions of the jump transformations and those of the original systems. In order to find a good relation between them, we make an assumption which guarantees a good connection between their periodic orbits and we construct towers over Markov systems.

In particular, when the jump transformations are Bernoulli (i.e. their symbolic dynamics are full shifts), we may have a nice product formula for the zeta functions which gives meromorphic extensions to the entire plane. In fact we see this for specific cases in §4.

(3) Decompose the recurrent part into mutually disjoint subshifts which are Bernoulli.

We proceed with this program in §§3 and 4. In section 2, we prepare preliminary results on the symbolic dynamics of piecewise invertible systems with FRS. Further, we give the definition of Markov cylinders and study their properties from which our results follow. The existence of Markov cylinders is shown in §3 under the FRS condition, so that we can construct jump transformations which are Markov systems.

Our topological Markov approximation is closely related to the method based on *orbit basis* which was introduced by Takahashi in [**20**] and we see a relation between the orbit basis and our topological Markov approximations in §3. We should mention that the approximation arguments still work for maps like $\beta$ transformations, which are finite state but do not satisfy the FRS condition. In fact, even if no Markov cylinder exists for $\beta$ transformations, we can find a collection of cylinders over which we can construct a jump transformation which is Bernoulli (see [**20**]). In the appendix, we study countable state shifts described by labelled graphs with finitely many vertices and countably many edges. We call them *subshifts of finite range* and explain the reason why the shifts can be considered naturally as countable state sofic shifts. Further, we observe that the existence of a magic word (which corresponds to the Markov cylinder) allows us to see that the factor map is one-to-one on the set of all periodic points up to an exceptional set. This

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fact allows us to have a simple proof of rationality of zeta functions for finite state sofic shifts when the dynamics of the exceptional sets of the Markov approximations have a nice structure.

## 2. *Piecewise invertible systems with FRS*

In this section, we give definitions and preliminary results on piecewise invertible systems with FRS. Let $T$ be a transformation on a bounded domain $X(\subset \mathbb{R}^d)$ and $Q = \{X_a\}_{a\in I}$ be a countable partition of $X$ satisfying the following conditions:

(1) $Q$ is a generating partition (i.e. $\bigvee_{m=0}^{\infty} T^{-m}Q$ is the partition into points).

(2) Each $X_a$ is a connected subset of $X$ and $\bigcup_{a\in I} \operatorname{int} X_a$ is an open dense subset of $X$. $X_a$ cannot intersect any $X_b$ with $b \neq a$.

(3) For each $X_a$, $T|_{X_a} : X_a \to TX_a$ is a homeomorphism.

We call $T$ satisfying the above conditions, a *piecewise invertible map* and call the triple $(T, X, Q)$ a *piecewise invertible system*. If $\operatorname{int}(X_a \cap TX_b) \neq \emptyset$ implies $TX_b \supset \operatorname{int}(X_a)$ we say that $Q$ is the Markov partition for $T$ and $Q$ satisfies the Markov property with respect to $T$. If $\operatorname{int}(X_{a_1} \cap T^{-1}X_{a_2} \ldots \cap T^{-(n-1)}X_{a_n}) \neq \emptyset$, then we say that the sequence $(a_1 \ldots a_n)$ is *admissible* and denote $X_{a_1} \cap T^{-1}X_{a_2} \ldots \cap T^{-(n-1)}X_{a_n}$ by $X_{a_1\ldots a_n}$. We call $X_{a_1\ldots a_n}$ *a cylinder of rank* $n$. $\mathcal{L}^n$ denotes the set of all cylinders of rank $n$. $|a_1 \ldots a_n|$ stands for the length of the sequence $(a_1 \ldots a_n)$. We denote by $\mathcal{A}_n$ the set of all admissible sequences of length $n$ and $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$. Put $\mathcal{L} = \bigcup_{n=1}^{\infty} \mathcal{L}^n$. Let $\Sigma^*$ be the set of all right semi-infinite sequences $(a_1a_2 \ldots a_n \ldots)$ satisfying $(a_1 \ldots a_n) \in \mathcal{A}$ for all $n > 0$. For $(a_1a_2 \ldots a_n \ldots) \in \mathcal{A}$, put $\rho(a_1a_2 \ldots) = \bigcap_{i=0}^{\infty} T^{-i}X_{a_{i+1}}$. It follows from condition (1) that $\rho(a_1a_2 \ldots)$ is at most a single point of $X$. Put $\Sigma_\emptyset = \{(a_1a_2 \ldots) \in \Sigma^* : \rho(a_1a_2 \ldots) = \emptyset\}$. Define

$$\Sigma^{*'} = \bigcap_{i=0}^{\infty} \sigma^{*^{-i}}\{(a_1a_2 \ldots) \in \Sigma^* \backslash \Sigma_\emptyset : \rho(a_1a_2 \ldots) \in X_{a_1}\},$$

where $\sigma^*$ is the left shift on $\Sigma^*$, i.e. $\sigma^*(a_1a_2 \ldots) = (a_2a_3 \ldots)$. We remark that $\Sigma^*$ and $\Sigma^{*'}$ are $\sigma^*$-invariant.

*Remark A.* $\rho : \Sigma^{*'} \to X$ is a bijective, continuous shift commuting map, i.e. $T\rho = \rho\sigma^*$ (see [**25**]).

We call the triple $(\Sigma^{*'}, \sigma^*, \rho)$ a *realization* of $(T, X, Q)$ or a *symbolic dynamics* of $(T, X, Q)$.

Denote $\mathcal{U}^k = \{T^k X_{a_1\ldots a_k} : (a_1 \ldots a_k) \in \mathcal{A}_k\}$ and put $\mathcal{U} = \cup_{k=1}^{\infty} \mathcal{U}^k$.

*Definition.* Let $(T, X, Q)$ be a piecewise invertible system such that $\mathcal{U}$ consists of only finitely many subsets of $X$ with positive Lebesgue measure. Then we say that $T$ guarantees a *finite range structure* (FRS) and we call the quadruple $(T, X, Q, \mathcal{U})$ a *piecewise invertible system with FRS*.

The FRS condition is well-known in the case of interval maps (cf. [**2**]).

The next theorem which was obtained in [**25**] shows a relation between the FRS condition and countable state sofic shifts, that is, piecewise invertible systems with FRS provide countable state sofic shifts as their symbolic dynamics.

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THEOREM 2.1. *Let $(T, X, Q = \{X_a\}_{a\in I}, \mathcal{U})$ be a piecewise invertible system with FRS. Suppose that $\mathcal{U} = \{U_0, U_1, \ldots, U_N\}$ and $X = U_0$. Then there exists a countable state sofic shift which realizes $T$ in the following sense. Define a directed graph whose vertex set is the finite set $\mathcal{U}$, and whose arc set is the countable alphabet $I$, where there is an edge $c$ from $U_i$ to $U_j$ if for any $(a_1 \ldots a_l)$ such that $T^l X_{a_1\ldots a_l} = U_i$, $X_{a_1\ldots a_l c} \in \mathcal{L}$ and $T^{l+1} X_{a_1\ldots a_l c} = U_j$.*

(1) *This labelled graph defines a one-sided edge subshift of finite type $\sigma$ with countable alphabet and a one-block map $\pi : \sigma \to \sigma'$, where $\sigma'$ is a subshift with the countable alphabet $I$.*

(2) *Let $(\Sigma', \sigma')$ be the one-sided sofic shift in the above, and for $(a_0 a_1 \ldots) \in \Sigma'$ put*

$$\rho(a_0 a_1 \ldots) = \bigcap_{i=0}^{\infty} T^{-i} X_{a_i}.$$

*Then the map $\rho : \Sigma' \to X$ is defined a.e. and $\rho$ is a bijective continuous conjugacy map, i.e. $T\rho = \rho\sigma'$.*

(3) *Let $\mathcal{V}$ be a disjoint partition generated by $\mathcal{U}$. Then $Q \vee \mathcal{V}$ is the Markov partition for $T$. In particular, we can see*

$$T(Q \vee \mathcal{V}) \subseteq \sigma(\mathcal{V}),$$

*where $\sigma(\mathcal{V})$ denotes the finite $\sigma$-algebra generated by $\mathcal{V}$.*

We remark that typical examples which arise from number theory have no periodic points on $\cup_{a\in I} \partial X_a$ but admit them on $\cup_{U\in\mathcal{U}} \partial U \setminus \cup_{a\in I} \partial X_a$. Thus the choice of the symbolic dynamics with respect to $Q \vee \mathcal{V}$ has a disadvantage, i.e. a loss of some information on periodic orbits. For this reason we consider that the definite partition $Q$ is more appropriate than $Q \vee \mathcal{V}$ for obtaining the symbolic dynamics even if it is not Markov.

We will define a version of the Markov property as follows. For convenience of notation, we denote $a_1 \ldots a_n$ by $a(n)$.

*Definition.* We say that $X_{a(n)} \in \mathcal{L}$ is *a Markov cylinder* if for any $U \in \mathcal{U}$ satisfying $\mathrm{int}(U \cap X_{a(n)}) \neq \emptyset$, $\mathrm{int}\, X_{a(n)} \subset U$.

*Remark B.* Markov cylinders give right resolving blocks (*magic words*) (see the appendix).

Let $\mathcal{M}^n$ be the set of all Markov cylinders of rank $n$ and put $\mathcal{M} = \bigcup_{i=1}^{\infty} \mathcal{M}^n$.

*Definition.* We say that $(T, X, Q)$ has the $k$-Markov property if $\mathcal{M}^k = \mathcal{L}^k$.

*Remark C.* The property $\mathcal{U} = \mathcal{U}^k$ is equivalent to the $k$-Markov property of the partition $Q$. If $T$ satisfies the $k$-Markov property, then we call a quadruple $(T, X, Q, \mathcal{U})$ a *piecewise invertible $k$-Markov system*. Its symbolic dynamics is exactly the countable state '$k$-Markov shift' which is defined in the next section.

The following properties will be important for our purpose.

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LEMMA 2.1. *The subclass $\mathcal{M}$ of $\mathcal{L}$ has the 'strong playback property', i.e. for any $X_{a(n)} \in \mathcal{M}$ and for any $X_{b(m)} \in \mathcal{L}$ such that $X_{b(m)a(n)} \in \mathcal{L}$, $X_{b(m)a(n)}$ also belongs to $\mathcal{M}$ (cf. [**17**]).*

*Proof.* In fact, as the admissibility of the sequence $c(l)\cdot b(m)a(n)$ implies the admissibility of $c(l)b(m)\cdot a(n)$, the above assertion is obtained immediately. □

LEMMA 2.2. *If $X_{a(n)}$ is a Markov cylinder, then for any $X_{b(m)} \in \mathcal{L}$ such that $X_{a(n)b(m)} \in \mathcal{L}$, $X_{a(n)b(m)}$ also belongs to $\mathcal{M}$.*

*Proof.* If $X_{a(n)b(m)} \notin \mathcal{M}$, there exists a $U \in \mathcal{U}$ s.t. $U^c \cap (\operatorname{int} X_{a(n)b(m)}) \neq \emptyset$. This contradicts the fact that $X_{a(n)} \in \mathcal{M}$. □

## 3. *Topological Markov approximations and orbit basis*

In this section, we first approximate piecewise invertible systems with FRS by Markov systems along the lines of Schweiger's technique [**17**], which is based on jump transformations over appropriate classes of cylinders. Next we show the existence of an orbit basis which is based on Takahashi's orbit basis and in the last part of this section we discuss a relation between our Markov approximations and the existence of orbit bases. In fact, the orbit basis consists of Markov cylinders satisfying a special property.

Let us consider a new partition of $X$ as follows. Put $\mathcal{B}_1 = \{X_b \in \mathcal{L}^1 : X_b \in \mathcal{M}\}$ and $\mathcal{D}_1 = \{X_d \in \mathcal{L}^1 : X_d \notin \mathcal{M}\}$. Denote $B_1 = \bigcup_{X_b \in \mathcal{B}_1} X_b$ and $D_1 = \bigcup_{X_d \in \mathcal{D}_1} X_d$. Next we divide $D_1$ into two disjoint subsets, $B_2$ and $D_2$ which consist of all cylinders of rank two belonging to $\mathcal{M}^2$ and $\mathcal{L}^2 \backslash \mathcal{M}^2$ respectively. More precisely, put $\mathcal{B}_2 = \{X_{b_1b_2} \in \mathcal{L}^2 : X_{b_1} \in \mathcal{D}_1, X_{b_1b_2} \in \mathcal{M}\}$ and $\mathcal{D}_2 = \{X_{d_1d_2} \in \mathcal{L}^2 : X_{d_1} \in \mathcal{D}_1, X_{d_1d_2} \notin \mathcal{M}\}$. Then $B_2 = \bigcup_{X_{b_1b_2} \in \mathcal{B}_2} X_{b_1b_2}$ and $D_2 = \bigcup_{X_{d_1d_2} \in \mathcal{D}_2} X_{d_1d_2}$. Inductively, we define

$$\mathcal{B}_i = \{X_{b(i)} \in \mathcal{L}^i : X_{b_1 \ldots b_{i-1}} \in \mathcal{D}_{i-1}, X_{b(i)} \in \mathcal{M}\},$$
$$\mathcal{D}_i = \{X_{d(i)} \in \mathcal{L}^i : X_{d_1 \ldots d_{i-1}} \in \mathcal{D}_{i-1}, X_{d(i)} \notin \mathcal{M}\}.$$

Put $B_i = \bigcup_{X_{b(i)} \in \mathcal{B}_i} X_{b(i)}$ and $D_i = \bigcup_{X_{d(i)} \in \mathcal{D}_i} X_{d(i)}$. For each $n > 0$, we have the following disjoint partition of $X$: $X = \bigcup_{i=1}^{n} B_i \cup D_n$. Let us consider a new index set $I_M$ defined by $I_M = \bigcup_{j=1}^{\infty} \{(b_1 \ldots b_j) \in \mathcal{A} : X_{b_1 \ldots b_j} \in \mathcal{B}_j\}$. Let us define a jump transformation $T_M$ on $\bigcup_{j=1}^{\infty} B_j$ by $T_M(x) = T^j(x)$ for $x \in B_j$. Put $X_M = X \backslash (\bigcup_{m \geq 0} T_M^{-m}(\bigcap_{n \geq 0} D_n))$. Then $T_M$ is a transformation of $X_M$ and $Q_M = \{X_\alpha\}_{\alpha \in I_M}$ is a generating partition of $X_M$. The generating property relies on the strong playback property of $\mathcal{M}$ and the argument in the proof of Theorem 6.1 in [**24**] (cf. [**5**]) still works, so we omit the proof. Denote by $\mathcal{U}_M$ the set of all range sets for $(T_M, X_M, Q_M)$. It is easy to see that $\mathcal{U}_M \subset \mathcal{U}$.

THEOREM 3.1. *Let $(T, X, Q, \mathcal{U})$ be a piecewise invertible system with FRS. Then $\mathcal{M} \neq \emptyset$ and $(T_M, X_M, Q_M, \mathcal{U}_M)$ is a piecewise invertible Markov system with FRS. In fact, $T_M$ has the (simple) Markov property, i.e.* $\operatorname{int}(X_\alpha \cap T_M X_\gamma) \neq \emptyset$ $(\alpha, \gamma \in I_M)$ *implies* $T_M X_\gamma \supset \operatorname{int} X_\alpha$.

*Proof.* First we observe that for $X_{a_1 \ldots a_n} \in \mathcal{M}^c$, there exists a cylinder $X_{b_1 \ldots b_l}$ such that $X_{b_1 \ldots b_l a_1 \ldots a_n} \in \mathcal{M}$. In fact, if we cannot obtain such a sequence $(b_1 \ldots b_l) \in \mathcal{A}$,

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then we have a strictly nested sequence of range sets, i.e. if $X_{a_1\dots a_n} \notin \mathcal{M}$, there exists $(b_1\dots b_l) \in \mathcal{A}$ such that $(b_1\dots b_l a_1\dots a_n) \in \mathcal{A}$ and $T^{n+l}X_{b_1\dots b_l a_1\dots a_n} \subset T^n X_{a_1\dots a_n}$. If $X_{b_1\dots b_l a_1\dots a_n}$ is also not a Markov cylinder, again we have some $(c_1\dots c_j) \in \mathcal{A}$ such that $(c_1\dots c_j b_1\dots b_l a_1\dots a_n) \in \mathcal{A}$ and

$$T^{n+l+j}X_{c(j)b(l)a(n)} \subset T^{n+l}X_{b(l)a(n)} \subset T^n X_{a(n)}.$$

Inductively, we can obtain an infinite nested sequence of range sets: $T^n X_{a(n)} = U_{i_0} \supset U_{i_1} \supset \cdots$. This contradicts the 'FRS' condition. Thus if $\mathcal{M} = \emptyset$, we have a contradiction. The rest of the assertions are an immediate consequence of the definition of the jump transformation $T_M$. □

LEMMA 3.1. $\bigcap_{n\geq 0} D_n \subseteq \bigcup_{U\in\mathcal{U}} \partial U$ *and* $\cap_{n\geq 0} D_n, X_M$ *are* $T$*-invariant. Further,* $\cup_{m=1}^{\infty} T_M^{-m}(\cap_{n\geq 0} D_n)$ *contains no periodic points.*

*Proof.* The first assertion follows from the generating property of $Q$. In fact, if $x \in \cap_{n\geq 0} D_n$ satisfies $x \in \operatorname{int} V$ for some $V \in \mathcal{V}$, then we have immediately a contradiction to the assumption. The rest of assertions are immediate consequences of the strong playback property of $\mathcal{M}$. □

Let $\mathcal{P}_n(X, T)$ be the set of all periodic points with minimal period $n$ and put $\mathcal{P}(X, T) = \cup_{n=0}^{\infty} \mathcal{P}_n(X, T)$. It follows from Lemma 3.1 that

$$\mathcal{P}(X, T) = \mathcal{P}(X_M, T) \cup \mathcal{P}(\cap_{n\geq 0} D_n, T).$$

Let $(\Sigma, \sigma)$ be the symbolic dynamics of $(T, X, Q, \mathcal{U})$ in Theorem 2.1. $(\Sigma|_{X_M}, \sigma)$ and $(\Sigma|_{\cap_{n\geq 0} D_n}, \sigma)$ denote the symbolic dynamics of $(X_M, T)$ and $(\cap_{n\geq 0} D_n, T)$ repectively. The strong playback property of $\mathcal{M}$ allows us to have two subsystems, i.e. on $\cap_{n=0}^{\infty} D_n$ which consists of boundaries of range sets in $\mathcal{U}$ as mentioned in Lemma 3.1 and on $X_M$ where we can construct Markov systems. Further, any points in $\Sigma|_{X_M}$ can be written as an infinite sequence of $I_M$ which is a point of $\Sigma_M$. The following condition allows us to have a one-to-one correspondence between $\mathcal{P}(\Sigma_M, \sigma_M)$ and $\mathcal{P}(\Sigma|_{X_M}, \sigma)$ which will be important for obtaining a nice formula of zeta functions in the next section.

(C) $\mathcal{M}^1 \neq \emptyset$ and $B_{n+1} = D_n \cap T^{-n} B_1$.
(Equivalently, if $X_{d_1\dots d_{k-1}} \in \mathcal{D}_{k-1}$, $X_{d_k} \in \mathcal{D}_1$ and $X_{d_1\dots d_k} \in \mathcal{L}^k$, then $X_{d_1\dots d_k} \in \mathcal{D}_k$.)

For typical examples (see §6), it is easy to verify the second part of condition (C) which can be described as follows: for $(d_1\dots d_k) \in \mathcal{A}$, $\operatorname{int} X_{d_1\dots d_i} \cap (\bigcup_{U\in\mathcal{U}} \partial U) \neq \emptyset$ $(i = 1\dots k-1)$ and $\operatorname{int} X_{d_k} \cap (\bigcup_{U\in\mathcal{U}} \partial U) \neq \emptyset$ if and only if $\operatorname{int} X_{d_1\dots d_k} \cap (\bigcup_{U\in\mathcal{U}} \partial U) \neq \emptyset$. We remark that condition (C) also leads us to a relation between the Markov system $T_M$ and an induced transformation on the set of Markov cylinders of rank 1.

PROPOSITION 3.1. *Let* $T$ *satisfy condition (C). Then we can form the induced transformation* $T_{B_1}$ *of* $T$ *with respect to* $B_1$*, which satisfies*

$$T|_{B_1} \circ T_{B_1} = T_M \circ T|_{B_1}.$$

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*Proof.* We can apply the same argument as used in the proof in [**17**] to our situation. Let $F_k = \{x \in B_1 : T^i x \in D_1 \text{ for } i = 1 \dots k-1, T^k x \in B_1\}$. Then

$$F_k = B_1 \cap T^{-1} D_{k-1} \cap T^{-k} B_1 = B_1 \cap T^{-1}(D_{k-1} \cap T^{-(k-1)} B_1) = B_1 \cap T^{-1} B_k.$$

It follows from (C) and the above that

$$\begin{aligned} B_1 &= \bigcup_{k=1}^{\infty} F_k = B_1 \cap \left( \bigcup_{k=1}^{\infty} T^{-1} B_k \right) = B_1 \cap T^{-1} \left( \bigcup_{k=1}^{\infty} B_k \right) \\ &= B_1 \cap T^{-1} \left( \bigcup_{k=1}^{\infty} (D_{k-1} \cap T^{-(k-1)} B_1) \right) \subseteq B_1 \cap \bigcup_{k=1}^{\infty} T^{-k} B_1 \subseteq \bigcup_{k=1}^{\infty} T^{-k} B_1. \end{aligned}$$

Hence we can define the induced transformation of $T$ with respect to $B_1$. If $x \in B_1$ belongs to $F_k$ for a $k > 0$, then $Tx \in B_k$, so that we have

$$T_M T|_{B_1} x = T^{k+1} x = T|_{B_1} T_{B_1} x. \qquad \square$$

In the rest of this section, we show a relation between systems with FRS and shifts with orbit bases which were defined by Takahashi in [**20**]. We prepare some notation on shifts with countable states. Let $(X, \sigma)$ be a one-sided shift with a countable state set $I$. For $(a_1 \dots a_n) \in \mathcal{A}_n$, $[a_1 \dots a_n]$ denotes the cylinder set

$$\{(x_1 x_2 \dots) \in I^{\mathbb{N}} : (x_1 \dots x_n) = (a_1 \dots a_n)\}.$$

For $V \subseteq \mathcal{A}$, denote $[V] = \bigcup_{(a_1 \dots a_n) \in V} [a_1 \dots a_n]$. We define for $V \subseteq \mathcal{A}$ *the Markov hull of* $V$ as follows:

$$M(V) = \{(a_1 a_2 \dots) \in I^{\mathbb{N}} : (a_i a_{i+1} \dots) \in [V] \ (\forall i \geq 1)\}.$$

The shift $(M(V), \sigma)$ for $V \subseteq \mathcal{A}_k$ is called *a $k$-Markov shift.* For a given one-sided shift $(X, \sigma)$, define $X^* = \{(a_n)_{n \in \mathbb{Z}} = (\dots a_{-1} a_0 a_1 \dots) \in I^{\mathbb{Z}} : (a_{m+n})_{m \in \mathbb{N}} \in X \ (\forall n \in \mathbb{Z})\}$. Then $(X^*, \sigma^*)$ is called *the natural extension* of $(X, \sigma)$. Let $\pi : I^{\mathbb{Z}} \to I^{\mathbb{N}}$ be a natural projection, i.e. $\pi(\dots a_{-1} a_0 a_1 \dots) = (a_0 a_1 \dots)$.

Let us define *an orbit basis* of a shift which is a version of Takahashi's definition.

*Defintion.* Let $(\Sigma, \sigma)$ be a one-sided shift over the index set $I$. The shift $(\Sigma, \sigma)$ is said to admit *an orbit basis* if there exist a subset $J$ of $\mathcal{A}$, a positive integer $r > 0$, and a subset $V$ of $J^{r+1}$ satisfying the following conditions.

(O1) For each $n \geq 1$, put $V_n = \{(\mathbf{b}_1 \dots \mathbf{b}_n) \in J^n : (\mathbf{b}_1 \dots \mathbf{b}_n) \in \mathcal{A}\}$. Then for $n > r$, $V_n = \{(\mathbf{b}_1 \dots \mathbf{b}_n) \in J^n : (\mathbf{b}_i \dots \mathbf{b}_{i+r}) \in V \ (1 \leq i \leq n-r)\}$.

(O2) For each $x \in \Sigma$, there exists a sequence $(\mathbf{b}_n)_{n \geq 0} \in M(V)$ and some $0 \leq i \leq |\mathbf{b}_0|$ such that

$$x = (\sigma^i \mathbf{b}_0) \mathbf{b}_1 \mathbf{b}_2 \dots.$$

(O3) Conversely, the sequence defined by $(\sigma^i \mathbf{b}_0) \mathbf{b}_1 \mathbf{b}_2 \dots$ for some $(\mathbf{b}_n)_{n \geq 0} \in M(V)$ and $i > 0$, belongs to $\Sigma$.

Let $\sigma_V$ be a shift transformation on $J^{\mathbb{N}}$, so that $(M(V), \sigma_V)$ is an $r$-Markov shift with structure set $V$. Let $(M(V)^*, \sigma_V^*)$ be a natural extension of $(M(V), \sigma_V)$ i.e.

$$M(V)^* = \{(\dots \mathbf{b}_{-1} \mathbf{b}_0 \mathbf{b}_1 \dots) \in J^{\mathbb{Z}} : (\mathbf{b}_n \mathbf{b}_{n+1} \dots) \in M(V) \ (\forall n \in \mathbb{Z})\},$$

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which coincides with the set

$$\{(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots) \in J^{\mathbb{Z}} : (\mathbf{b}_n \dots \mathbf{b}_{n+r}) \in V \ (\forall n \in \mathbb{Z})\}.$$

Let $\theta(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots) = |\mathbf{b}_0|$ and put

$$\Sigma^\theta = \{(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots, n) \in M(V)^* \times \mathbb{N} : 0 \le n < \theta(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots)\}.$$

Define a map $\tau : \Sigma^\theta \to \Sigma^\theta$ by

$$\tau(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots, n) = \begin{cases} (\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots, n+1), & \text{if } (\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots, n+1) \in \Sigma^\theta \\ (\sigma_V^*(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots), 0), & \text{otherwise.} \end{cases}$$

We call $(\Sigma^\theta, \tau)$ *a tower over* $(M(V)^*, \sigma_V^*)$ *with the ceiling function* $\theta$. Let us define a map $p_0 : M(V)^* \to I^{\mathbb{Z}}$ by

$$p_0(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots)(n) = \begin{cases} \mathbf{b}_k(n - |\mathbf{b}_0| - \dots - |\mathbf{b}_{k-1}|), & \text{if } 0 \le n - (|\mathbf{b}_0| + \dots \\ & +|\mathbf{b}_{k-1}|) < |\mathbf{b}_k| \text{ and } n \ge 0 \\ \mathbf{b}_{-k}(n + |\mathbf{b}_{-1}| + \dots + |\mathbf{b}_{-k}|), & \text{if } 0 > n + (|\mathbf{b}_{-1}| + \dots \\ & +|\mathbf{b}_{-k+1}|) > -|\mathbf{b}_{-k}| \ (n < 0). \end{cases}$$

$p_0$ gives a decoding from sequences with the index set $J$ to sequences with the index set $I$.

Let $p : \Sigma^\theta \to I^{\mathbb{Z}}$ be a map defined by

$$p(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots, n) = \sigma^{*n} p_0(\dots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\dots).$$

(O4) The map $p$ is a homeomorphism of $\Sigma^\theta$ into $(X^*, \sigma^*)$ such that $\pi p(\Sigma^\theta) = \Sigma$ and $p\tau = \sigma^* p$.

We call $(J, V)$ an orbit basis of order $r$ for $(\Sigma, \sigma)$. In particular, if $r = 0$ and $V = J$ then we say that a shift $(\Sigma, \sigma)$ admits a *free orbit basis* $J$ and in the case of $r = 1$ we say that the orbit basis $(J, V)$ is *simple Markov*.

*Remark D.* If $(\Sigma, \sigma)$ admits an orbit basis $(J, V)$, then $(\Sigma, \sigma)$ is a factor image of the tower over the $(r+1)$-Markov shift. In fact, $\pi p$ is an onto map satisfying $(\pi p)\tau = \sigma(\pi p)$.

Let $(T, X, Q = \{X_a\}_{a \in I}, \mathcal{U})$ be a piecewise invertible system with FRS.

*Definition.* We call $X_{a_1 \dots a_n} \in \mathcal{M}^n$ $(n \ge 2)$ *a special Markov cylinder* if $X_{a_i \dots a_n} \in \mathcal{M}^c$ $(\forall i \in \{2, \dots, n\})$.

Let $\mathcal{SM}^n$ $(n \ge 2)$ be the set of all special Markov cylinders of rank $n$.

*Remark E.* From Lemma 2.1 we have for $X_{a_1 \dots a_n} \in \mathcal{M}^c$, $X_{a_i \dots a_n} \in \mathcal{M}^c$ for all $i \in \{2, \dots, n\}$. From Lemma 2.2 we also have $\mathcal{L}^n \setminus \mathcal{M}^n = \mathcal{D}_n$.

Denote $\mathcal{SM} = \cup_{n=2}^\infty \mathcal{SM}^n \bigcup \mathcal{M}^1$ and let $I_{SM} = \{(a_1 \dots a_n) \in \mathcal{A} : X_{a_1 \dots a_n} \in \mathcal{SM}\}$.

THEOREM 3.2. *Under condition (C), the symbolic dynamics of* $(T, X_M, Q, \mathcal{U})$ *with FRS admits the following orbit basis of order 1:*

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$$J = I_{SM},$$

$$V = \{(\mathbf{b}_1\mathbf{b}_2) \in I_{SM}^2 : \mathbf{b}_1 \in Q(i) \text{ and } \mathbf{b}_2 \in \mathcal{M}(i) \text{ for } i \in \{0, 1, \dots N\}\},$$

*where*

$$Q(i) = \bigcup_{n=1}^{\infty} \{(a_1 \dots a_n) \in \mathcal{A}^n : T^n X_{a_1 \dots a_n} = U_i\}$$

*and*

$$\mathcal{M}(i) = \bigcup_{n=1}^{\infty} \{(a_1 \dots a_n) \in \mathcal{A}^n : X_{a_1 \dots a_n} \subseteq U_i\}.$$

*Remark F.* For $n \geq 2$, $I_{SM}^n \cap \mathcal{A} = \{(\mathbf{b}_1 \dots \mathbf{b}_n) \in I_{SM}^n : \mathbf{b}_i\mathbf{b}_{i+1} \in V\}$.

*Proof of Theorem 3.2.* From Theorem 3.1 it is easy to see that any point in $\Sigma|_{X_M}$ can be written as an infinite sequence of orbit bases. In fact, an infinite sequence of words in $I_M$ of the form $(\underbrace{d_1 \dots d_n}_{\mathcal{D}_n}\underbrace{b}_{\mathcal{M}})$ can be written as an infinite sequence of words of the form $(\underbrace{b}_{\mathcal{M}}\underbrace{d_1 \dots d_n}_{\mathcal{D}_n})$, which just correspond to cylinders in $\mathcal{SM}$. For $(\mathbf{b}_n)_{n\geq 0}, (\mathbf{b}'_n)_{n\geq 0} \in M(V)$, if there are some $i, j$ with $0 \leq i \leq |\mathbf{b}_0|, 0 \leq j \leq |\mathbf{b}'_0|$ satisfying

$$(\sigma^i \mathbf{b}_0)\mathbf{b}_1 \dots = (\sigma^j \mathbf{b}'_0)\mathbf{b}'_1 \dots,$$

then $\mathbf{b}_n = \mathbf{b}'_n$ $(n \geq 1)$, $|\mathbf{b}_0| - i = |\mathbf{b}'_0| - j$ and

$$(\mathbf{b}_0(i+1), \dots, \mathbf{b}_0(|\mathbf{b}_0|)) = (\mathbf{b}'_0(j+1), \dots, \mathbf{b}'_0(|\mathbf{b}'_0|)) \in \mathcal{D}_{|\mathbf{b}_0|-i}.$$

This is an immediate consequence of condition (C). It follows from this fact that $p$ is injective. Thus all the conditions (O1)–(O4) are valid. □

## 4. *Product formulas for zeta functions*

Let $T : X \to X$ be a map and write

$$\mathrm{Fix}(X, T^n) = \{x \in X; T^n x = x\}.$$

Let $f : X \to (-\infty, \infty]$ be a potential function such that

$$Z_n(X, f) = \sum_{x \in \mathrm{Fix}(X, T^n)} \exp\left(-\sum_{i=0}^{n-1} f(T^i x)\right)$$

is well-defined for all $n > 0$. Define the formal power series

$$\zeta_{X,f}(t) = \exp\left(\sum_{n=1}^{\infty} \frac{t^n}{n} Z_n(X, f)\right),$$

which is called the *Artin–Mazur–Ruelle* zeta function. In particular, for the case $f = 0$ it is called the *Artin–Mazur zeta function*. It is known that the radius of convergence of $\zeta_{X,f}$ is given by

$$\exp\left(-\limsup_{n\to\infty} \frac{1}{n} \log Z_n(X, f)\right).$$

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Let $(\Sigma, \sigma)$ be the symbolic dynamics of the $(T, X, Q, \mathcal{U})$ in Theorem 2.1. As mentioned in §3, we have a disjoint union:

$$P(\Sigma, \sigma) = P(\Sigma|_{X_M}, \sigma) \cup P(\Sigma|_{\cap_{n\geq 0} D_n}, \sigma)$$

so that

$$\zeta_{\Sigma, f}(t) = \zeta_{\Sigma|_{X_M}, f}(t)\zeta_{\Sigma|_{\cap_{n\geq 0} D_n}, f}(t)$$

for a function $f : \Sigma \to (-\infty, \infty]$. We first study $\zeta_{\Sigma|_{X_M}, f}(t)$. Let $(\Sigma^\theta, \tau)$ be a tower over $(M(V)^*, \sigma_V^*)$ with the ceiling function $\theta$ as defined in §3. Define $f^* : M(V^*) \to (-\infty, \infty]$ by

$$f^*(\ldots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1 \ldots) = \sum_{0\leq n<\theta(\ldots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1 \ldots)} f\sigma^n(\pi p_0(\ldots \mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1 \ldots)).$$

We have already seen in §3 that under condition (C), $(\Sigma|_{X_M}, \sigma)$ admits the orbit basis of order 1 and we can define a tower over $(\Sigma_M, \sigma_M)$ with the ceiling function $\theta$ which is defined by the length of the first coordinate. In this case, $f^* : \Sigma_M \to (-\infty, \infty)$ is defined by

$$f^*(\mathbf{b}_0\mathbf{b}_1 \ldots) = \sum_{0\leq n<\theta(\mathbf{b}_0\mathbf{b}_1 \ldots)} f\sigma^n p_0(\mathbf{b}_0\mathbf{b}_1 \ldots),$$

where $p_0 : \Sigma_M \to I^{\mathbb{N}}$ is the decoding as defined before. The following equality shows how $\zeta_{M(V)^*, f^*}$ relates to $\zeta_{\Sigma|_{X_M}, f}$ and also we have the same formula even if we take $(\Sigma_M, \sigma_M)$ instead of $(M(V)^*, \sigma_V^*)$.

LEMMA 4.1.

$$\sum_{n=1}^{\infty} \frac{t^n}{n} Z_n(\Sigma|_{X_M}, f) = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{y\in \mathrm{Fix}(M(V)^*, \sigma_V^{*k})} t^{\sum_{m=0}^{k-1} \theta(\sigma_V^{*m} y)} \exp\left\{ -\sum_{m=0}^{k-1} f^*(\sigma_V^{*m} y)\right\}.$$

*Proof.* As we have mentioned in the previous section, condition (C) gives a one-to-one correspondence between $\mathcal{P}(\Sigma_M, \sigma_M)$ and $\mathcal{P}(\Sigma|_{X_M}, \sigma)$. This also gives a one-to-one correspondence between $\mathcal{P}(M(V)^*, \sigma_V^*)$ and $\mathcal{P}(\Sigma|_{X_M}, \sigma)$ as we have seen in the proof of Theorem 3.2 (i.e. $p$ is injective). Then it is easy to observe the following equalities:

$$\begin{aligned}
&\sum_{n=1}^{\infty} t^{n-1} \sum_{x\in \mathrm{Fix}(\Sigma|_{X_M}, \sigma^n)} \exp\left( -\sum_{i=0}^{n-1} f(\sigma^i x)\right) \\
&\quad = \sum_{k=1}^{\infty} \sum_{y=(\mathbf{b}_0\ldots \mathbf{b}_{k-1})\in \mathrm{Fix}(M(V)^*, \sigma_V^{*k})} \sum_{m=0}^{k-1} \frac{|\mathbf{b}_m|}{k} t^{\sum_{m=0}^{k-1} |\mathbf{b}_m| - 1} \\
&\qquad \times \exp\left( -\sum_{m=0}^{k-1} \sum_{j=0}^{|\mathbf{b}_m|-1} f\sigma^j \pi p_0(\sigma_V^{*m} y, j)\right) \\
&\quad = \sum_{k=1}^{\infty} \sum_{y\in \mathrm{Fix}(M(V)^*, \sigma_V^{*k})} \frac{1}{k} \sum_{m=0}^{k-1} \theta(\sigma_V^{*m} y) t^{\sum_{m=0}^{k-1} \theta(\sigma_V^{*m} y) - 1} \exp\left( -\sum_{m=0}^{k-1} f^*(\sigma_V^{*m} y)\right).
\end{aligned}$$

By integrating in $t$, we have the desired inequality. □

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LEMMA 4.2. *If $(\Sigma, \sigma)$ is a shift with orbit basis $(J, V)$, then:*

(1) $\zeta_{\Sigma,f}(t) = \zeta_{M(V)^*, f^* - (\log t)\theta}(1)$;

(2) *if $M(V)^* = J^{\mathbb{Z}}$, and $g : M(V)^* \to (-\infty, \infty]$ is a function depending only on the zeroth coordinate, then*

$$\zeta_{M(V)^*,g}(t) = \left(1 - \sum_{\mathbf{b}\in J} \exp(-g(\mathbf{b}))t\right)^{-1}.$$

*Proof.* The first assertion is an immediate consequence of Lemma 4.1. We have the second assertion as follows:

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{y=(\mathbf{b}_0\mathbf{b}_1\ldots\mathbf{b}_{n-1})\in \mathrm{Fix}(M(V)^*, \sigma_V^{*n})} \exp\left(-\sum_{m=0}^{n-1} g(\sigma_V^{*m} y)\right) \\
&\quad = \sum_{n=1}^{\infty} \frac{t^n}{n} \sum_{(\mathbf{b}_0\ldots\mathbf{b}_{n-1})\in J^n} \prod_{m=0}^{n-1} \exp(-g(\mathbf{b}_m)) \\
&\quad = \sum_{n=1}^{\infty} \frac{(t\sum_{\mathbf{b}\in J} \exp(-g(\mathbf{b})))^n}{n}.
\end{aligned}$$

□

Along the line of the proof in [**20**], we can show that the following formulae are valid as formal power series.

LEMMA 4.3. *If $(\Sigma, \sigma)$ admits a free orbit basis $J$ and if the function $f : \Sigma \to (-\infty, \infty]$ depends only on the zeroth coordinate, then*

$$\zeta_{\Sigma,f}(t)^{-1} = 1 - \sum_{\mathbf{b}\in J} \exp\left(-\sum_{n=0}^{|\mathbf{b}|-1} f(\sigma^n p_0\mathbf{b})\right) t^{|\mathbf{b}|}.$$

*In particular,*

$$\zeta_{\Sigma,0}(t)^{-1} = 1 - \sum_{\mathbf{b}\in J} t^{|\mathbf{b}|}.$$

*Proof.* From Lemmas 4.1 and 4.2, we see easily that

$$\zeta_{\Sigma,f}(t)^{-1} = 1 - \sum_{\mathbf{b}\in J} \exp(-f^*(\mathbf{b}) + (\log t)\theta(\mathbf{b}))$$

which gives the desired formula. □

Let

$$\begin{aligned}
\Sigma^*_{\mathrm{rec}} \quad = \quad & \{x = (\ldots x_{-1}x_0x_1\ldots) \in \Sigma^* : \text{ for each word } w \text{ appearing in } x, \\
& (x_n \ldots x_{n+|w|-1}) = w \text{ both for infinitely many } n \geq 0 \text{ and } n \leq 0\}
\end{aligned}$$

and define for $(b_1 \ldots b_n) \in \mathcal{A}$, $\Sigma^*(b_1 \ldots b_n) = \{(a_k)_{k\in\mathbb{Z}} \in \Sigma^* : (a_k \ldots a_{k+n-1}) = (b_1 \ldots b_n) \text{ both for infinitely many } k \geq 0 \text{ and } k < 0\}$. Put $\Sigma(b_0 \ldots b_r) = \pi(\Sigma^*_{\mathrm{rec}}(b_0 \ldots b_r))$. From Theorems 3.1 and 3.2, we see

$$(\Sigma|_{X_M})^*_{\mathrm{rec}} = \bigcup_{\mathbf{b}\in I_{SM}} (\Sigma|_{X_M})^*_{\mathrm{rec}}(\mathbf{b}).$$

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PROPOSITION 4.1. *$(\pi((\Sigma|_{X_M})^*_{\rm rec}), \sigma)$ is a disjoint union of subshifts $(\Sigma_k, \sigma)$ $(k \geq 1)$ which admit free orbit bases so that*

$$P(\Sigma|_{X_M}, \sigma) = \cup_{k=1}^{\infty} P(\Sigma_k, \sigma).$$

*Proof.* By a diagonal argument we can write $I_{SM}$ as a countable union of words belonging to $I_{SM}$:

$$I_{SM} = \bigcup_{k=1}^{\sharp I_{SM}} \mathbf{v}(k).$$

Further, we can decompose $(\Sigma|_{X_M})^*_{\rm rec}$ into subshifts $\Sigma^*_k$ $(k \geq 1)$ which are mutually disjoint as follows:

$$(\Sigma|_{X_M})^*_{\rm rec} = \bigcup_{k=1}^{\infty} \Sigma^*_k,$$

where

$$\Sigma^*_k = \{(\ldots \mathbf{v}(k_{-1})\mathbf{v}(k_0)\mathbf{v}(k_1)\ldots) \in (\Sigma|_{X_M})^*_{\rm rec}(\mathbf{v}(k)) : (\mathbf{v}(k_j)\mathbf{v}(k_{j+1})) \in V, \forall k_j, k \leq k_j \leq \sharp I_{SM}\}.$$

Putting $\Sigma_k = \pi(\Sigma^*_k)$ gives the first assertion. For each $k \geqq 1$, $\Sigma_k$ admits a free orbit basis:

$$J(k) = \cup_{n\geq 1}\{(\mathbf{v}(k)\mathbf{v}(k_1)\ldots\mathbf{v}(k_{n-1})) \in \mathcal{A} : k < k_1, \ldots, k_{n-1} \leq \sharp I_{SM}\}.$$

We can also write $\Sigma^*_k = (\Sigma|_{X_M})^*_{\rm rec}(\mathbf{v}(k)) \setminus \bigcup_{l=1}^{k-1}(\Sigma|_{X_M})^*_{\rm rec}(\mathbf{v}(l))$ with $(k \geq 2)$ and the free orbit basis $J(k)$ of $\Sigma_k$ is just the set of all words $(\mathbf{c}_1 \ldots \mathbf{c}_m)$ $(m \geq 2)$ over $I_{SM}$ with $\mathbf{c}_i \in \cup_{l\geqq k}\mathbf{v}(l)$ satisfying:

(1) $(\mathbf{c}_1 \ldots \mathbf{c}_m) = (\mathbf{v}(k)\mathbf{c}_2 \ldots \mathbf{c}_m)$;
(2) $\mathbf{c}_l \neq \mathbf{v}(k)$ $(l = 2, \ldots, m)$;
(3) $(\mathbf{c}_m\mathbf{v}(k)) \in V$ and $(\mathbf{c}_{l-1}\mathbf{c}_l) \in V$ $(l = 2, \ldots, m)$.

We have finished the proof (cf. Lemma 2 in [**20**]). □

It follows from Lemma 4.3 that we have a formula of the *Artin–Mazur–Ruelle zeta function* associated to $f$. Let $S = (S_{ij})$ be a $\sharp I_{SM} \times \sharp I_{SM}$ matrix defined by

$$\begin{aligned} S_{ij} &= 1 \quad \text{if } (\mathbf{v}(i)\mathbf{v}(j)) \in V \\ S_{ij} &= 0 \quad \text{otherwise.} \end{aligned}$$

THEOREM 4.1. *Suppose that $(T, X, Q, \mathcal{U})$ satisfies all assumptions of Theorem 3.2. Then we have:*

(1) $\zeta_{\Sigma,f}(t) = \zeta_{\Sigma|_{X_M},f}(t)\zeta_{\Sigma|_{\cap_{n\geq 0}D_n},f}(t)$.
(2) *If $f$ depends only on the zeroth coordinate,*

$$\begin{aligned} \zeta_{\Sigma|_{X_M},f}(t)^{-1} &= \prod_{k=1}^{\sharp I_{SM}}\left(1 - \sum_{\mathbf{b}\in J(k)} \exp(-f^*(\mathbf{b}))t^{|\mathbf{b}|}\right) \\ &= \prod_{k=1}^{\sharp I_{SM}}\left(1 - \exp(-f^*(\mathbf{v}(k)))t^{|\mathbf{v}(k)|} - \sum_{n=2}^{\infty} \sum_{k<k_1\ldots k_{n-1}\leq \sharp I_{SM}} S_{kk_1}S_{k_1k_2}\ldots\right. \end{aligned}$$

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$$\dots S_{k_{n-1}k} \exp\left(-\sum_{j=0}^{|\mathbf{v}(k)|+\dots+|\mathbf{v}(k_{n-1})|-1} f\sigma^j p_0(\mathbf{v}(k)\mathbf{v}(k_1)\dots\mathbf{v}(k_{n-1}))\right) \times t^{|\mathbf{v}(k)|+|\mathbf{v}(k_1)|+\dots+|\mathbf{v}(k_{n-1})|}\Bigg),$$

*where $J(k)$ are free orbit bases of subshifts $\Sigma_k$ which give a disjoint decomposition of $\pi((\Sigma|_{X_M})^*_{\mathrm{rec}})$ and $f^*(\mathbf{v}(k)) = \sum_{0\le j\le|\mathbf{v}(k)|-1} f\sigma^j p_0(\mathbf{v}(k))$.*

(3) *Let $R = (R_{ij})$ be a $\sharp I_{SM} \times \sharp I_{SM}$ matrix defined by $R_{ij} = \exp(-f^*(\mathbf{v}(j)))S_{ij}$. If $f$ depends only on the zeroth coordinate, then*

$$\zeta_{\Sigma|_{X_M},f}(t)^{-1} = \prod_{k=1}^{\sharp I_{SM}} \Bigg(1 - \exp(-f^*(\mathbf{v}(k)))t^{|\mathbf{v}(k)|} - \sum_{n=2}^{\infty} \sum_{k<k_1\dots k_{n-1}\le\sharp I_{SM}} R_{kk_1}\dots R_{k_{n-1}k} t^{|\mathbf{v}(k)|+|\mathbf{v}(k_1)|+\dots+|\mathbf{v}(k_{n-1})|}\Bigg).$$

COROLLARY 4.1. *If $S_{ij} = 1$ $(\forall i, j)$, $J = \mathcal{U}^1$ and $f$ depends only on the zeroth coordinate, then*

$$\zeta_{\Sigma|_{X_M},f}(t) = \left(1 - t\sum_{k=1}^{\sharp I_{SM}} \exp(-f^*(\mathbf{v}(k)))\right)^{-1} = \det(I - tR)^{-1}.$$

*Remark G.* For a general function $f : X \to (-\infty, \infty]$, define piecewise constant functions $\underline{f}$ and $\overline{f}$ by

$$\underline{f}(x) = \inf_{y\in X_a} f(y) \quad \text{for } x \in X_a,$$
$$\overline{f}(x) = \sup_{y\in X_a} f(y) \quad \text{for } x \in X_a$$

respectively. Replacing $f$ by $\underline{f}(\overline{f})$ gives upper (lower) bounds of $\zeta_{X,f}$ which can be written as product formulas in Theorem 4.1. So it is easy to see that if $\sum_{a\in I} \exp(-\underline{f}(a)) < \infty$, then $\zeta_{\Sigma_{X_M},f}$ has the radius of convergence $\rho$ satisfying

$$\rho \ge \frac{1}{\sum_{a\in I} \exp(-\underline{f}(a))}.$$

*Remark H.* Let us consider the even system which is a typical example of a finite state sofic shift. In the space of sequences with the index set $\{a, b\}$, $b$'s are required to appear in blocks of even length between two $a$'s. For the system, non-Markov words are $b, bb, bbb, \dots$ and it is easy to see that

$$\zeta_{\Sigma|_{X_M},0}(t)^{-1} = \frac{1 - t - t^2}{1 - t^2}$$

which gives a meromorphic extension to $\mathbb{C}$. For finite state systems (i.e. $\sharp I < \infty$), we have another observation of the analycity property of zeta functions in §7.

Lastly, we refer to $\zeta_{\Sigma_{\cap_{n\ge0}D_n},f}(t)$. It follows from Remark E that $\Sigma_{\cap_{n\ge0}D_n}$ consists of sequences $(a_1 \dots a_n \dots)$ s.t. $X_{a_i} \notin \mathcal{M}^1$ $(\forall i > 0)$. Further, condition (C) implies that

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$\Sigma_{\cap_{n\geq 0}D_n}$ is a subshift of $(\Sigma, \sigma)$ with the index set consisting of symbols for cylinders in $\mathcal{D}_1 (= \mathcal{L}^1 \backslash \mathcal{M}^1)$. In typical examples, we see that $\Sigma_{\cap_{n\geq 0}D_n}$ is extremely simple. In fact we will see this in §6 where we study a multi-dimensional non-Markov (non-hyperbolic) map which is an interesting example in view of number theoretical applications [**5, 25, 28**]. Further, we observe that $(\cap_{n\geq 0} D_n, T)$ is a lower-dimensional subsystem of $(T, X)$.

## 5. *Pressure and the radius of convergence*

*Definition.* A probability measure $\nu$ which is positive on open sets is called a *weak Gibbs measure* for a function $f$ with a constant $P$ if there exists a sequence $\{K_n\}_{n>0}$ of positive numbers with $\lim_{n\to\infty}(1/n)\log K_n = 0$ such that $\nu$ a.e. $x$,

$$K_n^{-1} \leq \frac{\nu(X_{a_1\ldots a_n}(x))}{\exp(\sum_{i=0}^{n-1} fT^i(x) + nP)} \leq K_n,$$

where $X_{a_1\ldots a_n}(x)$ denotes the cylinder of rank $n$ containing $x$.

*Definition.* We say that a function $f$ is of *weak bounded variation* (WBV) if there exists a sequence $\{C_n\}_{n>0}$ satisfying $\lim_{n\to\infty}(1/n)\log C_n = 0$ and for each $n > 0$,

$$\frac{\sup_{x\in X_{a_1\ldots a_n}} \exp(\sum_{i=0}^{n-1} f(T^i x))}{\inf_{x\in X_{a_1\ldots a_n}} \exp(\sum_{i=0}^{n-1} f(T^i x))} \leq C_n \quad (\forall X_{a_1\ldots a_n} \in \mathcal{L}^n).$$

In order to discuss the analogy with thermodynamic formalism, we assume piecewise smoothness on $(T, X, Q, \mathcal{U})$, i.e. $T|_{X_a}$ is a $C^1$-diffeomorphism for all $a \in I$. Beautiful analogies with thermodynamic formalism were known for certain expanding interval maps, and for piecewise expanding maps like Gauss transformations nice analyticity properties of the dynamical zeta functions were established [**9–11**, **14–16**]. The standard technique used there is based on the (generalized) Perron–Frobenius operator and severely relies on Renyi's condition (i.e. $\exists C \geq C_n$ a uniform constant in $n$ for $f(x) = \log|\det DT(x)|^{-1}$). For establishing some specific properties, the Markov property of the definite partition was required. In this section, we direct our attention to certain non-hyperbolic systems; more precisely piecewise invertible multi-dimensional maps which do not satisfy Renyi's condition. As mentioned in §1, we cannot apply the standard method for such systems. Typical examples of such systems come from number theory and admit non-hyperbolic periodic orbits. The next result gives a characterization of weak Gibbs measures.

LEMMA 5.1. *Let $\mathcal{L}_f$ be the generalized Perron–Frobenius operator defined by $\mathcal{L}_f g(x) = \sum_{y\in T^{-1}x} \exp f(y) g(y)$ and let $\mathcal{L}_f^*$ be the dual of $\mathcal{L}_f$. Suppose that $f$ is of weak bounded variation. If $(\nu, P)$ is a solution of the equation $\mathcal{L}_f{}^*\nu = (\exp P)\nu$, then $\nu$ is a weak Gibbs measure for $-P$. Conversely, if $\nu$ is a weak Gibbs measure for $f$ with $-P$ then*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\nu(X_{a_1\ldots a_n}(x))}{\exp(\sum_{i=0}^{n-1} fT^i(x))} = -P$$

*([**29**, Proposition 2.2]).*

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For convenience of notation, we denote $\sum_{x\in \mathrm{Fix}(T^n,X)} \exp(\sum_{i=0}^{n-1} fT^i(x))$ by $Z_n(X, f)$ instead of $Z_n(X, -f)$ in this section. $\zeta_{X,f}(t)$ is defined by $\zeta_{X,f}(t) = \exp(\sum_{n=1}^{\infty}(t^n/n)Z_n(X, f))$ as before. Define for each $n > 0$,

$$\mathcal{A}_n^0 = \{(a_1 \dots a_n) \in \mathcal{A}_n : T^n X_{a_1 \dots a_n} = X\}.$$

THEOREM 5.1. *Let $(T, X, Q = \{X_a\}, \mathcal{U} = \{U_0, \dots U_N\})$ be a piecewise $C^1$-smooth invertible system with FRS. We assume further that:*

(1) *(weak Renyi condition) $\log|\det DT(x)|^{-1}$ satisfies the weak bounded variation property;*

(2) *(transitivity condition) for each $U_j \in \mathcal{U}$ there exists a cylinder $X_{a_1 \dots a_{s_j}} \subset U_j$ such that $T^{s_j} X_{a_1 \dots a_{s_j}} = X$.*

*Suppose that $f$ satisfies the weak bounded variation property and the normalized Lebesgue measure $\lambda$ of $X$ is a weak Gibbs measure for $f$ with a constant $-P$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(X, f) = P.$$

*In particular, $\lambda$ is a weak Gibbs measure for $f(x) = \log|\det DT(x)|^{-1}$ with constant zero so that*

$$\lim_{n\to\infty} \frac{1}{n} \log Z_n(X, f) = 0.$$

Let $\mathcal{F}$ be the $\sigma$-algebra of Borel subsets of $X$ and let $M(X, T)$ denote the collection of all $T$-invariant probability measures on $\mathcal{F}$. $E(X, T)$ denotes the collection of all $T$-invariant ergodic probability measures on $\mathcal{F}$. For a probability measure $m$ and a subalgebra $\mathcal{F}'$ of $\mathcal{F}$, $I_m(\mathcal{F}|\mathcal{F}')$ denotes the conditional information of $\mathcal{F}$ with respect to $\mathcal{F}'$.

*Definition.* We say that $\mu \in M(X, T)$ is an *equilibrium state* for a function $\phi$ if

$$\int (I_\mu(\mathcal{F}|T^{-1}\mathcal{F}) + \phi)\, d\mu \geq \int (I_m(\mathcal{F}|T^{-1}\mathcal{F}) + \phi)\, dm$$

$$(\forall m \in M(X, T) \text{ s.t. } I_m(\mathcal{F}|T^{-1}\mathcal{F}) + \phi \in L^1(m)).$$

The left-hand side is called the *pressure* of $\phi$ and is denoted by $P(\phi)$ (cf. [**22**]).

Now we ask:

*Question 1.* When does the radius of convergence of $\zeta_{X,f}(t)$ relate to $P(f)$?

In order to give a partial answer to this question, first we note the following fact.

PROPOSITION 5.1. *Suppose that all assumptions of Theorem 5.1 are satisfied. If there exists a $T$-invariant ergodic measure $\mu$ satisfying $H_\mu(Q) < \infty$ and $f \in L^1(\mu)$, then the property*

$$\lim_{n\to\infty} \frac{1}{n} \log \frac{\mu(X_{a_1 \dots a_n}(x))}{\lambda(X_{a_1 \dots a_n}(x))} = 0 \quad (\lambda\text{-a.e. } x)$$

*implies that*

$$h_\mu(T) + \int_X f\, d\mu = \lim_{n\to\infty} \frac{1}{n} \log Z_n(X, f) = P.$$

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Next we show the existence of an equilibrium state for $\log|\det DT(x)|^{-1}$ which is ergodic. Assume that $\det DT(x) \neq 0$ for all $x \in X$. Define for $n > 0$ and $(a_1 \dots a_n) \in \mathcal{A}_n$,

$$C(a_1 \dots a_n) \equiv \frac{\sup_{x \in X_{a_1 \dots a_n}} |\det DT^n(x)|}{\inf_{x \in X_{a_1 \dots a_n}} |\det DT^n(x)|}.$$

(Note that $\sup_n \sup_{(a_1 \dots a_n) \in \mathcal{A}_n} C(a_1 \dots a_n) < \infty$ implies the validity of Renyi's condition.)

The following two theorems were established in [**5, 23, 25, 26**].

THEOREM 5.2. (The existence of an ergodic invariant measure equivalent to $\lambda$) *Let $(T, X, Q, \mathcal{U})$ be a piecewise $C^1$-smooth invertible system with FRS and satisfying the transitivity condition. Suppose that there exists $C > 1$ satisfying the following conditions:*

(1) *(The local Renyi condition) If $C(a_1 \dots a_n) \leq C$ and $(b_1 \dots b_m a_1 \dots a_n) \in \mathcal{A}$, then*

$$C(b_1 \dots b_m a_1 \dots a_n) \leq C.$$

(2) *There is a symbol $a \in \mathcal{A}_1^0$ satisfying $C(a) \leq C$.*

(3) *For each $n > 0$ define $\mathcal{D}_n{}^{(C)} = \{X_{d_1 \dots d_n} \in \mathcal{L}^n : C(d_1 \dots d_i) \geq C (1 \leq i \leq n)\}$ and $D_n{}^{(C)} = \bigcup_{X_{d_1 \dots d_n} \in \mathcal{D}_n{}^{(C)}} X_{d_1 \dots d_n}$. Then $\lim_{n \to \infty} \lambda(D_n{}^{(C)}) = 0$.*

*Under the above conditions, $(T, X, Q, \mathcal{U})$ admits an ergodic $\sigma$-finite invariant measure $\mu$ equivalent to $\lambda$ and $(T, \mu)$ is exact. Further, if we assume that $\sum_{n=0}^{\infty} \lambda(D_n^{(C)}) < \infty$, then $\mu$ is finite.*

THEOREM 5.3. (Variational principle) *When Rohlin's formula is valid, the $T$-invariant probability measure $\mu_0 = \mu/\mu(X)$ in Theorem 5.2 satisfies the following variational principle: $\forall m \in E(X, T)$ s.t. $I_m(\mathcal{F}|T^{-1}\mathcal{F}) - \log|\det DT| \in L^1(m)$*

$$h_{\mu_0}(T) - \int_X \log|\det DT(x)|\, d\mu_0(x) \geq \int_X \{I_m(\mathcal{F}|T^{-1}\mathcal{F}) - \log|\det DT(x)|\}\, dm(x).$$

*Remark K.* The validity of Rohlin's entropy formula was established in [**24**].

We assume Theorems 5.2 and 5.3. Then we can apply Theorem 5.1 and Proposition 5.1 to the specific case $f(x) = \log|\det DT(x)|^{-1}$ satisfying the WBV property.

COROLLARY 5.1.

$$\lim_{n \to \infty} \frac{1}{n} \log Z_n(X, f) = P(f) = 0.$$

COROLLARY 5.2. $P(f) = \sup\{\int_X (I_m(\mathcal{F}|T^{-1}\mathcal{F}) + f)\, dm(x) : m \in E(X, T)$ s.t. $I_m(\mathcal{F}|T^{-1}\mathcal{F}) + f \in L^1(m)\}$.

(See [**3**].) We remark that [**29**] contains an answer to Question 1 for $f$ which is not necessarily $-\log|\det DT|$ and the reference measure is not necessarily Lebesgue measure.

In the rest of this section we give proofs of Theorem 5.1 and Proposition 5.1. To prove Theorem 5.1 we need the following lemma (cf. [**13**]).

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LEMMA 5.2. *There exists a sequence* $\{\gamma_m\}_{m\geqq 1}$ *of positive numbers and* $\exists s > 0$ *s.t.*

$$\sum_{(b_1\dots b_m)\in\mathcal{A}_m^0} \lambda(X_{b_1\dots b_m}) \geq \gamma_m \quad (\forall m \geq s)$$

*and*

$$\lim_{m\to\infty} \frac{1}{m} \log \gamma_m = 0.$$

*Proof.* Put $s = \mathrm{lcm}\{s_j : 0 \leq j \leq N\}$. Then for each $U_j \in \mathcal{U}$, there exists a cylinder of rank $s$, $X^{(j)}_{c_1\dots c_5}$ such that $X^{(j)}_{c_1\dots c_s} \subset U_j$ and $T^s X^{(j)}_{c_1\dots c_s} = X$.

First we show that there exists a $\delta_1(k) > 0$ s.t. for all $X_{a(k)}$,

$$\sum_{\substack{b(s)\in\mathcal{A},\\ a(k)b(s)\in\mathcal{A}^0_{k+s}}} \lambda(X_{a(k)b(s)}) \geq \delta_1(k)\lambda(X_{a(k)}).$$

It follows from the weak Renyi condition that

$$\begin{aligned}\sum_{\substack{b(s)\in\mathcal{A},\\ a(k)b(s)\in\mathcal{A}^0_{k+s}}} \lambda(X_{a(k)b(s)}) &\geq \sum_{\substack{b(s)\in\mathcal{A}^0_s,\\ T^kX_{a(k)}\supseteq X_{b(s)}}} \lambda(X_{a(k)b(s)}) \\ &\geq C_k^{-1} \sum_{\substack{X_{b(s)}\in\mathcal{A}^0_s,\\ T^kX_{a(k)}\supseteq X_{b(s)}}} \sup_{x\in T^kX_{a(k)}} |\det D\psi_{a(k)}(x)|\lambda(X_{b(s)}),\end{aligned}$$

where $\psi_{a(k)}$ denotes $(T^k|_{X_{a(k)}})^{-1}$.

Note that

$$\sum_{\substack{b(s)\in\mathcal{A}^0_s,\\ T^kX_{a(k)}\supseteq X_{b(s)}}} \lambda(X_{b(s)}) \geq \min_{0\leq j\leq N} \lambda(X^{(j)}_{c(s)}),$$

where $X^{(j)}_{c(s)} \subseteq U_j$ are the cylinders satisfying $T^s X^{(j)}_{c(s)} = X$. So it is suffices to choose

$$\delta_1(k) = \frac{\min_{0\leq j\leq N} \lambda(X^{(j)}_{c(s)})}{C_k \max_{U\in\mathcal{U}} \lambda(U)}.$$

In particular,

$$\sum_{b(s)\in\mathcal{A}^0_s} \lambda(X_{b(s)}) \geq (N+1) \min_{0\leq j\leq N} \lambda(X^{(j)}_{c(s)}).$$

Let $k > 0$. Then

$$\sum_{b(s+k)\in\mathcal{A}^0_{s+k}} \lambda(X_{b(s+k)}) = \sum_{(b_1\dots b_k)\in\mathcal{A}_k} \sum_{\substack{(b_{k+1}\dots b_{k+s})\in\mathcal{A}^0_s,\\ T^kX_{b_1\dots b_k}\supseteq X_{b_{k+1}\dots b_{k+s}}}} \lambda(X_{b_1\dots b_k b_{k+1}\dots b_{k+s}})$$

so that we again have the same lower bound $\delta_1(k)$. We have finished the proof. □

*Proof of Theorem 5.1.* First we observe that

$$(a_1\dots a_n) \in \mathcal{A}^0_n \Leftrightarrow \underbrace{(a_1\dots a_n)\dots(a_1\dots a_n)}_{k} \in \mathcal{A}^0_{kn}.$$

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Then it follows from the weak bounded variation property of $f$ that we have

$$\begin{aligned} Z_n(X, f) &\geq \sum_{(a_1 \dots a_n) \in \mathcal{A}_n^0} \inf_{x \in X_{a_1 \dots a_n}} \exp\left( \sum_{i=0}^{n-1} f(T^i x) \right) \\ &\geq C_n^{-1} \sum_{(a_1 \dots a_n) \in \mathcal{A}_n^0} \sup_{x \in X_{a_1 \dots a_n}} \exp\left( \sum_{i=0}^{n-1} f(T^i x) \right). \end{aligned}$$

On the other hand, by the generator condition each $X_{a_1 \dots a_n}$ contains only one fixed point of $T^n$ so that we have

$$Z_n(X, f) \leq C_n \sum_{(a_1 \dots a_n) \in \mathcal{A}_n} \inf_{x \in X_{a_1 \dots a_n}} \exp\left( \sum_{i=0}^{n-1} f(T^i x) \right).$$

Then the desired result follows from the definition of the weak Gibbs measure. □

*Proof of Proposition 5.1.* The first assertion is an immediate consequence of the ergodic theorem and the Shannon–McMillan theorem. It follows from Theorem 5.1 that the second assertion is obtained. □

## 6. *Multi-dimensional examples*

We apply the results from the previous sections to some examples of piecewise invertible system with FRS. All of these examples relate to number theory.

*Example 1 (Two-dimensional map related to uniform distribution of cylinder flows).* Let $X = \{(x, y) : 0 < x, y \leq 1\}$ and $T$ be defined by

$$T(x, y) = \left( -\frac{1}{x} - \left[ -\frac{1}{x} \right]_{(-)}, \frac{y}{x} - \left[ \frac{y}{x} \right] \right)$$

where $[x]_{(-)} = \max\{n \in \mathbb{Z} : n < x\}$ and $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$. We remark that we must restrict the domain $X$ to the set

$$\{(x, y) \in X : T^n(x, y) \in X \ (\forall n > 0)\}.$$

However, the Lebesgue measure of this set is equal to one, so we denote the restricted set by $X$ for convenience. Let $a(x) = -[-1/x]_{(-)}$ and $b(x, y) = [y/x]$. The index set $I$ is given by $I = \{(a, b) \in \mathbb{N} \times (\mathbb{N} \cup \{0\}) : a \geq 2, a > b\}$. The partition $Q = \{X_{(a,b)}\}_{(a,b) \in I}$ is defined as

$$(x, y) \in X_{(a,b)} \text{ iff } a(x) = a \text{ and } b(x, y) = b$$

(see Figure 1). Let $U_0 = X$ and $U_1 = \{(x, y) \in X : x + y \leq 1\}$. The appearance of $U_1$ does not allow us to have the Markov property of $Q$. The first component of $T$ is known as a map related to a negative continued fraction (Hirzebruch's continued fraction) which admits a $\sigma$-finite infinite absolutely continuous ergodic invariant measure. The map is also conjugate to the map $S(x) = x/(1-x) \pmod 1$ after applying the conjugacy $\Psi(x) = 1 - x$. $T$ itself also provides a $\sigma$-finite infinite ergodic invariant measure with

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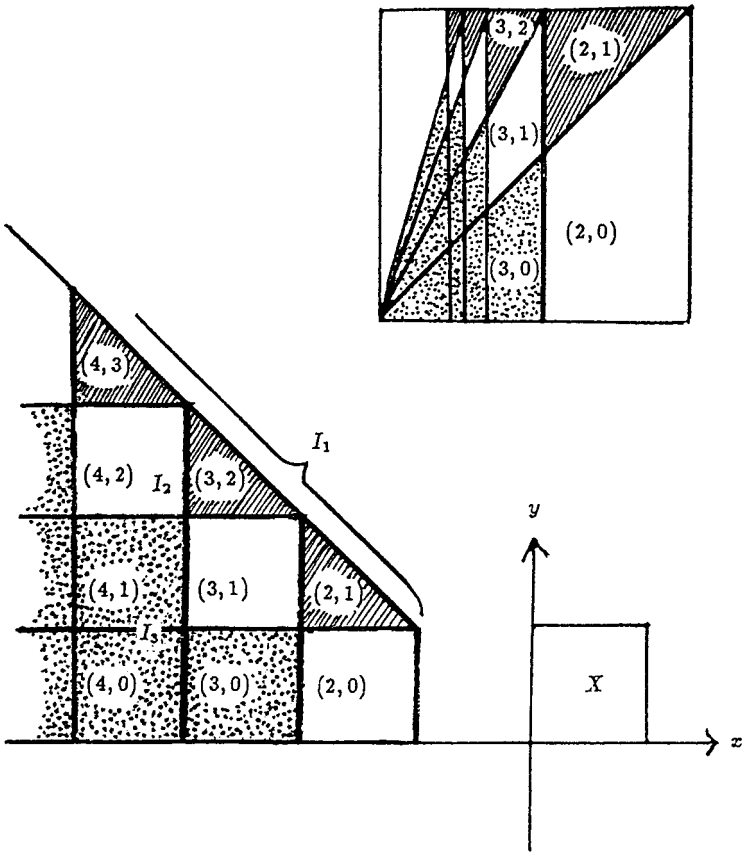



FIGURE 1.

the density $h(x, y)$:

$$h(x, y) = \begin{cases} \dfrac{2-x}{2(1-x)^2} & \text{if } x + y < 1 \\ \dfrac{1}{2(1-x)} & \text{if } x + y > 1 \end{cases}$$

(see [**3, 28**]). For $n \geq 1$, we define $a_n(x) = a(r_{n-1}(x))$, $b_n(x, y) = b(r_{n-1}(x), s_{n-1}(x, y))$, where $(r_n(x), s_n(x, y)) = T^n(x, y)$. Define $c(x, y) = a(x) - b(x, y)$ and $c_n(x, y) = a_n(x) - b_n(x, y)$ inductively. Then the admissibility rule of symbols of $I$ is given as follows. If $c_i = 1$, then $c_{i+1} \neq 1$. Divide $I$ into three groups as follows:

$$I_1 = \{(a, b) \in I : c = 1\}, \quad I_2 = \{(a, b) \in I : c = 2\}, \quad I_3 = \{(a, b) \in I : c > 2\}$$

(Figure 1). Then we can obtain a quotient labelled graph $G^*$ with two vertices and finitely many bundles consisting of countably many edges [**25**] (see Figure 2). From now on, for the set of cylinders whose symbols belong to $I_i$ $(i = 1, 2, 3)$ we use the

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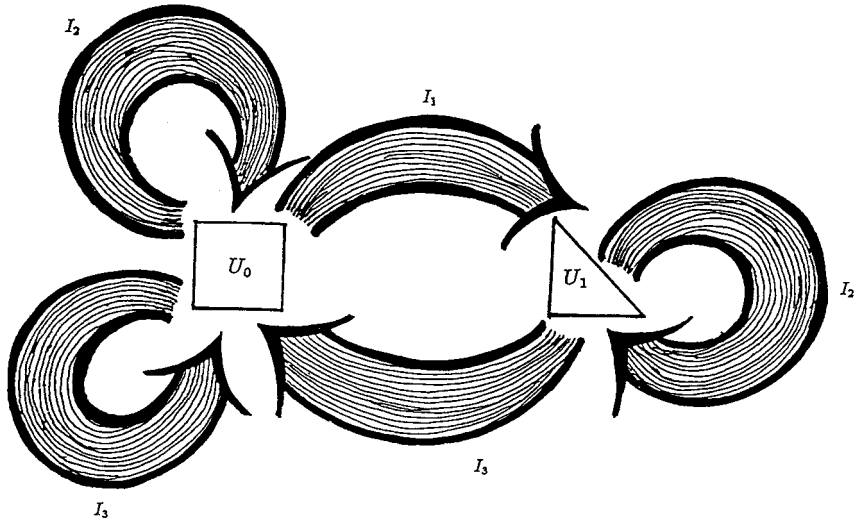



FIGURE 2.

same notation. It is easy to check the following facts:

$$\begin{aligned}
\mathcal{M}(0) &= \mathcal{A} \\
\mathcal{M}(1) &= \bigcup_{n=1}^{\infty}\{((a_1,b_1)\dots(a_n,b_n)) \in \mathcal{A}_n : c_1 \in I_3 \text{ or } c_1 \in I_2 \text{ and } c_i \in I_3 \\
&\qquad \text{for } \exists 1 < i \le n\} \\
Q(0) &= \{((a_1,b_1)\dots(a_n,b_n)) \in \mathcal{A}_n : c_n \notin I_1 \text{ and if } i < n \text{ is the maximal} \\
&\qquad \text{number such that } c_i \in I_1, \text{ then } \exists c_{i+k} \notin I_2\} \\
Q(1) &= \{((a_1,b_1)\dots(a_n,b_n)) \in \mathcal{A}_n : \exists 1 \leqq i \leqq n \text{ such that } c_i \in I_1 \text{ and} \\
&\qquad c_{i+1},\dots,c_n \in I_2\}.
\end{aligned}$$

(In particular, in the case of $c_n \in I_1$, $((a_1,b_1)\dots(a_n,b_n)) \in Q(1)$.)

$$\mathcal{M} = \mathcal{A} \Big\backslash \bigcup_{n=1}^{\infty}\{((a_1,b_1)\dots(a_n,b_n)) \in \mathcal{A}_n : c_i \in I_2 \ \forall i > 0\}.$$

So, $\mathcal{M}$ consists of cylinders which do not intersect $\partial U_1$. As $I_1, I_3 \subset \mathcal{M}$, we have

$$\mathcal{S}\mathcal{M} = I_1 \cup I_3 \cup \bigcup_{k>0}\{I_1 \times \underbrace{I_2 \times \cdots \times I_2}_{k \text{ times}}, I_3 \times \underbrace{I_2 \times \cdots \times I_2}_{k \text{ times}}\},$$

$$V = \bigcup_{m,n>0}\{I_k \times \underbrace{I_2 \times \cdots \times I_2}_{n-1} \times I_l \times \underbrace{I_2 \times \cdots \times I_2}_{m-1} : (k,l) = (1,3) \text{ or } (3,1) \text{ or } (3,3)\}.$$

Similarly, the index set $I_M$ for the jump system $(T_M, X_M, Q_M, \mathcal{U}_M)$ consists of sequences belonging to

$$I_1 \cup I_3 \cup \underbrace{I_2 \times \cdots \times I_2}_{k} \times I_l \quad (l \in \{1,3\}, k > 0).$$

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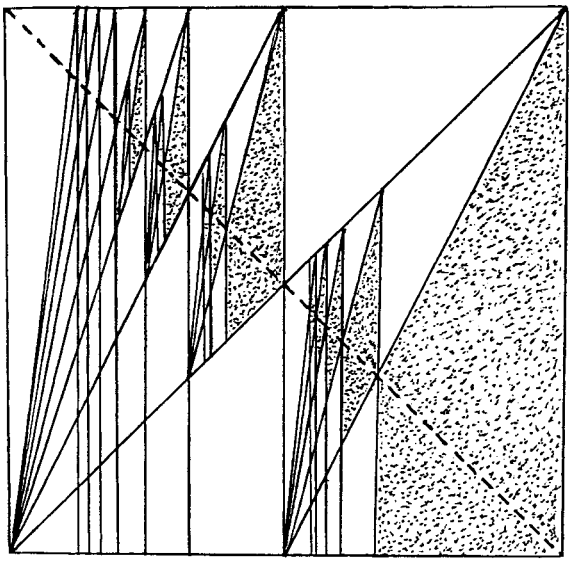


FIGURE 3. The shaded cylinders are non-Markov cylinders of rank 2, i.e. cylinders belonging to $\mathcal{D}_2$.

(see Figure 3).

Next we observe the set of all periodic points $P(X, T)$. We can easily check the following facts.

(1) $\{1\} \times [0, 1]$ consists of indifferent fixed points.

(2) All fixed points in $(0, 1) \times [0, 1]$ belong to $\operatorname{int} X_a$ for some $a \in I$. In fact, each $X_{(a,b)} = \{(x, y) \in X : a^{-1} \leq x \leq (a-1)^{-1}, bx \leq y < (b+1)x\}$ with $a - b > 1$ contains a fixed point

$$\left(\frac{a - \sqrt{a^2 - 4}}{2}, \frac{b}{2}\left(\sqrt{\frac{a+2}{a-2}} - 1\right)\right) \in \operatorname{int} X_{(a,b)}.$$

When $a - b = 1$, $X(a, b)$ does not contain any fixed points.

(3) Put $\Gamma = \{(x, y) \in X : x + y = 1\}$. Then $\Gamma = \cap_{n \geq 0} D_n = \cup_{U \in \mathcal{U}} \partial U \backslash \partial X$ is a $T$-invariant set.

(4) $T((0, 1] \times \{1\}) = \Gamma$.

(5) $(0, 1] \times \{0\}$ is $T$-invariant so $T|_{(0,1]\times\{0\}}$ gives just the one-dimensional map related to the negative continued fraction as mentioned above.

(6) Let $x_0 \in \operatorname{Fix}(X, T^p)$. If $x_0 \in \partial X_{a_1 \ldots a_p}$ for some $X_{a_1 \ldots a_p}$, then $x_0 \in \partial X_{a_1 \ldots a_p} \cap (\cup_{U \in \mathcal{U}} \partial U)$. Thus $x_0$ must belong to $\Gamma \cup ((0, 1] \times \{0\})$. Both subsystems $(\Gamma, T)$ and $((0, 1] \times \{0\}, T)$ admit countable full (Bernoulli) shifts as their symbolic dynamics.

Thus all periodic points on $X \backslash (\{1\} \times [0, 1])$ belong to $\Gamma \cup ((0, 1] \times \{0\}) \cup X_M$.

The following examples admit finite ergodic invariant measures equivalent to $\lambda$ of which densities are unbounded on indifferent periodic points. A sufficient condition for the validity of Rohlin's entropy formula was given in [**24**]. All assumptions for obtained results in §5 are included in those conditions and both of the examples satisfy them.

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Further details about these examples can be found in [**23**–**26**].

*Example 2 (A locally skew product two-dimensional map which is related to Diophantine approximation in inhomogeneous linear class).* Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_2 \le 1, -x_2 \le x_1 \le -x_2 + 1\}$. Define $T$ on $X$ by

$$T(x_1, x_2) = (1/x_1 - [(1 - x_2)/x_1] + [-(x_2/x_1)], -[-(x_2/x_1)] - (x_2/x_1)).$$

The invariant density of the finite invariant measure of $T$ is

$$h(x_1, x_2) = \frac{1}{(2\log 2)(1 - {x_1}^2)},$$

so the singular points of the density $h(x_1, x_2)$ are $(1, 0)$ and $(-1, 1)$. These points are periodic points with period 2 and are indifferent (cf. [**24**]). Let

$$\begin{aligned} a(x_1, x_2) &= [(1 - x_2)/x_1] - [-(x_2/x_1)], \quad b(x_1, x_2) = -[-(x_2/x_1)], \\ a_n(x_1, x_2) &= a(T^{n-1}(x_1, x_2)), \quad b_n(x_1, x_2) = b(T^{n-1}(x_1, x_2)). \end{aligned}$$

On a cylinder $X_{(a,b)}$, $T(x_1, x_2) = ((1/x_1) - a, -(x_2/x_1) + b)$, and for any $X_{a_1 \dots a_n} \in \mathcal{L}^n$, $T^n X_{a_1 \dots a_n} = X$; that is $T$ is a Bernoulli map.

*Example 3 (A real two-dimensional map which is related to a complex continued fraction expansion by Tanaka [**21**]).* Let $X = \{z = x_1\alpha + x_2\overline{\alpha} : -(1/2) \le x_1, x_2 \le 1/2\}$ ($\alpha = 1 + i$) and define $T$ on $X$ by $Tz = 1/z - [1/z]_1$, where $[z]_1$ denotes $[x_1 + 1/2]\alpha + [x_2 + 1/2]\overline{\alpha}$ for a complex number $z = x_1\alpha + x_2\overline{\alpha}$. Let the index set $I$ be $I = \{n\alpha + m\overline{\alpha} : m, n \in \mathbb{Z}\} \setminus \{0\}$. $T$ induces a continued fraction expansion of $z \in X$:

$$z = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots \cfrac{1}{a_n + \cdots}}}},$$

where each $a_i$ is contained in $I$. $T$ has indifferent periodic points $\pm 1$, $\pm i$ and the invariant density admits singularities at these points and no others [**24**].

*Appendix. Subshift of finite range*

Let $G$ be a labelled graph with finitely many vertices and infinitely many edges, satisfying the following properties.

(1) The successor set of the vertices are distinct. (Here a successor set $\mathcal{F}(v)$ of a vertex $v$ is defined by the set of infinite sequences $a_1 a_2 \dots$ which label paths beginning at $v$.)
(2) All vertices are terminal vertices of paths in $G$ (i.e. for any vertex $v$ there exists a path in $G$ ending at $v$).
(3) $G$ is a right resolving graph (i.e. no two edges beginning at the same vertex have the same label).

Let $\overline{\Sigma}$ be the space of all infinite sequences $y = \dots y_{-1} y_0 y_1 \dots$ which label paths in $G$. Let $\overline{\Sigma}_{(-)}$ be the set of all left semi-infinite sequences appearing in points of

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$\overline{\Sigma}$. For $(\ldots a_{-n}\ldots a_{-1}) \in \overline{\Sigma}_{(-)}$, $\mathcal{F}(\ldots a_{-n}\ldots a_{-1})$ denotes the set of right semi-infinite sequences $(a_0a_1\ldots)$ such that $(\ldots a_{n-1}\ldots a_{-1})(a_0a_1\ldots) \in \overline{\Sigma}$. Let $\mathcal{F}(a_{-n}\ldots a_{-1})$ be the set of right semi-infinite sequences $(a_0a_1\ldots)$ such that $(a_{-n}\ldots a_{-1})(a_0a_1\ldots)$ appears in points of $\overline{\Sigma}$. Observe that

$$\mathcal{F}(a_{-n}\ldots a_{-1}) \supseteq \mathcal{F}(a_{-(n+1)}a_{-n}\ldots a_{-1}).$$

Then we have

$$\lim_{n\to\infty} \mathcal{F}(a_{-n}\ldots a_{-1}) = \cap_{n>0}\mathcal{F}(a_{-n}\ldots a_{-1}) = \mathcal{F}(\ldots a_{-n}\ldots a_{-1}).$$

Define an equivalence relation over $\overline{\Sigma}_{(-)}$ by

$$(\ldots a_{-n}\ldots a_{-1}) \sim (\ldots b_{-n}\ldots b_{-1}) \Leftrightarrow \mathcal{F}(\ldots a_{-n}\ldots a_{-1}) = \mathcal{F}(\ldots b_{-n}\ldots b_{-1}).$$

As the number of vertices of $G$ is finite, there are only finitely many equivalence classes. This is one of the well-known characterizations of (finite state) sofic shifts (cf. [**8**]).

*Definition.* We call a subshift which is obtained from a labelled graph satisfying (1)–(3) a *subshift of finite range*.

From the above observation, it is reasonable to call such a shift a countable state sofic shift. As we have already shown in Theorem 2.1, piecewise invertible systems with FRS naturally provide their Markov covers. Similarly, it is easy to find Markov covers of subshifts of finite range. In fact, let $\mathcal{V} = \{v_0\ldots v_N\}$ be vertices and let $I$ be the arc set. Then the Markov cover of the shift $(\overline{\Sigma}, \overline{\sigma})$ determined by $G$ has the state set $\{(v_i, a) \in \mathcal{V}\times I$ : there is a path starting at $v_i$ which is labelled by $a \in I\}$. On the other hand, a Markov word for a subshift of finite range can be characterized as follows: all paths labelled by it end at the same vertex. The next theorem shows the existence of a Markov word and insists that a Markov word is just the *right resolving block* in symbolic dynamics.

THEOREM A.1. *Assume that $G$ satisfies (1)–(3). Then there exists a word $m = m_1\ldots m_q$ such that if $y$ is an infinite sequence which labels an infinite path in $G$, and $y_1\ldots y_q = m$, then for any infinite paths $x, x'$ in $G$ which are labelled by $y$, we have $x_i = x'_i$ for $i > q$. (The word $m$ is sometimes called a* magic word.*) Let $y = \ldots y_{-1}y_0y_1\ldots \in \overline{\Sigma}$ and $y_{n+1}\ldots y_{n+q} = m$ for infinitely many negative integers $n$. Then there is only one infinite path in $G$ labelled by $y$.*

*Proof.* We can prove the existence along the lines of Theorem 3.1. The last assertion is an immediate consequence of the definition of the Markov word. □

Combining the above observation and the fact that

$$P(\Sigma, \sigma) = P(\Sigma|_{X_M}, \sigma) \cup P(\Sigma|_{\cap_{n\geq 0}D_n}, \sigma)$$

we can easily see the rationality of zeta functions of finite state sofic shifts when the zeta function for $\sigma|_{\cap_{n\geq 0}D_n}$ is a rational function. Let the Markov cover be determined by a matrix $A$. For typical examples, the subsystem $(\Sigma|_{\cap_{n\geq 0}D_n}, \sigma)$ is just SFT and they admit a matrix $A'$ s.t. $\sharp\,\mathrm{Fix}(\Sigma|_{\cap_{n\geq 0}D_n}, \sigma^m) = \mathrm{Trace}(A'^m)$ which gives the rationality of $\zeta_{\Sigma,0}(t)$.

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