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A rigidity theorem for Haken manifolds

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A compact, orientable 3-manifold M is called *hyperbolic* if $\mathrm{int}M$ admits a complete hyperbolic structure (Riemannian metric of constant curvature -1) of finite volume. Any hyperbolic 3-manifold M is irreducible, and each component of ∂M is an incompressible torus. Let $f: M \to N$ be a proper, continuous map between hyperbolic 3-manifolds. By Mostow's Rigidity Theorem [**8**], if f is π_1-isomorphic then f is properly homotopic to a diffeomorphism $g: M \to N$ such that $g|\mathrm{int}M: \mathrm{int}M \to \mathrm{int}N$ is isometric. In particular, the topological type of $\mathrm{int}M$ determines uniquely the hyperbolic structure on $\mathrm{int}M$ up to isometry, so the volume $\mathrm{vol}(\mathrm{int}M)$ of $\mathrm{int}M$ is well-defined. This Rigidity Theorem is generalized by Thurston [**11**, theorem 6·4] so that any proper, continuous map $f: M \to N$ between hyperbolic 3-manifolds with $\mathrm{vol}(\mathrm{int}M) = \deg(f)\,\mathrm{vol}(\mathrm{int}N)$ is properly homotopic to a $\deg(f)$-fold covering $g: M \to N$ such that $g|\mathrm{int}M: \mathrm{int}M \to \mathrm{int}N$ is locally isometric.

The aim of this paper is to obtain the corresponding result for a proper, continuous map $f: M \to N$ between Haken manifolds with toral boundary. Here M *Haken* means that M is a connected, orientable and irreducible 3-manifold containing an incompressible surface. The 'volume' for a Haken manifold M with toral boundary is Gromov's simplicial volume (or the Gromov invariant) $\|M\|$, see Gromov [**3**] and Thurston [**11**], and the 'hyperbolic structure' on M is the union $\mathscr{H}(M)$ of hyperbolic pieces for a torus decomposition of M, see Jaco-Shalen [**6**], Johannson [**7**] and Thurston [**12**]. We always assume that any two hyperbolic pieces H_i, H_j of $\mathscr{H}(M)$ are mutually disjoint. In fact, when $\partial H_i \cap \partial H_j \neq \varnothing$, we insert a union of new Seifert pieces diffeomorphic to $(\partial H_i \cap \partial H_j) \times [0,1]$ between H_i and H_j. Similarly we may assume that $\mathscr{H}(M) \cap \partial M = \varnothing$. Then, our main theorem is stated as follows.

Theorem. *Let M and N be Haken manifolds with (possibly empty) toral boundary, and let $f: M \to N$ be a proper, continuous map of* $\deg(f) > 0$. *Then, f is properly homotopic to a map $g: M \to N$ with $g(\mathscr{H}(M)) = \mathscr{H}(N)$ and such that the restriction $g|\mathscr{H}(M): \mathscr{H}(M) \to \mathscr{H}(N)$ is a* $\deg(f)$*-fold covering if and only if* $\|M\| = \deg(f)\,\|N\|$.

This theorem is especially significant in the case of $\deg(f) = 1$. Then the equality $\|M\| = \|N\|$ implies that f is homotopic to a map g such that $g|\mathscr{H}(M): \mathscr{H}(M) \to \mathscr{H}(N)$ is a diffeomorphism. This property is useful for investigating degree-one maps between Haken manifolds. For example, in [**9**, §2·3], Rong gives some necessary conditions that a degree-one, continuous map $f: M \to N$ between closed Haken manifolds is homotopic to a homeomorphism under the assumption that $\mathscr{H}(M)$ is homeomorphic to $\mathscr{H}(N)$. By our theorem, one can replace this assumption by that on the Gromov invariant which is a general invariant defined directly without respecting torus decompositions on M and N.

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The following corollary is immediate from the Theorem.

COROLLARY. *Let $f: M \to N$ be a proper, continuous map of* $\deg(f) > 0$ *between a hyperbolic 3-manifold M and a Haken manifold N. Then f is properly homotopic to a* $\deg(f)$*-fold covering if and only if* $\|M\| = \deg(f)\,\|N\|$.

First of all, we present an outline of the proof of our main theorem. Let $f: M \to N$ be a proper map of $\deg(f) > 0$ as in the Theorem, and let $\mathcal{S}(M), \mathcal{S}(N)$ be the unions of Seifert pieces in M and N respectively. By Gromov's Cutting-off Theorem [3] and Thurston's results in [11], $\|M\| = \|\mathcal{H}(M)\| = \mathrm{vol}(\mathrm{int}\,\mathcal{H}(M))/v_3$ and $\|N\| = \|\mathcal{H}(N)\| = \mathrm{vol}(\mathrm{int}\,\mathcal{H}(N))/v_3$, where v_3 is the volume of a regular ideal simplex in $\mathbb{H}^3$. The proof of the 'only if' side is immediate from these facts. Consider a minimal union $\mathcal{U}_N$ of tori determining a torus decomposition on N. In Section 2, Lemma 3 shows that if $\|M\| = \deg(f)\,\|N\|$ then f can be deformed so that each component of $f^{-1}(\mathcal{U}_N)$ is an incompressible surface in M disjoint from $\partial\mathcal{H}(M)$.

The essential part of the proof of the 'if' side is to prove $f^{-1}(\mathcal{U}_N) \cap \mathrm{int}\,\mathcal{H}(M) = \varnothing$ if necessary after deforming f by a proper homotopy. We shall suppose that $f^{-1}(\mathcal{U}_N) \cap \mathrm{int}\,\mathcal{H}(M)$ contains a surface F of genus $(F) > 1$ and reach a contradiction by using the condition that f 'collapses' F to a tours $f(F) \subset \mathcal{U}_N$.

In Section 3, we define certain 3-chains z_H on $\mathcal{H}(H)$ and z_S on $\mathcal{S}(M)$ with $\|z_H\| \approx \|\mathcal{H}(M)\|, \|z_S\| \approx 0$. This z_H is a usual (finite) 3-chain approximating a Borel measure 3-chain z_L on $\mathcal{H}(M)$ defined in [11, chapter 6] to determine the value of $\|\mathcal{H}(M)\|$. Note that z_H is fairly easily treated comparing with z_L. The fundamental 3-cycle z_M on M is defined by joining z_H and z_S in $\partial\mathcal{H}(M)$, so that

$$\|z_M\| \approx \|z_H\| + \|z_S\| \approx \|\mathcal{H}(M)\| = \|M\|.$$

In Section 4, a Riemannian metric g_N on N is defined so that the restriction $g_N | \mathcal{H}(N)$ is isometric to a thick part of $\mathrm{int}\,\mathcal{H}(N)$, and $g_N | \mathcal{S}(N)$ has a very small volume. In particular, we have $\mathrm{vol}(N) \approx \mathrm{vol}(\mathrm{int}\,\mathcal{H}(N)) = \|N\|\,v_3$. The hyperbolic structure $g_N | \mathcal{H}(N)$ is used to define straight singular simplices in N.

In Section 5, we rearrange z_M so that, by using the resulting 3-cycle $z_{M,\mathrm{red}}$, the 'straight' 3-cycle $z_N = \Phi(z_{M,\mathrm{red}}) + d$ on N is well defined, where $\Phi = \mathrm{straight}_H \circ f_*$ is the straightening map defined in Section 5 and d is a small 3-chain on $\partial\mathcal{H}(N)$ added only to make z_N a cycle, and z_N satisfies the following.

For any singular 3-simplex σ in z_N

$$\mathrm{vol}(\sigma) \lesssim v_3, \quad \text{and} \quad \mathrm{vol}(\sigma) \approx 0 \text{ if } \sigma \text{ meets } f(F) \text{ 'essentially'}. \qquad (*), (**)$$

In Section 6, it is shown that the existence of F as above implies $\|M\| > \deg(f)\,\|N\|$. This is done by estimating the integral $\langle z_N, \Omega_N \rangle$ from above, where Ω_N is the volume form on (N, g_N). Since z_N represents $\deg(f)\,[N, \partial N] \in H_3(N, \partial N; \mathbb{R})$,

$$\langle z_N, \Omega_N \rangle = \deg(f)\,\mathrm{vol}(N) \approx \deg(f)\,\|N\|\,v_3.$$

Let $\Phi(q)$ is the part of z_N meeting $f(F)$ essentially, and let $\Phi(p) = z_N - \Phi(q) - d$. Since $\|z_N\| \lesssim \|z_{M,\mathrm{red}}\| \leqslant \|z_M\| \approx \|M\|, \|\Phi(p)\| \approx \|z_N\| - \|\Phi(q)\|$ is strictly less than $\|M\|$. Since, by $(*)$ and $(**)$, $\langle z_N, \Omega_N \rangle \lesssim \|\Phi(p)\|\,v_3$, $\langle z_N, \Omega_N \rangle$ is strictly less than $\|M\|\,v_3$. This implies our desired inequality.

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1. *Preliminaries*

In this section, we briefly present some definitions and properties on 3-manifold topology and hyperbolic geometry which will be used later. We refer to Hempel[**4**] and Jaco[**5**] for other fundamental definitions and notation on 3-manifold topology, and to Thurston[**11**] for those on hyperbolic geometry.

For a subspace Y of a topological space X, we denote by $\mathrm{cl}(X-Y)$ the closure of $X-Y$ in X.

For a proper submanifold S of a smooth manifold M, let $\mathcal{N}(S,M)$ (for short $\mathcal{N}(S)$) denote an open regular neighbourhood of S in M. In the case where S is a union of components ∂M, let $\mathcal{N}_{\mathrm{collar}}(S,M)$ denote an open collar neighbourhood of S in M.

We suppose that M is a Haken manifold with toral boundary. For a torus T^2 and an annulus A, a proper, continuous map $f: T^2 \to M$ (respectively $f: A \to M$) is *essential* if f is π_1-injective and if f is not properly homotopic to any map g with $\mathrm{image}(g) \subset \partial M$. We say that M is *homotopically atoroidal* (respectively *homotopically acylindrical*) if M does not admit any essential maps $f: T^2 \to M$ (respectively $f: A \to M$). If an essential map f is an embedding, then the image $f(T^2)$ (respectively $f(A)$) is called an *essential torus* (respectively *essential annulus*). A union $\mathcal{U} = T_1 \cup \ldots \cup T_n$ is called *essential* in M if it is a union of mutually disjoint, mutually non-parallel and essential tori T_i in M.

According to Thurston's Uniformization Theorem [**12**] and the Torus Decomposition Theorem (Jaco-Shalen[**6**], Johannson[**7**]), there exists an essential union $\mathcal{U}$ of tori in M such that, for the closure P of each component of $M-\mathcal{U}$, P is either Seifert-fibered or hyperbolic. In the former case P is called a *Seifert piece*, and in the latter a *hyperbolic piece*. We say that $\mathcal{U}$ is a union *determining a torus decomposition* on M. Let $\mathcal{H}(M)$ (respectively $\mathcal{S}(M)$) be the union of all hyperbolic pieces (respectively Seifert pieces). As assumed in the introduction, $\mathcal{H}(M)$ is a disjoint union of hyperbolic pieces with $\mathcal{H}(M) \cap \partial M = \varnothing$. In particular, $\partial\mathcal{H}(M)$ is contained in $\partial\mathcal{S}(M)$. Here, for any Seifert piece S_j diffeomorphic to $T^2 \times I$ between hyperbolic pieces H_i and H_k with $\partial S_j \subset \partial H_i \cup \partial H_j$, we suppose that $S_j \cap \mathcal{U}$ is a single torus in $\mathrm{int}\, S_j$. An essential union $\mathcal{U}_M$ is called *minimal* if it is one with minimal number of tori among all unions determining a torus decomposition on M. A minimal union of every Haken manifold with toral boundary is unique up to ambient isotopy.

We briefly review the second definition of the Gromov invariant given in [**11**, 6·1·8]. Let M be a compact, oriented, smooth manifold with possibly non-empty boundary. Let Δ^k be a regular k-simplex of edge length 1 in the Euclidean k-space. A *k-chain on M* is a Borel measure μ on $C^1(\Delta^k, M)$ with compact support and bounded total variation. Here the *total variation* $\|\mu\|$ of μ is

$$\|\mu\| = \int_{C^1(\Delta^k, M)} d\mu_+ + \int_{C^1(\Delta^k, M)} d\mu_-,$$

where μ_+, μ_- are the positive and negative parts of μ respectively, that is, $\mu_+ \geqq 0, \mu_- \geqq 0$ and $\mu = \mu_+ - \mu_-$. We denote by $\mathcal{C}_k(M)$ the set of k-chains on M. This set $\mathcal{C}_k(M)$ is an $\mathbb{R}$-vector space. The boundary operator $\partial_k: \mathcal{C}_k(M, \partial M) \to \mathcal{C}_{k-1}(M, \partial M)$ is naturally defined, where $\mathcal{C}_k(M, \partial M) = \mathcal{C}_k(M)/\mathcal{C}_k(\partial M)$. We denote the set of k-cycles by $\mathcal{Z}_k(M, \partial M)$. The homology determined by $(\mathcal{C}_*(M, \partial M), \partial_*)$ is equal to the standard

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homology $H_*(M, \partial M; \mathbb{R})$ for M. Here, the duality between the homology and cohomology for M is given by

$$\langle \mu, \omega \rangle = \int_{C^1(\Delta^k, M)} \left(\int_{\Delta^k} \sigma^*(\omega) \right) d\mu(\sigma),$$

where $\mu \in \mathscr{Z}_k(M, \partial M)$ and ω is a closed k-form on M with $\sigma^*(\omega) = 0$ for all $\sigma \in C^1(\Delta^k, \partial M)$. The usual singular C^1-chain $\sum_{i=1}^n r_i \sigma_i$ with $r_i \in \mathbb{R}$ and $\sigma_i \in C^1(\Delta^k, M)$ is naturally identified with

$$\sum_{i=1}^n r_i \delta_{\sigma_i} \in \mathscr{C}_k(M),$$

where δ_{σ_i} is the Dirac measure on $C^1(\Delta^k, M)$ with $\delta_{\sigma_i}(\{\sigma_i\}) = 1$. We say that such a chain is a *finite element* of $\mathscr{C}_*(M)$. Let $\mathscr{C}^f_*(M)$ denote the subset of $\mathscr{C}_*(M)$ consisting of all finite elements. For any $c = \sum_{i=1}^n r_i \sigma_i \in \mathscr{C}^f_k(M)$, we set $\mathrm{basis}(c) = \{\sigma_1, \ldots, \sigma_n\}$. *Gromov's norm* $\|\alpha\|$ for $\alpha \in H_k(M, \partial M; \mathbb{R})$ is defined by

$$\|\alpha\| = \inf\{\|\mu\| ; \mu \in \mathscr{C}_k(M) \text{ is a } k\text{-cycle in } \mathscr{Z}_k(M, \partial M) \text{ representing } \alpha\}.$$

If M is a compact, oriented n-manifold, then the *Gromov invariant* $\|M\|$ for M is Gromov's norm $\|[M, \partial M]\|$ for the fundamental class $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$.

From now on, we consider only the case where M is a compact, oriented, Riemannian 3-manifold with toral boundary. If $\sigma \in C^1(\Delta^3, M)$, then the volume

$$\mathrm{vol}(\sigma) = \int_{\Delta^3} \sigma^*(\Omega_M)$$

is well defined, where Ω_M is the volume form for M. Later, we will also use the *absolute volume* of σ defined by

$$|\mathrm{vol}|(\sigma) = \int_{\Delta^3} |\sigma^*(\Omega_M)|.$$

Obviously, we have $|\mathrm{vol}|(\sigma) \geqslant |\mathrm{vol}(\sigma)|$, and both the maps $\mathrm{vol}: C^1(\Delta^3, M) \to \mathbb{R}$ and $|\mathrm{vol}|: C^1(\Delta^3, M) \to \mathbb{R}$ are continuous. The duality between $\mu \in \mathscr{Z}_3(M, \partial M)$ and Ω_M is given by

$$\langle \mu, \Omega_M \rangle = \int_{C^1(\Delta^3, M)} \mathrm{vol}(\sigma)\, d\mu(\sigma).$$

A *Kleinian group* Γ is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$, the group of orientation preserving isometries on the hyperbolic 3-space $\mathbb{H}^3$. Γ acts conformally on the sphere $S^2_\infty = \mathbb{C} \cup \{\infty\}$ at infinity. In this paper, any Kleinian groups are always assumed to be torsion free. Let $p: \mathbb{H}^3 \to H = \mathbb{H}^3/\Gamma$ be the universal covering. If G_0 is the isotropy subgroup of the identity in $\mathrm{Isom}^+(\mathbb{H}^3)$, then $G_0 \backslash \mathrm{Isom}^+(\mathbb{H}^3)$ is naturally identified with $\mathbb{H}^3$. Since $\mathrm{Isom}^+(\mathbb{H}^3)$ is semi-simple, there exists a Haar measure μ_0 on $\mathrm{Isom}^+(\mathbb{H}^3)$ invariant under left and right translations which is normalized so that

$$\mu_0(q^{-1}(W)) = \mathrm{vol}(W) \tag{1·1}$$

for any measurable subset W on $\mathbb{H}^3$, where $q: \mathrm{Isom}^+(\mathbb{H}^3) \to \mathbb{H}^3$ is the natural quotient map. For any $\sigma \in C^1(\Delta^k, \mathbb{H}^3)$, let $\mathrm{smear}_H(\sigma)$ be the k-chain on H defined in [11, §6·2] with $\mathrm{supp}(\mathrm{smear}_H(\sigma)) = \{p \circ h \circ \sigma ; h \in \mathrm{Isom}^+(\mathbb{H}^3)\}$ and such that, for any open subset U of $\mathrm{Isom}^+(\mathbb{H}^3)$ with $r | U$ injective,

$$\mathrm{smear}_H(\sigma)(C) = \mu_0(U), \tag{1·2}$$

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where $r\colon \mathrm{Isom}^+(\mathbb{H}^3) \to \mathrm{Isom}^+(\mathbb{H}^3)/\Gamma$ is the natural quotient map and $C = \{p \circ h \circ \sigma;\ h \in U\}$. If σ is a regular, straight 3-simplex, then

$$\frac{1}{2}\,\mathrm{smear}_H(\sigma - \sigma_-) = \frac{1}{2}\,(\mathrm{smear}_H(\sigma) - \mathrm{smear}_H(\sigma_-))$$

is a 3-cycle on H, where σ_- is the mirror image of σ in $\mathbb{H}^3$. By (1·1) and (1·2),

$$\|\frac{1}{2}\,\mathrm{smear}_H(\sigma - \sigma_-)\| = \mathrm{vol}(H).$$

Note that, when H is non-compact, the support of $\frac{1}{2}\mathrm{smear}(\sigma - \sigma_-)$ is also non-compact and hence not an element of $\mathscr{Z}_3(H)$.

For a smooth map (at least in C^1-class) $f\colon M \to N$ between Riemannian 3-manifolds (M, g_M) and (N, g_N), the *Lipschitz constant* $\mathrm{Lip}(f)$ of f is defined by

$$\mathrm{Lip}(f) = \max\{\sqrt{[g_N(f_*(v), f_*(v))]};\ x \in M \quad \text{and} \quad v \in TU_x(M)\},$$

where $TU(M)$ is the unit tangent bundle over M. If $\mathrm{Lip}(f) = \alpha^{1/3}$, then, for any $\sigma \in C^1(\Delta^3, M)$, $|\mathrm{vol}|(f \circ \sigma) \leqslant \alpha\,|\mathrm{vol}|(\sigma)$, but in general, the inequality

$$|\mathrm{vol}(f \circ \sigma)| \leqslant \alpha\,|\mathrm{vol}(\sigma)|$$

does not hold. So, in certain cases, we prefer to use the absolute volume $|\mathrm{vol}|(\sigma)$ rather than the usual volume $\mathrm{vol}(\sigma)$.

2. *Elementary lemmas*

Throughout the remainder of this paper, we suppose that $f\colon M \to N$ is a proper, smooth map of $\deg(f) > 0$ between (oriented) Haken manifolds with (possibly empty) toral boundary. By the Cutting-off Theorem of Gromov [**3**, §4·2], for any incompressible torus T in M, the Gromov invariant $\|\hat{M}\|$ for the compact 3-manifold $\hat{M}$ obtained by splitting M along T is equal to $\|M\|$. By [**11**, theorem 6·5·4], if H is a hyperbolic 3-manifold, then $\|H\| = \mathrm{vol}(\mathrm{int}\,H)/v_3$, where v_3 is the volume of a regular ideal simplex in $\mathbb{H}^3 \cup S^2_\infty$. By [**11**, corollary 6·5·7], if S is a Seifert fibered space, then $\|S\| = 0$. Therefore, for any Haken manifold M with toral boundary, we have

$$\|M\| = \|\mathscr{H}(M)\| = \frac{\mathrm{vol}(\mathrm{int}\,\mathscr{H}(M))}{v_3}.$$

If the restriction $f|\mathscr{H}(M)\colon \mathscr{H}(M) \to \mathscr{H}(N)$ is a $\deg(f)$-fold covering, then by the definition of the Gromov invariant, $\|\mathscr{H}(M)\| = \deg(f)\,\|\mathscr{H}(N)\|$. This implies the 'only if' part of our main theorem.

From now on, we always assume that $\|M\| = \deg(f)\,\|N\|$ and will prove the 'if' part. If $\|M\| = \deg(f)\,\|N\| = 0$, then $\mathscr{H}(M) = \mathscr{H}(N) = \varnothing$, and hence the proof is obvious. So we may also assume that $\|M\| = \deg(f)\,\|N\| > 0$.

In this section we will give some elementary lemmas. These will be presented in forms suitable to prove the main theorem.

LEMMA 1. *Let $g\colon N_1 \to N_2$ be a proper, continuous map of $\deg(g) > 0$ between a surface bundle N_1 over S^1 with fiber F of genus $(F) \geqslant 1$ and a Haken manifold N_2 such that g is transverse to an incompressible, connected, closed surface G in $\mathrm{int}\,N_2$. If $g^{-1}(G)$ contains F, then N_2 is finitely covered by a surface bundle over S^1 with fibre corresponding to G.*

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Proof. By [**4**, lemma 6·5], we may assume that each component of $g^{-1}(G)$ is incompressible in N_1. Since G is a closed surface, F is also closed. Set $R_2 = \mathrm{cl}\,(N_2 - \mathcal{N}(G))$ and $R_1 = g^{-1}(R_2)$. Note that each component C_1 of R_1 is homeomorphic to $F_0 \times [0,1]$, where F_0 is a component of ∂C_1. For each component C_2 of R_2, $g^{-1}(C_2)$ contains a component C_1 with $\deg(g|C_1) > 0$. For the lift $\widetilde{g|C_1}$ of $g|C_1$ to the covering $p: \tilde{C}_2 \to C_2$ corresponding to $g_*(\pi_1(C_1))$, since $p \circ \widetilde{g|C_1} = g|C_1$ has positive degree, $\tilde{C}_2$ is compact, or equivalently the index $[\pi_1(C_2): g_*(\pi_1(C_1))]$ is finite. For each component G_0 of ∂C_2, there exists a component F_0 of $\partial C_1 \cap g^{-1}(G_0)$ with $\deg(g|F_0) > 0$. This also implies that $[\pi_1(G_0): g_*(\pi_1(F_0))] < \infty$. Since $g_*(\pi_1(F_0)) = g_*(\pi_1(C_1))$,

$$[\pi_1(C_2): \pi_1(G_0)] < \infty.$$

By [**4**, theorem 10·5], C_2 is either homeomorphic to $G_0 \times [0,1]$ or doubly covered by $G_0 \times [0,1]$. The conclusion is easily obtained from this fact. ∎

Let N_1, N_2 be oriented, compact 3-manifolds, and let $y \in \mathrm{int}\, N_2$ be a regular value for a proper, smooth map $g: N_1 \to N_2$. The local degree $\deg(g|x)$ of g at $x \in g^{-1}(y)$ is defined to be 1 if the restriction of g on a small neighbourhood of x in N_1 is an orientation preserving diffeomorphism onto a neighbourhood of y in N_2, and otherwise $\deg(g|x) = -1$. For a subset A of $g^{-1}(y)$, we set

$$\deg(g|A) = \sum_{x \in A} \deg(g|x).$$

It is well known that $\deg(g) = \deg(g|g^{-1}(y))$.

LEMMA 2. *For any component B_0 of $\partial\mathcal{H}(M)$, the restriction $f|B_0: B_0 \to N$ is π_1-injective.*

Proof. Let $\phi: \pi_1(B_0) \to \pi_1(N)$ be the homomorphism induced from $f|B_0$. If ϕ were not injective, then there would exist a non-trivial element $\mu \in \pi_1(B_0)$ such that $\phi(\mu) = 1$ in $\pi_1(N)$. Then there exists the element $\mu_0 \in \pi_1(B_0)$ represented by a simple, non-contractible, oriented loop l_0 in B_0 with $\mu = \mu_0^n$ for some $n \in \mathbb{N}$. Since N is an irreducible 3-manifold with $\pi_1(N)$ infinite, by [**4**, corollary 9·9] $\pi_1(N)$ is torsion-free. Thus, $\phi(\mu_0)^n = \phi(\mu) = 1$ implies that $\phi(\mu_0) = 1$. Let $\hat{M}$ be the compact manifold obtained by splitting M along B_0, and let $q: \hat{M} \to M$ be the quotient map. We set $q^{-1}(B_0) = B_0^+ \cup B_0^-$, $l_0^+ = q^{-1}(l_0) \cap B_0^+$ and $l_0^- = q^{-1}(l_0) \cap B_0^-$. The 3-manifold M_1 is obtained by attaching two copies V^+, V^- of a solid torus V to $\hat{M}$ by diffeomorphisms $h^\pm: \partial V^\pm \to B_0^\pm$ with $h^\pm(\partial D^\pm) = l_0^\pm$, where $D^\pm$ are meridian discs for $V^\pm$. The composite map $\hat{f} = f \circ q: \hat{M} \to N$ extends to a proper, continuous map $f_1: M_1 \to N$ with $f_1|V^+ = f_1|V^-$. We may assume that $N - f(B_0)$ contains a regular value y for f_1. Since V^+ and V^- have mutually opposite orientations induced from $\hat{M}$,

$$\deg(f_1|V^+ \cap f_1^{-1}(y)) = -\deg(f_1|V^- \cap f_1^{-1}(y)).$$

Hence

$$\begin{aligned}\deg(f_1) &= \deg(\hat{f}|\hat{f}^{-1}(y)) + \deg(f_1|V^+ \cap f_1^{-1}(y)) + \deg(f_1|V^- \cap f_1^{-1}(y)) \\ &= \deg(f|f^{-1}(y)) = \deg(f).\end{aligned}$$

Since $\partial\mathcal{H}(M) \supset B_0$, we may assume that $\mathcal{H}(\hat{M}) = q^{-1}(\mathcal{H}(M))$ contains a component H^+ with $\partial H^+ \supset B_0^+$. By Rong[**9**, lemma 2·12], $\|H^+ \cup_{h^+} V^+\| < \|H^+\| = \|q(H^+)\|$.

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Therefore, we have

$$\begin{aligned}\|M_1\| &\leqslant \|\mathrm{cl}\,(\mathscr{H}(\hat{M})-H^+)\|+\|H^+\cup_{h^+}V^+\|+\|V^-\|+\|\mathrm{cl}\,(\hat{M}-\mathscr{H}(\hat{M}))\|\\ &=\|\mathrm{cl}\,(\mathscr{H}(\hat{M})-H^+)\|+\|H^+\cup_{h^+}V^+\|\\ &<\|\mathrm{cl}\,(\mathscr{H}(M)-q(H^+))\|+\|q(H^+)\|\\ &=\|M\|.\end{aligned}$$

Since the degree of $f_1: M_1\to N$ is equal to $\deg(f)$, $\|M\|>\|M_1\|\geqslant\deg(f)\,\|N\|$. This contradiction implies that $f|B_0: B_0\to N$ is π_1-injective. ∎

Since $\mathscr{H}(N)$ is homotopically atoroidal and acylindrical, by Lemma 2, we may assume that

$$f(\partial\mathscr{H}(M))\subset\mathscr{S}(N). \tag{2·1}$$

For a Kleinian group Γ, the *limit set* $\Lambda_\Gamma\subset S^2_\infty$ is the set of accumulation points of the orbit Γ_x of $x\in\mathbb{H}^3$ in the closed 3-ball $\mathbb{H}^3\cup S^2_\infty$. Note that Λ_Γ is independent of the choice of $x\in\mathbb{H}^3$. A Kleinian group is said to be *non-elementary* if it contains no abelian subgroups of finite index. For the convex hull $H(\Gamma)$ of Λ_Γ in $\mathbb{H}^3$, the quotient $C_\Gamma=H(\Gamma)/\Gamma$ is the smallest, closed, convex core of $\mathbb{H}^3/\Gamma$. We say that Γ is *geometrically finite* if Γ is non-elementary and if, for any (or some) $r>0$, the r-neighbourhood of C_Γ in $\mathbb{H}^3/\Gamma$ has finite volume.

Since each component H_i of $\mathscr{H}(M)$ is a hyperbolic manifold, there exists a Kleinian group Γ_i such that $\mathrm{int}\,H_i$ is identified with $\mathbb{H}^3/\Gamma_i$. So, for any closed, incompressible surface F in $\mathrm{int}\,H_i$, $\pi_1(F)$ can be regarded as a subgroup of Γ_i. In particular, $\pi_1(F)$ is a Kleinian group.

Lemma 3. *The map f is properly homotopic to a smooth map $f': M\to N$ such that each component F of $f'^{-1}(\mathscr{U}_N)$ is a closed, incompressible surface in M, and F is either contained in $\partial\mathscr{H}(M)$ or disjoint from $\partial\mathscr{H}(M)$. Furthermore, if F is contained in $\mathrm{int}\,\mathscr{H}(M)$, then $\pi_1(F)$ is geometrically finite.*

Proof. Deforming f by a proper homotopy, we may assume that the retriction $\phi=f|\partial\mathscr{H}(M)$ is transverse to $\mathscr{U}_N$ and $\phi^{-1}(\mathscr{U}_N)$ has the minimum number of components among $(h|\partial\mathscr{H}(M))^{-1}(\mathscr{U}_N)$ for all proper maps $h: M\to N$ properly homotopic to f and such that $h|\partial\mathscr{H}(M)$ is transverse to $\mathscr{U}_N$. If $\phi^{-1}(\mathscr{U}_N)$ were not empty, then there would exist a component T'_j of $\mathscr{U}_N$ such that $f(\partial\mathscr{H}(M))\cap T'_j\neq\varnothing$. Since $f(\partial\mathscr{H}(M))\subset\mathscr{S}(N)$ by (2·1), both the sides of T'_j in N are Seifert pieces of $\mathscr{S}(N)$. By the minimality of $\phi^{-1}(\mathscr{U}_N)$, each component l of $\phi^{-1}(T'_j)$ is a non-contractible loop in $\partial\mathscr{H}(M)$. By Lemma 2, $f|l$ is π_1-injective. So we may assume that the image $f(l)$ is a simple, non-contractible loop in T'_j such that $f|l: l\to f(l)$ is a cyclic covering. For the closure A_t of each component of $f^{-1}(\mathrm{int}\,S'_t)\cap\partial\mathscr{H}(M)$ with $\partial A_t\supset l$, $\hat{f}|A_t: A_t\to S'_t$ is homotopically essential, where S'_t is any Seifert piece in $\mathscr{S}(N)$ with $\partial S_t\supset T'_j$. By the Torus Theorem (see [6] and [7]), S'_t contains a properly embedded, essential annulus $\bar{A}_t$ with $\partial\bar{A}_t\supset f(l)$. By [5, VI·3·4], $\bar{A}_t$ is a saturated annulus in S'_t. There $f(l)$ would be a common fibre of the Seifert fibrations on both sides of T'_j in $S(N)$. So, $\mathscr{U}_N-T'_j$ would determine a torus decomposition for N. This contradicts that $\mathscr{U}_N$ is a minimal union. Thus we have $f^{-1}(\mathscr{U}_N)\cap\partial\mathscr{H}(M)=\varnothing$. By [4, lemma 6·5], f is properly homotopic to a smooth map $f': M\to N$ such that each component of $f'^{-1}(\mathscr{U}_N)$ is an incompressible

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surface in M disjoint from $\partial\mathscr{H}(M)$. We suppose that F is a component of $f'^{-1}(\mathscr{U}_N)$ contained in $\operatorname{int}\mathscr{H}(M)$. Since $\mathscr{H}(M)$ is homotopically atoroidal, in the case where F is a torus, F is parallel to some component B of $\partial\mathscr{H}(M)$ in $\mathscr{H}(M)$. Then the interior of any hyperbolic piece of the torus decomposition for M with respect to the minimal union $\mathscr{U}'_M = (\mathscr{U}_M - B) \cup F$ does not contain F. Therefore we may assume that F is not a torus. In particular, $\pi_1(F)$ is a non-elementary, indecomposable, Kleinian group. By Thurston[**11**, proposition 8·4·3] and Bonahon [**2**, theorem A] (see also Soma [**10**, lemma 4] in the case of $\partial\mathscr{H}(M) \neq \varnothing$), either $\pi_1(F)$ is geometrically finite or M is covered by a fibre bundle over S^1 with fibre corresponding to F. In the latter case, by Lemma 1, N would be covered by a torus bundle over S^1. Then $\|N\|$ would be zero, a contradiction. Thus $\pi_1(F)$ is geometrically finite. ∎

For a component F of $f'^{-1}(\partial\mathscr{H}(N)) \cap \operatorname{int}\mathscr{H}(M)$, note that, in general, $\pi_1(F)$ is not quasi-Fuchsian, that is, the limit set Λ_F of $\pi_1(F)$ is not a Jordan curve. Let H_j be the component of $\mathscr{H}(M)$ containing F, and let $p_j : \mathbb{H}^3 \to \operatorname{int} H_j$ be the universal covering. Since any component $\tilde{F}$ of $p_j^{-1}(F)$ is a properly embedded plane in $\mathbb{H}^3$, $\tilde{F}$ separates $\mathbb{H}^3$ into just two components R_+, R_-. Since $\pi_1(F)$ is geometrically finite, by Ahlfors[**1**], the limit set Λ_F is a closed subset of S^2_∞ with zero Lebesgue measure. Suppose that Ω_+ is the union of components of $S^2_\infty - \Lambda_F$ adjacent to R_+ in $\mathbb{H}^2 \cup S^2_\infty$, and $\Omega_- = S^2_\infty - \Lambda_F \cup \Omega_+$.

LEMMA 4. *There exists a regular ideal simplex σ in $\mathbb{H}^3 \cup S^2_\infty$ such that two of the four vertices of σ are in Ω_+ and the other two are in Ω_-.*

Proof. We may assume that $S^2_\infty = \mathbb{C} \cup \{\infty\}$ and $\infty \in \Omega_-$. Let D be a round disc in $\mathbb{C}$ with $\operatorname{int} D \subset \Omega_+$ and $D \cap \Lambda_F \neq \varnothing$. Let D_0 be a round subdisc of D such that $\{z_0\} = \partial D_0 \cap \partial D$ is a single point contained in Λ_F. Let Δ be the regular triangle inscribed to D_0 and with $\partial\Delta \ni z_0$. By sliding Δ on $\mathbb{C}$ slightly, one can obtain another regular triangle Δ_0 such that two v_1, v_2 of the three vertices of Δ_0 are contained in $\operatorname{int} D \subset \Omega_+$ and the other, v_3, is in Ω_-. Then, the regular ideal simplex with four vertices v_1, v_2, v_3, ∞ is a desired one. ∎

We label the four vertices of Δ^3 as v_1, v_2, v_3, v_4. Let $\sigma_0 : \Delta^3 \to \mathbb{H}^3 \cup S^2_\infty$ be the regular ideal simplex given in Lemma 4 such that $\sigma(v_1), \sigma(v_2) \subset \Omega_+$ and $\sigma(v_3), \sigma(v_4) \subset \Omega_-$. Let O_1, O_2, O_3, O_4 be mutually disjoint, open discs in S^2_∞ such that $\sigma(v_i) \in O_i$ $(i = 1, 2, 3.4)$, $O_1 \cup O_2 \subset \Omega_+$ and $O_3 \cup O_4 \subset \Omega_-$. Since the map

$$\phi : \operatorname{Isom}^+(\mathbb{H}^3) \to S^2_\infty \times S^2_\infty \times S^2_\infty \times S^2_\infty$$

defined by $\phi(h) = (h \circ \sigma(v_1), h \circ \sigma(v_2), h \circ \sigma(v_3), h \circ \sigma(v_4))$ for $h \in \operatorname{Isom}^+(\mathbb{H}^3)$ is continuous, $U_F = \phi^{-1}(O_1 \times O_2 \times O_3 \times O_4)$ is an open neighbourhood of the identity in $\operatorname{Isom}^+(\mathbb{H}^3)$, and hence in particular, $\mu_0(U_F) > 0$, where μ_0 is the Haar measure on $\operatorname{Isom}^+(\mathbb{H}^3)$ given in Section 1. Let $\sigma_L : \Delta^3 \to \mathbb{H}^3$ denote the regular, straight simplex of edge length L such that σ_L and σ have the common barycentre c, and each geodesic ray r_i in $\mathbb{H}^3$ emanating from c and passing through $\sigma_L(v_i)$ tends toward $\sigma(v_i) \in S^2_\infty$.

We suppose that B_i $(i = 1, 2, 3, 4)$ are closed 3-balls in $\mathbb{H}^3 \cup S^2_\infty$ such that each $B_i \cap S^2_\infty$ is a 2-disc with $\operatorname{int}(B_i \cap S^2_\infty) \supset O_i$ and such that

$$(B_1 \cup B_2 \cup B_3 \cup B_4) \cap (\tilde{F} \cup \Lambda_F) = \varnothing.$$

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Then $(B_1 \cup B_2) \cap \mathbb{H}^3 \subset R_+$ and $(B_3 \cup B_4) \cap \mathbb{H}^3 \subset R_-$. If L is sufficiently large, then each $h \circ \sigma_L(v_i)$ is contained in B_i for all $h \in U_F$. Thus, we have obtained

LEMMA 5. *With the notation as above, there exists an open neighbourhood U_F of the identity in* $\mathrm{Isom}^+(\mathbb{H}^3)$ *such that, for all sufficiently large $L > 0$ and all $h \in U_F$, $h \circ \sigma_L(v_1)$, $h \circ \sigma_L(v_2) \in R_-$ and $h \circ \sigma_L(v_3), h \circ \sigma_L(v_4) \in R_-$.*

For the projection $r_j\colon \mathrm{Isom}^+(\mathbb{H}^3) \to \mathrm{Isom}^+(\mathbb{H}^3)/\Gamma_j$, if necessary replacing each O_i by a smaller open disc containing $\sigma(v_i)$, we may assume that $r_j | U_F$ is injective. Let σ_{L-} be the mirror image of σ_L in $\mathbb{H}^3$ with $\sigma_{L-}(v_i) = \sigma_L(v_i)$ and $\sigma_{L-}(v_{i+2}) = \sigma_L(v_{5-i})$ for $i = 1, 2$. By (1·2), we have

$$\left\| \frac{1}{2} \left(\mathrm{smear}_{\mathrm{int} H_j}(\sigma_L) - \mathrm{smear}_{\mathrm{int} H_j}(\sigma_{L-}) \right) | C_{F\pm} \right\| = \frac{1}{2} \mu_0(U_F), \tag{2·2}$$

where $C_{F\pm} = \{p_j \circ h \circ (\sigma_L)_\pm ; h \in U_F\} \subset C^1(\Delta^3, \mathrm{int} H_j)$ and $p_j\colon \mathbb{H}^3 \to \mathrm{int} H_j$ is the universal covering defined in the paragraph preceding Lemma 4. For any two components F_k, F_l of $f'^{-1}(\partial \mathscr{H}(N)) \cap \mathrm{int} H_j$, we may choose the regular, ideal simplices σ_k, σ_l given in Lemma 4 so that $p_j \circ \sigma_k \neq p_j \circ \sigma_l$ and hence $p_j \circ \sigma_{k,L} \neq p_j \circ \sigma_{l,L}$ for all sufficiently large $L > 0$. So, for any distinct components F_k, F_l of

$$f'^{-1}(\partial \mathscr{H}(N)) \cap \mathrm{int}\, \mathscr{H}(M),$$

if necessary replacing U_{F_k}, U_{F_l} by sufficiently smaller, open neighbourhoods of the identity, we may assume that

$$(C_{F_k+} \cup C_{F_k-}) \cap (C_{F_l+} \cup C_{F_l-}) = \varnothing. \tag{2·3}$$

3. *A fundamental 3-cycle on* $(M, \partial M)$

For a Kleinian group Γ and $\delta > 0$, the *δ-thin part* $H_{(0,\delta]}$ of $H = \mathbb{H}^3/\Gamma$ is the set consisting of all points $x \in H$ such that there exist non-contractible loops l in H with $l \ni x$ and of length $\leqslant \delta$. The *δ-thick part* $H_{[\delta,\infty)}$ is the closure $\mathrm{cl}(H - H_{(0,\delta]})$. Let $p\colon \mathbb{H}^3 \to H$ be the universal covering. By Margulis' Lemma [11, corollary 5·10·2], there exists $\delta_0 > 0$ independent of Γ such that, for any $\delta > 0$ with $\delta < \delta_0$, each component of $H_{(0,\delta]}$ is either an embedded, tubular neighbourhood of a closed geodesic (called a *Margulis tube*), or a $\mathbb{Z}$-cusp or a $\mathbb{Z} \times \mathbb{Z}$-cusp C, that is, each component B of $p^{-1}(C)$ is a horoball centred at some point $x_0 \in S^2_\infty$ and the stabilizer of B (or equivalently of x_0) in Γ is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. If $\mathrm{vol}(H) < \infty$, then H has at most finitely many Margulis tubes. So, replacing δ by a sufficiently smaller positive number if necessary, we may assume that $H_{(0,\delta]}$ has no Margulis tube components. Then each component of $H_{(0,\delta]}$ is a $\mathbb{Z} \times \mathbb{Z}$-cusp, and $H_{[\delta,\infty)}$ is a strong deformation retract of H.

Let $\mathscr{F}$ denote the interaction $f'^{-1}(\partial \mathscr{H}(N)) \cap \mathrm{int}\, \mathscr{H}(M)$. For any ϵ with $0 < \epsilon < 1/2$, we will define constants depending on the value of ϵ.

Let $\{H_1, \ldots, H_n\}$ be the set of components of $\mathscr{H}(M)$ and let $\Gamma_1, \ldots, \Gamma_n$ be Kleinian groups such that $\mathrm{int} H_i$ is diffeomorphic to $\mathbb{H}^3/\Gamma_i$ for $i = 1, \ldots, n$. According to [11, proposition 6·5·1], there exists a constant $K > 0$ such that any null-homologous 2-cycle c on T^2 is bounded by a 3-chain z on T^2 with $\|\partial z\| \leqslant K\|c\|$. Such 3-chains will be used later to join certain 3-chains on $\mathscr{H}(M)$ and on $\mathscr{S}(M)$ in $\partial \mathscr{H}(M)$. Let $\epsilon_1 > 0$ be a constant satisfying

$$\epsilon_1 < \frac{\epsilon}{37 + 16K}. \tag{3·1}$$

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We set $\mathscr{H} = \operatorname{int}\mathscr{H}(M)$ and choose a constant $\delta > 0$ less than the Margulis constant such that $\mathscr{H}_{(0,\delta]}$ contains no Margulis tubes and $\operatorname{vol}(\mathscr{H}_{(0,\delta]}) < \epsilon_1$. The positive number L is chosen so that

$$\operatorname{vol}(\sigma_L) > v_3 - \frac{\epsilon}{2}.$$

The thin part $\mathscr{H}_{(0,\delta]}$ is divided into the two parts P and Q such that P is the set of all points in $\mathscr{H}_{(0,\delta]}$ whose distances from $\partial\mathscr{H}_{(0,\delta]}$ are not greater than $3L$ and such that $Q = \operatorname{cl}(\mathscr{H}_{(0,\delta]} - P)$. Note that Q is also a thin part of $\mathscr{H}$. The support of the 3-cycle

$$z_L = \frac{1}{2}\sum_{i=1}^{n}(\operatorname{smear}_{\operatorname{int}H_i}(\sigma_L) - \operatorname{smear}_{\operatorname{int}H_i}(\sigma_{L^-})) \tag{3·2}$$

on $C^1(\Delta^3, \mathscr{H})$ is not compact. Let $z'_L \in \mathscr{C}3(\mathscr{H})$ be the restriction of z_L to the regular 3-simplices in $\mathscr{H}$ of edge length L whose images are disjoint from $\operatorname{int}\mathscr{N}_L(Q)$, where $\mathscr{N}_L(Q)$ is the L-neighbourhood of Q in $\mathscr{H}$. In particular, any regular 3-simplices of edge length L meeting $\mathscr{H}_{[\delta,\infty)}$ non-trivially are contained in $\operatorname{supp}(z'_L)$, and $\operatorname{supp}(\partial z'_L)$ is contained in $C^1(\Delta^2, P)$. We set $z''_L = z_L - z'_L$. Since $\operatorname{supp}(z''_L)$ is contained in $C^1(\Delta^3, \mathscr{H}_{(0,\delta]})$, by (1·1) and (1·2),

$$\|z''_L\| \leqslant \operatorname{vol}(\mathscr{H}_{(0,\delta]}) < \epsilon_1.$$

Since the 3-simplex Δ^3 has four faces and since $\partial z_L = 0$,

$$\|\partial z'_L\| = \|\partial z''_L\| \leqslant 4\|z''_L\| < 4\epsilon_1. \tag{3·3}$$

Let $\delta_1 > 0$ be a constant with $\mathscr{H}_{(0,\delta_1]} \subset Q$, and let $\mathscr{A} = \{(A_i, a_i); i = 1, \ldots, s\}$ be a finite set such that (i) $A_i (i = 1, \ldots, s)$ are mutually disjoint, simply-connected subsets of $\mathscr{H}_{[\delta,\infty)} \cup P$ with $A_1 \cup \ldots \cup A_s = \mathscr{H}_{[\delta,\infty)} \cup P$, (ii) the diameter of each A_i is less than $\delta_1/2$ and (iii) a_i is a point in $\operatorname{int}A_i$. Note that, in general, A_i are neither open nor closed subsets. Let $p_k\colon \mathbb{H}^3 = \mathbb{H}^3_k \to \operatorname{int}H_k \subset \mathscr{H}(k = 1, \ldots, n)$ be the universal coverings, and $\tilde{\mathscr{A}}$ the infinite set of pairs $(\tilde{A}_j, \tilde{a}_j)$ such that each $\tilde{A}_j$ is a component of $p_k^{-1}(A_i)$ for some $A_i \subset \operatorname{int}H_k$, and $\tilde{a}_j = \tilde{A}_j \cap p_k^{-1}(a_i)$. For any $\sigma \in \operatorname{supp}(z'_L)$, the vertices of its lift $\tilde{\sigma}\colon \Delta^3 \to \mathbb{H}^3_1 \cup \ldots \cup \mathbb{H}^3_k$ are contained in mutually distinct $\tilde{A}_j$'s. We suppose that $\tilde{\sigma}(v_i) \in \tilde{A}_{j_i} \subset \mathbb{H}^3_k (i = 1, 2, 3, 4)$, where $\{v_1, v_2, v_3, v_4\}$ is the set of vertices of Δ^3. For the straight simplex $\tilde{\sigma}_0$ in $\mathbb{H}^3$ with $\tilde{\sigma}_0(v_i) = \tilde{a}_{j_i} (i = 1, 2, 3, 4)$, we set $\sigma^{\mathscr{A}} = p_k \circ \tilde{\sigma}_0$. Since the $\tilde{A}_j$'s are mutually disjoint, this correspondence induces the (well-defined) map

$$\mathscr{A}_*\colon \operatorname{supp}(z'_L) \to C^1(\Delta^3, \mathscr{H}).$$

Since $\operatorname{vol}(\sigma) > v_3 - \epsilon/2$ for any $\sigma \in \operatorname{supp}(z'_L)$, we may choose $\delta_1 > 0$ so small that

$$\operatorname{vol}(\sigma^{\mathscr{A}}) > v_3 - \epsilon. \tag{3·4}$$

Since $\sigma(\Delta^3) \cap \operatorname{int}\mathscr{N}_L(Q) = \varnothing$, we may assume that $\sigma^{\mathscr{A}}(\Delta^3) \cap Q = \varnothing$, that is $\sigma^{\mathscr{A}} \in C^1(\Delta^3, \mathscr{H}_{[\delta,\infty)} \cup P)$. Note that the image $\mathscr{A}_*(\operatorname{supp}(z'_L))$ is a finite set $\{\sigma^{\mathscr{A}}_1, \ldots, \sigma^{\mathscr{A}}_m\}$. We set $z^{\mathscr{A}}_L = \sum_{i=1}^{m} r_i \sigma^{\mathscr{A}}_i \in \mathscr{C}^f_1(\mathscr{H}_{[\delta,\infty)} \cup P)$, where $r_i = z'_L(\mathscr{A}_*^{-1}(\sigma^{\mathscr{A}}_i)) = z_L(\mathscr{A}_*^{-1}(\sigma^{\mathscr{A}}_i))$ for $i = 1, \ldots, m$. Obviously, $\|z^{\mathscr{A}}_L\| = \sum_{i=1}^{m} |r_i| \leqslant \|z'_L\|$. The boundary 2-cycle $\partial z^{\mathscr{A}}_L$ is obtained from $\partial z'_L$ by the similar process. Then, by (3·3),

$$\|\partial z^{\mathscr{A}}_L\| \leqslant \|\partial z'_L\| < 4\epsilon_1. \tag{3·5}$$

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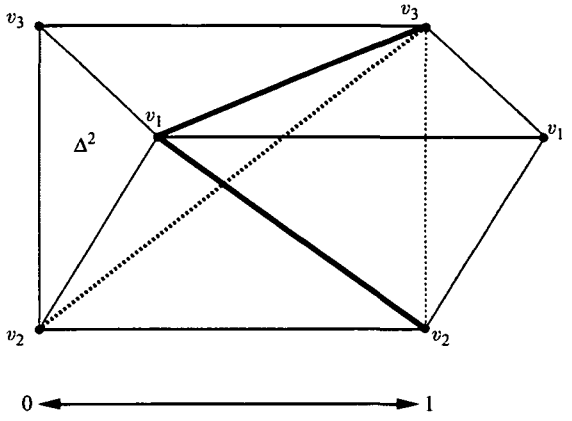


Fig. 1. The case where $v_1 < v_2 < v_3$.

Since the correspondence $\sigma \mapsto \sigma^{\mathscr{A}}$ neither reverses the orientation of simplices nor collapses them to lower dimensional ones,

$$\mathscr{A}_*(\mathrm{supp}_+(z'_L)) \cap \mathscr{A}_*(\mathrm{supp}_-(z'_L)) = \varnothing,$$

where $\mathrm{supp}_\pm(z'_L) = \mathrm{supp}\,(z'_L) \cap \mathrm{supp}\,(\mathrm{smear}_{\mathscr{H}}(\sigma_{L\pm}))$. Then, by (3·2), we have

$$|r_i| = |z_L(\mathscr{A}_*^{-1}(\sigma_i^{\mathscr{A}}))| = \|z_L \mid \mathscr{A}_*^{-1}(\sigma_i^{\mathscr{A}})\|, \tag{3·6}$$

and hence

$$\|z_L^{\mathscr{A}}\| = \|z'_L\|. \tag{3·7}$$

Since P is diffeomorphic to $\partial\mathscr{H}_{(0,\delta]} \times [0,1]$, the 2-cycle $\partial z_L^{\mathscr{A}}$ is homotopic in P to a 2-cycle in $\mathscr{C}_2^f(\partial P - \partial\mathscr{H}_{(0,\delta]})$. Therefore there exists a 3-chain $w_L \in \mathscr{C}_3^f(P)$ such that (i) $\partial(z_L^{\mathscr{A}} + w_L) \subset \mathscr{C}_2^f(\partial P - \partial\mathscr{H}_{(0,\delta]})$, (ii) $\|\partial(z_L^{\mathscr{A}} + w_L)\| = \|\partial w_L \mid (\partial P - \partial\mathscr{H}_{(0,\delta]})\| \leqslant \|\partial z_L^{\mathscr{A}}\|$ and (iii) $\|w_L\| \leqslant 3\|\partial z_L^{\mathscr{A}}\|$. The existence of such a 3-chain is immediate from the fact that the product $\Delta^2 \times [0,1]$ is divided into three 3-simplices as in Fig. 1, where the vertices of Δ^2 are labelled mutually distinct numbers v_1, v_2, v_3 and two vertices $(v_i, 0)$ and $(v_j, 1)$ are joined by an edge of one of the three simplices if $v_i < v_j$.

By (3·5) and (iii),

$$\|w_L\| \leqslant 12\epsilon_1. \tag{3·8}$$

The sum $z_L^{\mathscr{A}} + w_L$ is a finite 3-cycle on $(\mathscr{H}_{[\delta,\infty)} \cup P, \partial(\mathscr{H}_{[\delta,\infty)} \cup P))$.

Let $\psi = \mathrm{straight}: C^1(\Delta^k, \mathscr{H}) \to C^1(\Delta^3, \mathscr{H})$ be the straight map given in [**11**, §6·1]. Since $\mathscr{H}_{[\delta,\infty)} \cup P$ is not convex in $\mathscr{H}$, $\psi(C^1(\Delta^3, \mathscr{H}_{[\delta,\infty)} \cup P))$ is not contained in $C^1(\mathscr{H}_{[\delta,\infty)} \cup P)$. For a fixed, strong deformation retract $\rho: \mathscr{H} \to \mathscr{H}_{[\delta,\infty)} \cup P$, we set $\mathrm{straight}_\rho(\sigma) = \rho \circ \psi(\sigma)$. Intuitively, $\mathrm{straight}_\rho(\sigma)$ is obtained from $\psi(\sigma)$ by collapsing $\psi(\sigma)(\Delta^3) \cap Q$ into ∂Q. In general, $\mathrm{straight}_\rho(\sigma)$ is not a C^1-map. In fact, if $\psi(\sigma)(\Delta^3) \cap \mathrm{int}\, Q \neq \varnothing$, then $\mathrm{straight}_\rho(\sigma)$ is not differentiable on the boundary B of $\mathrm{cl}\,(\Delta^3 - \psi(\sigma)^{-1}(Q))$. Then, by deforming $\mathrm{straight}_\rho(\sigma)$ slightly in a small neighbourhood of B in Δ^3, we may assume that $\mathrm{straight}_\rho(\sigma) \in C^1(\Delta^3, \mathscr{H}_{[\delta,\infty)} \cup P)$. Such a smoothing process by small deformation will be used again in Section 5. Note that, for any $\sigma \in \mathrm{basis}\,(z_L^{\mathscr{A}} + w_L)$, $|\mathrm{vol}\,(\mathrm{straight}_\rho(\sigma))| = |\mathrm{vol}|\,(\mathrm{straight}_\rho(\sigma)) < v_3$.

Since the composition $\mathscr{H}_{[\delta,\infty)} \cup P \to \mathscr{H} \to \mathscr{H}(M)$ of the two inclusions is homotopic to a diffeomorphism, we may identify $\mathscr{H}_{[\delta,\infty)} \cup P$ with $\mathscr{H}(M)$. In other words, we equip $\mathscr{H}(M)$ with the metric induced from $\mathscr{H}_{[\delta,\infty)} \cup P$ by the diffeomorphism. We denote $\mathscr{H}(M)$ with this metric by $\mathscr{H}(M)^*$. Then,

$$\mathrm{vol}\,(\mathscr{H}(M)^*) = \mathrm{vol}\,(\mathscr{H}_{[\delta,\infty)} \cup P) < \mathrm{vol}\,(\mathscr{H}) = \|M\|\, v_3.$$

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Let Ω_H be the volume form on $\mathscr{H}(M)^*$. Since $\text{straight}_\rho(z_L^{\mathscr{A}}+w_L)=z_L^{\mathscr{A}}+\text{straight}_\rho(w_L)$,

$$\langle z_L^{\mathscr{A}}+w_L,\Omega_H\rangle=\langle z_L^{\mathscr{A}},\Omega_H\rangle+\langle\text{straight}_\rho(w_L),\Omega_H\rangle. \tag{3·9}$$

Since, for any $h\in\text{Isom}^+(\mathbb{H}^3)$, $\sigma_\pm=p_j\circ h\circ(\sigma_L)_\pm:\Delta^3\to\text{int}\,H_j$ with $\sigma_\pm(\Delta^3)\cap\mathscr{H}_{[\delta,\infty)}\neq\varnothing$ is contained in $\text{supp}\,(z_L')$, by (1·1) and (1·2), $\text{vol}\,(\mathscr{H}_{[\delta,\infty)})<\|z_L'\|$. Since $\text{supp}\,(z_L')\subset C^1(\Delta^3,\mathscr{H}(M))$, we have $\|z_L'\|\leqslant\text{vol}\,(\mathscr{H}(M)^*)$. Since $\text{vol}\,(\mathscr{H}_{[\delta,\infty)})<\epsilon_1$,

$$\text{vol}\,(\mathscr{H})-\epsilon_1<\text{vol}\,(\mathscr{H}_{[\delta,\infty)}).$$

Then, by (3·7), we have

$$\text{vol}\,(\mathscr{H}(M)^*)-\epsilon_1<\text{vol}\,(\mathscr{H})-\epsilon_1<\|z_L^{\mathscr{A}}\|<\text{vol}\,(\mathscr{H}(M)^*). \tag{3·10}$$

Since, by Milnor [**11**, §7·2], the volume of any straight simplex is less than v_3, by (3·4), (3·8), (3·9) and (3·10),

$$\langle z_L^{\mathscr{A}}+w_L,\Omega_H\rangle>(\text{vol}\,(\mathscr{H}(M)^*)-\epsilon_1)(v_3-\epsilon)-12\epsilon_1 v_3.$$

By (3·8) and (3·10), the fundamental 3-cycle

$$z_H=\frac{\text{vol}\,(\mathscr{H}(M)^*)}{\langle z_L^{\mathscr{A}}+w_L,\Omega_H\rangle}(z_L^{\mathscr{A}}+w_L)$$

on $(\mathscr{H}(M),\partial\mathscr{H}(M))$ has norm

$$\|z_H\|\leqslant K_0(\|z_L^{\mathscr{A}}\|+\|w_L\|)<K_0(\text{vol}\,(\mathscr{H}(M)^*)+12\epsilon_1), \tag{3·11}$$

where

$$K_0=\frac{\text{vol}\,(\mathscr{H}(M)^*)}{(\text{vol}\,(\mathscr{H}(M)^*)-\epsilon_1)(v_3-\epsilon)-12\epsilon_1 v_3}.$$

One can choose $\epsilon_1>0$ so small that

$$K_0<\frac{1+\epsilon}{v_3-\epsilon}.$$

Since $v_3>1$ and $\epsilon<1/2$, $v_3-\epsilon>1/2$, so that $K_0<3$. Since $\|\partial(z_L^{\mathscr{A}}+w_L)\|\leqslant\|\partial z_L^{\mathscr{A}}\|$, by (3·5),

$$\|\partial z_H\|\leqslant K_0\|\partial(z_L^{\mathscr{A}}+w_L)\|\leqslant K_0\|\partial z_L^{\mathscr{A}}\|<12\epsilon_1. \tag{3·12}$$

Let $\{S_1,\ldots,S_m\}$ be the set of Seifert pieces in M. Since $\|\mathscr{S}(M)\|=\|S_1\|+\ldots+\|S_m\|=0$, there exists a fundamental 3-cycle $z_S\in\mathscr{C}_3^f(\mathscr{S}(M),\partial\mathscr{S}(M))$ with $\|z_S\|<\epsilon_1$. Then $\|\partial z_S\|<4\epsilon_1$. By [**11**, proposition 6·5·1], there exists a 3-chain $c\in\mathscr{C}_3^f(\partial\mathscr{H}(M))$ such that $\partial c=-(\partial z_H\,|\,\partial\mathscr{H}(M)+\partial z_S\,|\,\partial\mathscr{H}(M))$ and

$$\|c\|\leqslant K(\|\partial z_H\,|\,\partial\mathscr{H}(M)\|+\|\partial z_S\,|\,\partial\mathscr{H}(M)\|)\leqslant K(\|\partial z_H\|+\|\partial z_S\|).$$

By (3·12),

$$\|c\|<K(12\epsilon_1+4\epsilon_1)=16K\epsilon_1. \tag{3·13}$$

The sum

$$z_M=z_H+z_S+c$$

is our desired 3-cycle in $\mathscr{C}_3^f(M,\partial M)$. Since $K_0<3$, by (3·1), (3·11) and (3·13),

$$\begin{aligned}\|z_M\|&\leqslant\|z_H\|+\|z_S\|+\|c\|\\&<K_0(\text{vol}\,(\mathscr{H}(M)^*)+12\epsilon_1)+\epsilon_1+16K\epsilon_1\\&<\frac{1+\epsilon}{v_3-\epsilon}\text{vol}\,(\mathscr{H}(M)^*)+\epsilon.\end{aligned}$$

Since $\text{vol}\,(\mathscr{H}(M)^*)<\|M\|\,v_3$,

$$\|z_M\|<\frac{v_3(1+\epsilon)}{v_3-\epsilon}\|M\|+\epsilon. \tag{3·14}$$

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4. *A metric on N*

In this section, we will define a Riemannian metric on N as stated in the introduction. Let $\{H'_1, \dots, H'_p\}$ be the set of components of $\mathscr{H}(N)$, and let δ' be any positive constant such that $2\delta'$ is less than the Margulis constant and such that each $(\operatorname{int} H'_s)_{(0,2\delta']}$ contains no Margulis tubes. For each H'_s, there exists a Kleinian group Γ'_s and a diffeomorphism $\psi_s : H'_s \to (\mathbb{H}^2/\Gamma_s)_{[\delta',\infty)}$. We equip $\mathscr{H}(N)$ with the metric induced from $(\mathbb{H}^2/\Gamma_s)_{[\delta',\infty)}$ by ψ_s, and denote the resulting Riemannian manifold by $\mathscr{H}(N)^*$. If the constant $\delta' > 0$ is sufficiently small, then

$$\deg(f)(\|N\| v_3 - \operatorname{vol}(\mathscr{H}(N)^*)) = \deg(f)(\operatorname{vol}(\operatorname{int}\mathscr{H}(N)) - \operatorname{vol}(\mathscr{H}(N)^*)) < \epsilon. \quad (4{\cdot}1)$$

We fix a Riemannian metric g_S on $\mathscr{S}(N) - \partial\mathscr{H}(N)$. Then the Riemannian metric $g_{\delta'}$ on $\mathscr{N}(\partial\mathscr{H}(N)) = \mathscr{N}(\partial\mathscr{H}(N), N)$ is defined so that, for any component B'_k of $\partial H'_s$, the restriction $g_{\delta'} | \mathscr{N}_{\text{collar}}(B'_k, H'_s)$ is equal to that of the metric on $\mathscr{H}(N)^*$ and extended to the metric on $\mathscr{N}(B'_k)$ such that $\mathscr{N}(B'_k)$ is isometric to the component of $(\mathbb{H}^3/\Gamma'_s)_{[\delta'/2,2\delta']}$ containing $\psi_s(B'_k)$. For a given constant $r > 0$, we define the Riemannian metric $g_S(r, \delta')$ on $\mathscr{S}(N)$ by connecting $g_{\delta'} | \mathscr{N}_{\text{collar}}(\partial\mathscr{H}(N), \mathscr{S}(N))$ with rg_S by partitions λ, $1-\lambda$ of unity. Let λ be a smooth function on $\mathscr{S}(N)$ such that $\operatorname{supp}(\lambda) \subset \mathscr{N}_{\text{collar}}(\partial\mathscr{H}(N), \mathscr{S}(N))$ and $\lambda = 1$ on a smaller collar neighbourhood of $\partial\mathscr{H}(N)$ in $\mathscr{S}(N)$. Then the metric on $\mathscr{S}(N)$ is defined by

$$g_S(r, \delta') = \lambda g_{\delta'} | \mathscr{N}_{\text{collar}}(\partial\mathscr{H}(N), \mathscr{S}(N)) + (1-\lambda) r g_S.$$

This extends the metric on $\mathscr{H}(N)^*$ smoothly to the metric $g(r, \delta')$ on N.

Next, we modify the map $f' : M \to N$ by a proper homotopy as follows. For any component B'_k of $\partial\mathscr{H}(N)$, let F_k be a component of $f'^{-1}(B'_k) \cap \operatorname{int}\mathscr{H}(M)$. Let H'_s and S'_t be the pieces of $\mathscr{H}(N)$ and $\mathscr{S}(N)$ respectively with $H'_k \cap S'_t \supset B'_k$. Suppose that, for the component H_i of $\mathscr{H}(M)$ with $\operatorname{int} H_i \supset F_k$, $\eta : F_k \times [-1,1] \to \operatorname{int} H_i$ is an embedding onto a closed, regular neighbourhood of F_k in $\operatorname{int} H_i$ such that

$$\eta(F_k \times \{0\}) = F_k,\ f' \circ \eta(F_k \times [-1,0]) \subset H'_s \text{ and } \eta(F_k \times [0,1]) \subset S'_t.$$

Here, we identify $F_k \times [-1,1]$ with $\eta(F_k \times [-1,1]) \subset M$. Modifying f' by a homotopy rel. $\operatorname{cl}(M - F_k \times [-1,1])$ if necessary, we may assume that $f'(F_k \times [-1,0]) = \mathscr{N}_k$ and $f(F_k \times [0,1]) = \mathscr{N}_{\text{collar}}(B'_k, S'_t)$, where $\mathscr{N}_k$ is the component of $\psi_s^{-1}((\mathbb{H}^3/\Gamma'_s)_{[\delta',2\delta']})$ containing B'_k. For any $x \in F_k$, $\alpha_x(t) = (x,t)(-1 \leqslant t \leqslant 1)$ is a parametrized arc in $F_k \times [-1,1] \subset M$. We modify f' only on $\operatorname{int}(F_k \times [-1,1])$ so that, for each $x \in F_k$, $f' \circ \alpha_x(t)$ $(-1 \leqslant t \leqslant 0)$ is the affinely parametrized, geodesic map in $\mathscr{N}_k$ emanating from $f'(x,-1)$ and meeting B'_k orthogonally. We modify f' similarly on regular neighbourhoods of all other components of $\operatorname{int}\mathscr{H}(M) \cap f'^{-1}(\partial\mathscr{H}(N))$. For any $x, y \in F_k$ and $-1 \leqslant s, t \leqslant 0$, the distance between $f' \circ \alpha_x(s)$ and $f' \circ \alpha_y(t)$ diverges as $\delta' \to 0$ if $s \neq t$, and it converges to zero if $s = t$. In particular, $\operatorname{Lip}(f' | F_k)$ converges to zero as $\delta' \to 0$. Therefore, for any constant $\eta > 0$, if both δ' and $r > 0$ are sufficiently small, then for the metric $g_S(r, \delta')$ on $\mathscr{S}(N)$, $\operatorname{Lip}(f' | f'^{-1}(\mathscr{S}(N))) < \eta$. Note that, under such deformations of f', the place of $f'^{-1}(\partial\mathscr{H}(N))$ in $\mathscr{H}(M)$ is not moved at all.

5. *Haken-straight simplices*

Throughout this section, suppose that $f' | \mathscr{U}_M$ is π_1-injective. Then, since $\mathscr{H}(N)$ is homotopically atoroidal and acylindrical, we may assume that $f'(\mathscr{S}(M))$ is contained in $\mathscr{S}(N)$.

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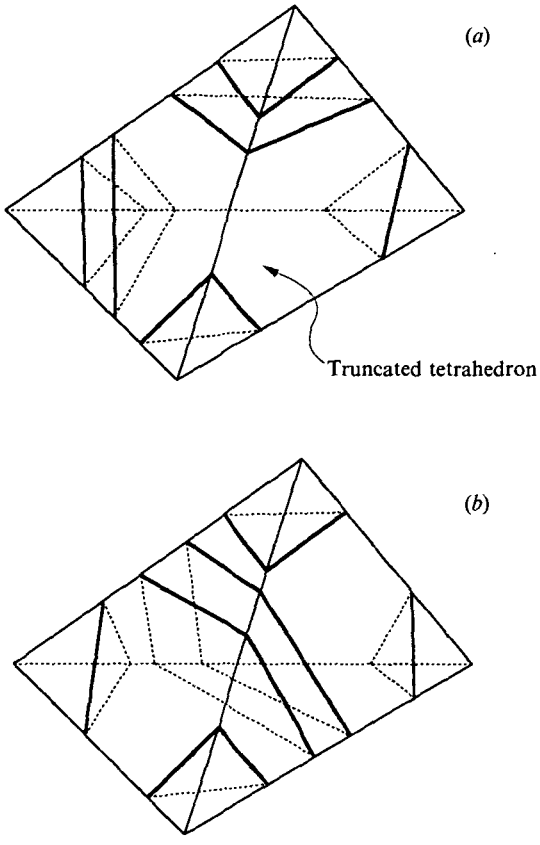


Fig. 2.

The sets of vertices, edges and faces of Δ^3 are denoted respectively by $V(\Delta^3)$, $E(\Delta^3)$ and $F(\Delta^3)$.

We set $\mathscr{F} = f'^{-1}(\partial\mathscr{H}(N)) \cap \operatorname{int}\mathscr{H}(M)$ and $\mathscr{G} = \mathscr{F} \cup \mathscr{U}_M \cup \partial P$. For $\sigma \in C^1(\Delta^3, M)$, let $\mathscr{F}_\sigma$ (respectively $\mathscr{G}_\sigma$) be the closure of $\sigma^{-1}(\mathscr{F}) \cap \operatorname{int}\Delta^3$ (respectively $\sigma^{-1}(\mathscr{G}) \cap \operatorname{int}\Delta^3$) in Δ^3. Since $\mathscr{G}$ is incompressible in M, it is easily seen that any singular 3-simplex $\sigma \in C^1(\Delta^3, M)$ is homotopic rel. $V(\Delta^3)$ to $\sigma_{\mathrm{red}} \in C^1(\Delta^3, M)$ such that either

$$\sigma_{\mathrm{red}}(\operatorname{int}\Delta^3) \cap \mathscr{G} = \varnothing$$

or each component of $\mathscr{G}_{\sigma_{\mathrm{red}}}$ and hence of $\mathscr{F}_{\sigma_{\mathrm{red}}}$ is either a triangle or a rectangle, as illustrated in Fig. 2(*a*) and (*b*). We say that σ_{red} is *reduced with respect to* $\mathscr{G}$.

Let $\Delta^3_{\sigma_{\mathrm{red}}}$ be the set of the closures of components of $\Delta^3 - \mathscr{F}_{\sigma_{\mathrm{red}}}$. An element $J_0 \in \Delta^3_{\sigma_{\mathrm{red}}}$ is called a *truncated tetrahedron* if there exists a homeomorphism h from the quotient space obtained from J_0 by pinching each face F of J_0 with $F \subset \mathscr{F}_{\sigma_{\mathrm{red}}}$ into a single point to a tetrahedron such that h takes collapsed faces to vertices and the edges to edges. If each component of $\mathscr{F}_{\sigma_{\mathrm{red}}}$ is a triangle, then $\Delta^3_{\sigma_{\mathrm{red}}}$ contains only one truncated-tetrahedral element. Otherwise, $\Delta^3_{\sigma_{\mathrm{red}}}$ contains no such elements. For any $\sigma_i \in$ basis (z_M), let $\sigma_{i,\mathrm{red}}$ $(i = 1, \ldots, n)$ be singular simplexes reduced with respect to $\mathscr{G}$ and homotopic to σ_i rel. $V(\Delta^3)$. We may assume that $\sigma_{i,\mathrm{red}}|D_1 = \sigma_{j,\mathrm{red}}|D_2$ if $\sigma_i|D_1 = \sigma_j|D_2$ for $D_1, D_2 \in E(\Delta^3) \cup F(\Delta^3)$. This correspondence red: basis $(z_M) \to C^1(\Delta^3, M)$ is linearly extended to the homomorphism, denoted by red_*, from the subspace

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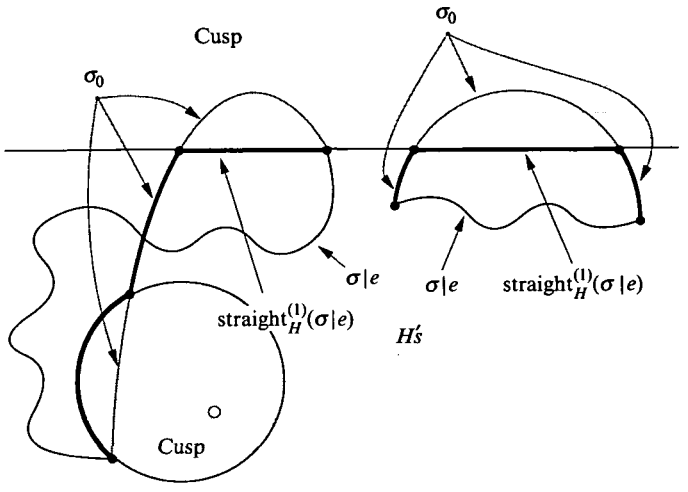


Fig. 3.

of $\mathscr{C}^f_3(M)$ generated by basis (z_M) to $C^f_3(M)$. We set $z_{M,\mathrm{red}} = \mathrm{red}_*(z_M)$, $z^{\mathscr{A}}_{L,\mathrm{red}} = \mathrm{red}_*(z^{\mathscr{A}}_L)$, $w_{L,\mathrm{red}} = \mathrm{red}_*(w_L)$, $z_{S,\mathrm{red}} = \mathrm{red}_*(z_S)$ and $c_{\mathrm{red}} = \mathrm{red}_*(c)$. Then $\|z_{M,\mathrm{red}}\| \leqslant \|z_M\|$ and $z_{M,\mathrm{red}}$ is homologous to z_M. So $z_{M,\mathrm{red}}$ also represents the fundamental class $[M, \partial M]$ and hence $f'_*(z_{M,\mathrm{red}})$ does $\deg(f)\,[N, \partial N]$. By these definitions, we have basis $(z^{\mathscr{A}}_{L,\mathrm{red}}) \in C^1(\Delta^3, \mathrm{int}\,\mathscr{H}(M))$, basis $(w_{L,\mathrm{red}}) \in C^1(\Delta^3, P)$ and basis $(z_{S,\mathrm{red}} + c_{\mathrm{red}}) \in C^1(\Delta^3, \mathscr{S}(M))$.

For any $\sigma \in \mathrm{basis}\,(f'_*(z_{M,\mathrm{red}}))$, we will define the *Haken-straight simplex* $\mathrm{straight}_H(\sigma) \in C^1(\Delta^3, N)$ as follows. If $\sigma \in \mathrm{basis}\,(f'_*(z_{S,\mathrm{red}} + c_{\mathrm{red}}))$, then we set $\mathrm{straight}_H(\sigma) = \sigma$. Since we may assume that $\mathscr{F} \cap P = \varnothing$ by choosing $\delta > 0$ sufficiently small, each element σ of basis $(f'_*(w_{L,\mathrm{red}}))$ is contained in either $C^1(\Delta^3, \mathscr{H}(N))$ or $C^1(\Delta^3, \mathscr{S}(N))$. If $\sigma \in C^1(\Delta^3, \mathscr{S}(N))$, then we set $\mathrm{straight}_H(\sigma) = \sigma$. Otherwise, $\mathrm{straight}_H(\sigma)$ is the ρ-straight simplex $\mathrm{straight}_\rho(\sigma)$ as defined in Section 3. The remainder is the case of $\sigma \in \mathrm{basis}\,(f'_*(z^{\mathscr{A}}_{L,\mathrm{red}}))$. For the closure $\hat{S}$ of $\sigma^{-1}(\mathrm{int}\,\mathscr{S}(N))$ in Δ^3, we set $\mathrm{straight}_H(\sigma)|\hat{S} = \sigma|\hat{S}$. For each component $\hat{H}_k$ of the closure $\hat{H}$ of $\sigma^{-1}(\mathrm{int}\,\mathscr{H}(N))$ in Δ^3, let H'_s be the component of $\mathscr{H}(N)$ with $\sigma(\hat{H}_k) \subset H'_s$. We regard H'_s as a submanifold of $\mathbb{H}^3/\Gamma'_s$ by identifying H'_s with the image $\psi_s(H'_s) = (\mathbb{H}^3/\Gamma'_s)_{[\delta', \infty)}$. For each component e of $\hat{H}_k \cap E(\Delta^3)$, $\sigma|e\colon e \to H'_s \subset \mathbb{H}^3/\Gamma'_s$ is homotopic rel. ∂e to the geodesic map $\sigma_0\colon e \to \mathbb{H}^3/\Gamma'_s$. Let $\mathrm{straight}^{(1)}_H(\sigma|e)\colon e \to H'_s$ be the map obtained from σ_0 replacing each $\sigma_0|\iota$ by the geodesic map $\iota \to \partial H'_s$ with respect to the Euclidean metric on $\partial H'_s$ which is homotopic in $\mathbb{H}^3/\Gamma'_s$ rel. $\partial\iota$ to $\sigma_0|\iota$, where ι are all subintervals of e with $\sigma_0(\iota) \cap \mathrm{int}\,H'_s = \varnothing$ and $\sigma_0(\partial\iota) \subset \partial H'_s$, see Fig. 3.

Here, we note that, since f' is possibly not π_1-injective, the degenerate case could occur. In other words, for $\sigma' \in \mathrm{basis}\,(z^{\mathscr{A}}_{L,\mathrm{red}})$ with $f' \circ \sigma' = \sigma$, though $\sigma'|e$ is not homotopic rel. ∂e into $\mathscr{F}$, $\sigma|e$ could be homotopic rel. ∂e into $\partial H'_s$ and hence $\mathrm{straight}^{(1)}_H(\sigma|e) \subset \partial H'_s$. Then, for any $\sigma \in \mathrm{basis}\,(f'_*(z^{\mathscr{A}}_{L,\mathrm{red}}))$, one can construct $\mathrm{straight}_H(\sigma)$ satisfying the following (1), (2) and (3).

(1) For any component e of $\hat{H} \cap E(\Delta^3)$, $\mathrm{straight}_H(\sigma)|e = \mathrm{straight}^{(1)}_H(\sigma|e)$.

(2) For the closure g_ι of each component of $\partial D_{\hat{G}} - \partial D_{\hat{G}} \cap E(\Delta^3)$ as shown in Fig. 4, $\mathrm{straight}_H(\sigma)|g_\iota\colon g_\iota \to \partial\mathscr{H}(N)$ is the geodesic map with respect to the Euclidean metric on $\partial\mathscr{H}(N)^*$, where $\hat{G} \subset \hat{H}$ is the closure of $\mathrm{straight}_H(\sigma)^{-1}(\mathrm{int}\,\mathscr{H}(N))$ in Δ^3 and $D_{\hat{G}} = \hat{G} \cap D$ for any D in $F(\Delta^3)$.

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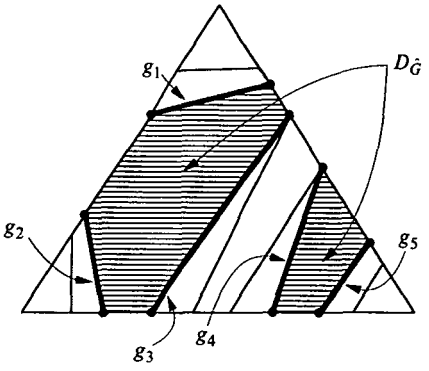


Fig. 4. Shaded regions represent $\hat{H} \cap D \supset D_{\hat{G}}$.

(3) $\text{straight}_H(\sigma_1)|D_1 = \text{straight}_H(\sigma_2)|D_2$ if $\sigma_1|D_1 = \sigma_2|D_2$ for $\sigma_1, \sigma_2 \in \text{basis}\,(f'_*(z^{\mathscr{A}}_{L,\text{red}}))$ and $D_1, D_2 \in E(\Delta^3) \cup F(\Delta^3)$.

By smoothing $\text{straight}_H(\sigma)$ in a small neighbourhood of the bent part $\text{fr}\,(\hat{H}) \cup \text{fr}\,(\hat{D})$ in Δ^3 if necessary, we may assume that $\text{straight}_H(\sigma) \in C^1(\Delta^3, N)$, where $\text{fr}\,(Y) = Y \cap \text{cl}\,(\Delta^3 - Y)$ for any closed subset Y of Δ^3. For any $\sigma_{\text{red}} \in \text{basis}\,(z_{M,\text{red}})$, we set

$$\Phi(\sigma_{\text{red}}) = \text{straight}_H(f' \circ \sigma_{\text{red}}).$$

This map is linearly extended to the homomorphism, still denoted by Φ, from the subspace of $\mathscr{C}^f_3(M)$ generated by $\text{basis}\,(z_{M,\text{red}})$ to $\mathscr{C}^f_3(N)$. Though the 2-cycle

$$\partial\Phi(w_{L,\text{red}})|\partial\mathscr{H}(N) + \partial\Phi(z_{S,\text{red}} + c_{\text{red}})|\partial\mathscr{H}(N)$$

is null-homologous in $\partial\mathscr{H}(N)$, in general it is non-zero, and hence $\text{straight}_H(f'_*(z_{M,\text{red}}))$ is not a cycle in $\mathscr{C}^f_3(N, \partial N)$. Let d be a 3-chain in $\mathscr{C}^f_3(\partial\mathscr{H}(N))$ such that

$$-\partial d = \partial\Phi(w_{L,\text{red}})|\partial\mathscr{H}(N) + \partial\Phi(z_{S,\text{red}} + c_{\text{red}})|\partial\mathscr{H}(N).$$

Then $z_N = \Phi(z_{M,\text{red}}) + d$ is a 3-cycle in $\mathscr{Z}^f_3(N, \partial N)$ representing

$$\deg(f)\,[N, \partial N] \in H_3(N, \partial N; \mathbb{R}).$$

Since, for any singular 3-simplex $\sigma \in \text{basis}\,(d)$, $|\text{vol}|\,(\sigma) = 0$, we will not need later the estimate for the norm $\|d\|$ from above.

Suppose that N admits the Riemannian metric $g(r, \delta')$ defined in Section 4 and

$$C_{\mathscr{F}} = C_{\mathscr{F}+} \cup C_{\mathscr{F}-} = \Big(\bigcup_{F_i \subset \mathscr{F}} C_{F_i+}\Big) \cup \Big(\bigcup_{F_i \subset \mathscr{F}} C_{F_i-}\Big) \subset C^1(\Delta^3, M),$$

where, for each component F_i of $\mathscr{F}$ with $F_i \subset \text{int}\,H_k \cap \mathscr{F}$, $C_{F_i\pm}$ is the subset of $C^1(\Delta^3, \text{int}\,H_k)$ defined in the last paragraph of Section 2. If $\delta > 0$ is sufficiently small, then the distance between $\mathscr{F}$ and ∂P in $\text{int}\,\mathscr{H}(M)$ is greater than $2L$. Since, for each $\sigma \in C_{\mathscr{F}} \subset \text{basis}\,(z_L)$, $\sigma(\Delta^3) \cap \mathscr{F} \neq \varnothing$, the intersection $\sigma(\Delta^3) \cap P$ is empty. Therefore, $C_{\mathscr{F}} \subset \text{supp}\,(z'_L)$ and hence

$$C^{\mathscr{A}}_{\mathscr{F}} = \{\sigma^{\mathscr{A}}; \sigma \in C_{\mathscr{F}}\} \subset \text{basis}\,(z^{\mathscr{A}}_L) \quad \text{and} \quad C^{\mathscr{A}}_{\mathscr{F},\text{red}} = \{\sigma^{\mathscr{A}}_{\text{red}}; \sigma^{\mathscr{A}} \in C^{\mathscr{A}}_{\mathscr{F}}\} \subset \text{basis}\,(z^{\mathscr{A}}_{L,\text{red}}).$$

If $\delta_1 > 0$ is sufficiently small, then by Lemma 5, for any $\sigma^{\mathscr{A}} \in C^{\mathscr{A}}_{\mathscr{F}}$, $\sigma^{\mathscr{A}}: \Delta^3 \to \text{int}\,H_k$ has a lift $\tilde{\sigma}^{\mathscr{A}}: \Delta^3 \to \mathbb{H}^3$ such that $\tilde{\sigma}^{\mathscr{A}}(v_1), \tilde{\sigma}^{\mathscr{A}}(v_2) \in R_+$, $\tilde{\sigma}^{\mathscr{A}}(v_3), \tilde{\sigma}^{\mathscr{A}}(v_4) \in R_-$. Therefore, for any

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$\sigma^{\mathscr{A}}_{\mathrm{red}} \in C^{\mathscr{A}}_{\mathscr{F},\mathrm{red}}$, the closure $\mathscr{F}_{\sigma^{\mathscr{A}}_{\mathrm{red}}}$ of $(\sigma^{\mathscr{A}}_{\mathrm{red}})^{-1}(\mathscr{F}) \cap \mathrm{int}\,\Delta^3$ in Δ^3 contains a rectangle component, so that the set $\Delta^3_{\sigma^{\mathscr{A}}_{\mathrm{red}}}$ of the closures of components of $\Delta^3 - \mathscr{F}_{\sigma^{\mathscr{A}}_{\mathrm{red}}}$ contains no truncated-tetrahedral elements, see Fig. 2(*b*).

LEMMA 6. *With the notation as above, one can reconstruct the Haken-straight simplices (if necessary) such that, for sufficiently small $r > 0$ and $\delta' > 0$, the following* (i) *and* (ii) *hold.*

(i) $|\mathrm{vol}|(\sigma) < v_3 + \epsilon$ *for any* $\sigma \in \mathrm{basis}(z_N)$.

(ii) $|\mathrm{vol}|(\sigma) < \epsilon$ *for any* $\sigma \in \Phi(C^{\mathscr{A}}_{\mathscr{F},\mathrm{red}})$.

Proof. We choose the positive constants r and δ' so small that

$$\mathrm{Lip}\,(f' | f'^{-1}(\mathscr{S}(N))) < \left(\frac{\epsilon}{2R}\right)^{\frac{1}{3}},$$

where $R = \max\{|\mathrm{vol}|(\sigma); \sigma \in \mathrm{basis}(z_{M,\mathrm{red}})\}$. If $\sigma \in \mathrm{basis}(d)$, then $|\mathrm{vol}|(\sigma) = 0$. Otherwise, there exists $\sigma_0 \in \mathrm{basis}(z_{M,\mathrm{red}})$ with $\Phi(\sigma_0) = \sigma$. By the definition of the Haken-straight map, for the closure $\hat{S}$ of $\sigma_0^{-1}(\mathrm{int}\,\mathscr{S}(N))$, $\sigma | \hat{S} = \Phi(\sigma_0) | \hat{S}$ is equal to $f' \circ \sigma_0 | \hat{S}$. Since $|\mathrm{vol}|(\sigma_0 | \hat{S}) \leqslant |\mathrm{vol}|(\sigma_0) < R$,

$$|\mathrm{vol}|(\sigma | \hat{S}) \leqslant \frac{\epsilon}{2R} \cdot R = \frac{\epsilon}{2}.$$

Since $f'(\mathscr{S}(M)) \subset \mathscr{S}(N)$, in the case of $\sigma_0 \in \mathrm{basis}(z_{S,\mathrm{red}})$, $|\mathrm{vol}|(\sigma)$ is less than $\epsilon/2$. If $\sigma_0 \in \mathrm{basis}(w_{L,\mathrm{red}})$, then σ is contained in $C^1(\Delta^3, \mathscr{H}(N)) \cup C^1(\Delta^3, \mathscr{S}(N)) \subset C^1(\Delta^3, N)$. If $\sigma \in C^1(\Delta^3, \mathscr{H}(N))$, then σ is equal to $\mathrm{straight}_\rho(f'_*(\sigma_0))$, so that $|\mathrm{vol}|(\sigma) < v_3$. If $\sigma \in C^1(\Delta^3, \mathscr{S}(N))$, then we have $\Delta^3 = \sigma^{-1}(\mathscr{S}(N))$, so that $|\mathrm{vol}(\sigma)| < \epsilon/2$. We suppose that $\sigma_0 \in \mathrm{basis}(z^{\mathscr{A}}_{L,\mathrm{red}})$ and $\hat{H} = \hat{H}_{\sigma_0}$ is the closure of $\sigma_0^{-1}(\mathrm{int}\,\mathscr{H}(N))$ in Δ^3. Then $\hat{H}$ contains at most one truncated-tetrahedral component J_0. We set

$$m = \max\{|\hat{H}_{\sigma_0}|; \sigma_0 \in \mathrm{basis}(z^{\mathscr{A}}_{L,\mathrm{red}})\},$$

where $|\hat{H}_{\sigma_0}|$ denotes the number of connected components of $\hat{H}_{\sigma_0}$. For each component J_* of $\hat{H}_{\sigma_0}$, we set $E(J_*) = \{J_* \cap e; e \in E(\Delta^3)\}$. For each e of $E(J_0)$, if $\sigma | e$ is non-degenerate or, equivalently, $\sigma(e) \notin \partial\mathscr{H}(N)$, then $\sigma | (e - e \cap \sigma^{-1}(\partial\mathscr{H}(N)))$ converges in C^1-topology as $\delta' \to 0$ to the geodesic line connecting toral cusps of $\mathbb{H}^3/\Gamma'_s \supset H'_s$, where H'_s is the component of $\mathscr{H}(N)$ with $H'_s \supset \sigma(e)$. If $\sigma | e$ is degenerate, then $\sigma(e)$ shrinks and diverges as $\delta' \to 0$ to an ideal point in $\mathbb{H}^3/\Gamma'_s$. Therefore, if $\sigma | e$ is non-degenerate for each $e \in E(J_0)$, then $\sigma | (E(J_0) - E(J_0) \cap \sigma^{-1}(\partial\mathscr{H}(N)))$ converges to the union of edges of a 3-dimensional ideal, straight simplex. So, one can reconstruct (if necessary) the Haken-straight simplex $\Phi(\sigma_0)$ so that $\sigma | (J_0 - J_0 \cap \sigma^{-1}(\partial\mathscr{H}(N)))$ converges in C^1-topology to a map onto a 3-dimensional ideal, straight simplex as $\delta' \to 0$. If $\sigma | e$ is degenerate for only one (respectively at least two but not all) $e \in E(J_0)$, then one can assume that $\sigma | (J_0 - J_0 \cap \sigma^{-1}(\partial\mathscr{H}(N)))$ converges to a map onto 2-dimensional ideal, straight simplex (respectively onto a geodesic line). Otherwise, $\sigma(J_0)$ shrinks and diverges to an ideal point. For any other component J_l of $\hat{H}_{\sigma_0}$, $\sigma | J_l$ behaves as $\delta' \to 0$ just as $\sigma | J_0$ in the degenerate cases does. Since the volume of any ideal, straight simplex is not greater than v_3, one can choose $\delta' > 0$ so small that, for all $\sigma \in \Phi(\mathrm{basis}(z^{\mathscr{A}}_{L,\mathrm{red}}))$, $|\mathrm{vol}|(\sigma | J_0) < v_3 + \epsilon/4$ and $|\mathrm{vol}|(\sigma | J_l) < \epsilon/4m$. Thus, we have

$$|\mathrm{vol}|(\sigma) = |\mathrm{vol}|(\sigma | J_0) + \sum_l |\mathrm{vol}|(\sigma | J_l) + |\mathrm{vol}|(\sigma | \hat{S}) < v_3 + \epsilon.$$

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This proves (i). If $\sigma_0 \in C^{\mathscr{A}}_{\mathscr{F},\mathrm{red}}$, then $\hat{H}$ contains no truncated-tetrahedral components. Therefore,

$$|\mathrm{vol}|(\sigma) = \sum_l |\mathrm{vol}|(\sigma \mid J_l) + |\mathrm{vol}|(\sigma \mid \hat{S}) < \epsilon.$$

This proves (ii).

6. *Proof of the theorem*

In the case of $\mathscr{F} = f'^{-1}(\partial\mathscr{H}(N)) \cap \mathrm{int}\,\mathscr{H}(M) \neq \varnothing$, for each component F_i of $\mathscr{F}$, let U_{F_i} be the open neighbourhood of the identity in $\mathrm{Isom}^+(\mathbb{H}^3)$ given in Lemma 5. We set

$$z_{M,\mathrm{red}} = \sum_{\sigma^{\mathscr{A}}_{\mathrm{red}} \in C^{\mathscr{A}}_{\mathscr{F},\mathrm{red}}} r_i\, \sigma^{\mathscr{A}}_{i,\mathrm{red}} + p,$$

where the basis of the 3-chain p is $\mathrm{basis}(z_{M,\mathrm{red}}) - C^{\mathscr{A}}_{\mathscr{F},\mathrm{red}}$. Since $\mathrm{red}^{-1}_*(\sigma^{\mathscr{A}}_{i,\mathrm{red}}) = \{\sigma^{\mathscr{A}}_i\}$, by (3·6), $|r_i| = \|z_L \mid \mathscr{A}^{-1}_*(\sigma^{\mathscr{A}}_i)\|$. Therefore

$$\Big\| \sum_{\sigma^{\mathscr{A}}_{\mathrm{red}} \in C^{\mathscr{A}}_{\mathscr{F},\mathrm{red}}} r_i\, \sigma^{\mathscr{A}}_{i,\mathrm{red}} \Big\| = \sum_i |r_i| \geqslant \|z_L \mid \mathscr{A}^{-1}_*(C^{\mathscr{A}}_{\mathscr{F}})\| \geqslant \|z_L \mid C_{\mathscr{F}}\|,$$

where $\mathscr{A}_*$ is the map defined in Section 3. Since $\|z_L \mid C_{\mathscr{F}}\| = \|z_L \mid C_{\mathscr{F}_+}\| + \|z_L \mid C_{\mathscr{F}_-}\|$, by (2·2), (2·3) and (3·2),

$$\sum_i |r_i| \geqslant \sum_{F_i \subset \mathscr{F}} \mu_0(U_{F_i}) > 0. \tag{6·1}$$

By Lemma 5, U_{F_i} is independent of the sufficiently large value of L or equivalently of the sufficiently small value of $\epsilon > 0$. Thus, the value of $\mu_0(\mathscr{F}) = \sum_{F_i \subset \mathscr{F}} \mu_0(U_{F_i})$ is independent of L and ϵ.

Now, we will prove the final lemma.

LEMMA 7. *If $f' \mid \mathscr{U}_M$ is π_1-injective, then $f'^{-1}(\partial\mathscr{H}(N)) \cap \mathrm{int}\,\mathscr{H}(M) = \varnothing$.*

Proof. We suppose that $\mathscr{F} = f'^{-1}(\partial\mathscr{H}(N)) \cap \mathrm{int}\,\mathscr{H}(M) \neq \varnothing$. Recall that the 3-cycle $z_N \in C^f_3(N, \partial N)$ represents $\deg(f)[N, \partial N]$. Let Ω_N be the volume form on N with respect to the metric defined in Section 4. Since the metric on N extends that on $\mathscr{H}(N)^*$, $\mathrm{vol}(N) > \mathrm{vol}(\mathscr{H}(N)^*)$. By (4·1),

$$\langle z_N, \Omega_N \rangle = \deg(f)\,\mathrm{vol}(N) > \deg(f)\,\mathrm{vol}(\mathscr{H}(N)^*) > \deg(f)\,\|N\|\,v_3 - \epsilon.$$

On the other hand, since

$$\|\Phi(\sum_{\sigma^{\mathscr{A}}_{\mathrm{red}} \in C^{\mathscr{A}}_{\mathscr{F},\mathrm{red}}} r_i\, \sigma^{\mathscr{A}}_{i,\mathrm{red}})\| \leqslant \sum_i |r_i| \quad \text{and} \quad \|\Phi(p)\| \leqslant \|p\| = \|z_{M,\mathrm{red}}\| - \sum_i |r_i|,$$

by (3·14), (6·1) and Lemma 6,

$$\begin{aligned}
\langle z_N, \Omega_N \rangle &= \Big\langle \sum_{\sigma^{\mathscr{A}}_{\mathrm{red}} \in C^{\mathscr{A}}_{\mathscr{F},\mathrm{red}}} r_i\, \Phi(\sigma^{\mathscr{A}}_{i,\mathrm{red}}), \Omega_N \Big\rangle + \langle \Phi(p), \Omega_N \rangle \\
&\leqslant (\sum_i |r_i|)\,\epsilon + (\|z_{M,\mathrm{red}}\| - \sum_i |r_i|)\,(v_3 + \epsilon) \\
&\leqslant \|z_M\|\,(v_3 + \epsilon) - (\sum_i |r_i|)\,v_3 \\
&< \frac{v_3(1+\epsilon)(v_3+\epsilon)}{v_3 - \epsilon}\,\|M\| + \epsilon(v_3 + \epsilon) - \mu_0(\mathscr{F})\,v_3.
\end{aligned}$$

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Therefore, we have

$$\deg(f)\,\|N\| - \frac{\epsilon}{v_3} < \frac{(1+\epsilon)(v_3+\epsilon)}{v_3-\epsilon}\,\|M\| + \frac{\epsilon(v_3+\epsilon)}{v_3} - \mu_0(\mathscr{F}).$$

The limit of this inequality as $\epsilon \to 0$ induces

$$\deg(f)\,\|N\| \leqslant \|M\| - \mu_0(\mathscr{F}) < \|M\|.$$

This contradiction implies $f'^{-1}(\partial\mathscr{H}(N)) \cap \operatorname{int}(M) = \varnothing$.

Proof of Theorem. Let $\mathscr{U}$ be the subunion of $\mathscr{U}_M = T_1 \cup \ldots \cup T_n$ consisting of components T_i such that $f|T_i : T_i \to N$ are not π_1-injective. We may set $\mathscr{U} = T_1 \cup \ldots \cup T_k$ $(0 \leqslant k < n)$. Let M_q be the compact 3-manifold obtained by splitting M along $\mathscr{U}$, and let $q : M_q \to M$ be the natural quotient map. As in the proof of Lemma 2, for the two components T_i^+, T_i^- of $q^{-1}(T_i)$ with $T_i \subset \mathscr{U}$, attach solid tori V_i^+, V_i^- to M_q by diffeomorphisms $h_i^\pm : \partial V_i^\pm \to T_i^\pm$, so that $f \circ q : M_q \to N$ extends to a proper map $f_q : M_q \cup_h \mathscr{V} \to N$ with $\deg(f_q) = \deg(f)$, where $\mathscr{V} = V_1^+ \cup V_1^- \cup \ldots \cup V_k^+ \cup V_k^-$ and $h = h_1^+ \cup h_1^- \cup \ldots \cup h_k^+ \cup h_k^-$. Let $\hat{M}$ be the union of components of $M_q \cup_h \mathscr{V}$ containing a component of $q^{-1}(\mathscr{H}(M))$ and set $\hat{f} = f_q|\hat{M}$.

Now, we show that $\hat{\mathscr{U}} = q^{-1}(\mathscr{U}_M - \mathscr{U}) \cap \hat{M}$ is a union of tori in $\hat{M}$ determining a torus decomposition and that $\mathscr{H}(\hat{M}) = q^{-1}(\mathscr{H}(M))$. Note that M_q admits the torus decomposition with respect to $\hat{\mathscr{U}}$ induced from that on M by q. For each component $\hat{T}$ of $\hat{\mathscr{U}}$, $\hat{f} \circ i = \hat{f}|\hat{T} = f \circ q|\hat{T}$ is π_1-injective, where $i : \hat{T} \to \hat{M}$ is the inclusion. This shows that $\hat{T}$ is incompressible in $\hat{M}$. Each component T of $\hat{M} \cap q^{-1}(\mathscr{U})$ is a boundary component of a Seifert piece S of M_q with $S \subset \hat{M}$. Let V_T be the component of $\mathscr{V}$ with $h(\partial V_T) = T$. If the Seifert fibration on S were not extended to $\hat{S} = S \cup_{h|\partial V_T} V_T$, the regular fibre l in T would be contractible in $\hat{S}$. For each component T' of $\partial S - T$, any regular fibre in T' is freely homotopic to l in S. Thus, T' is compressible in $\hat{S} \subset \hat{M}$. This proves that $\partial S \subset q^{-1}(\mathscr{U})$, and hence S is an (isolated) component of M_q with $S \cap q^{-1}(\mathscr{H}(M)) = \varnothing$. This contradicts that S is contained in $\hat{M}$. Hence, the Seifert fibrations on pieces in $\mathscr{S}(M_q \cap \hat{M})$ are naturally extended to those in $\mathscr{S}(\hat{M})$. Thus $\hat{\mathscr{U}}$ determines a torus decomposition on $\hat{M}$.

We may assume that a minimal union $\mathscr{U}_{\hat{M}}$ for $\hat{M}$ contained in $\hat{\mathscr{U}}$. If, for some component C of $M_q \cup_h \mathscr{V} - \hat{M}$, $\deg(f_q|C)$ were not zero, then

$$\deg(f_q|C)\,\|N\| \leqslant \|C\| = 0$$

would imply $\|N\| = 0$, a contradiction. Therefore we have

$$\deg(f_q|(M_q \cup_h \mathscr{V} - \hat{M})) = 0,$$

and hence $\deg(\hat{f}) = \deg(f_q) = \deg(f)$. Since, for each component $\hat{T}_j$ of $\mathscr{U}_{\hat{M}}$, $\hat{f}|\hat{T}_j$ is π_1-injective, by the argument as in the proof of Lemma 3, we may assume that either $\hat{f}(\hat{T}_j) \subset \partial\mathscr{H}(N)$ or $\hat{f}(\hat{T}_j) \cap \mathscr{U}_N = \varnothing$. By Lemma 7, each component F of $\hat{f}^{-1}(\mathscr{U}_N)$ is contained in a single Seifert piece $\hat{S}_t$ of $\mathscr{S}(\hat{M})$. By [**4**, lemma 6·5] and [**5**, theorem VI·3·4], we may assume that F is an incompressible saturated torus in $\hat{S}_t$. In particular, $\hat{f}^{-1}(\mathscr{H}(N))$ is a compact submanifold of $\hat{M}$ such that each component of $\partial\hat{f}^{-1}(\mathscr{H}(N))$ is an incompressible torus in $\hat{M}$.

If $\mathscr{H}(\hat{M})$ had a component $\hat{H}_j$ with $\hat{H}_j \not\subset \hat{f}^{-1}(\mathscr{H}(N))$, then by Lemma 7, we would have

$$\operatorname{int}\hat{H}_j \cap \hat{f}^{-1}(\mathscr{H}(N)) = \varnothing.$$

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So, we would have

$$\begin{aligned}\deg(\hat{f})\,\|N\| &= \deg(\hat{f})\,\|\mathscr{H}(N)\| \\ &\leqslant \|\hat{f}^{-1}(\mathscr{H}(N))\| \leqslant \|\hat{M}\| - \|\hat{H}_j\| < \|\hat{M}\|.\end{aligned}$$

Since $\deg(\hat{f}) = \deg(f)$ and $\|\hat{M}\| = \|\mathscr{H}(\hat{M})\| = \|\mathscr{H}(M)\| = \|M\|$, this inequality would imply $\deg(f)\,\|N\| < \|M\|$, a contradiction. Thus, we have $\mathscr{H}(\hat{M}) \subset \hat{f}^{-1}(\mathscr{H}(N))$. Since $\mathscr{H}(N)$ is homotopically atoroidal, we may assume that for each component $\hat{H}_j$ of $\mathscr{H}(\hat{M})$, $\hat{f}(\partial \hat{H}_j) \subset \partial \mathscr{H}(\hat{M})$. So, if necessary pushing off $f \mid \mathscr{N}_{\text{collar}}(\partial \mathscr{H}(\hat{M}), \text{cl}(\hat{M} - \mathscr{H}(\hat{M})))$ into $\mathscr{S}(N)$ by a homotopy rel. $\partial \mathscr{H}(\hat{M})$, we may assume that $\mathscr{H}(\hat{M})$ is a union of components of $\hat{f}^{-1}(\mathscr{H}(N))$. Since $\|\hat{f}^{-1}(\mathscr{H}(N)) - \mathscr{H}(\hat{M})\| = 0$, the degree of the proper map $\hat{f} \mid \hat{f}^{-1}(\mathscr{H}(N)) - \mathscr{H}(\hat{M})) : \hat{f}^{-1}(\mathscr{H}(N)) - \mathscr{H}(\hat{M}) \to \mathscr{H}(N)$ is zero. Therefore the degree of $\hat{f} \mid \mathscr{H}(\hat{M}) : \mathscr{H}(\hat{M}) \to \mathscr{H}(N)$ is equal to $\deg(f)$. So, $\|M\| = \deg(f)\,\|N\|$ implies $\text{vol}(\text{int}\,\mathscr{H}(\hat{M})) = \deg(\hat{f} \mid \mathscr{H}(\hat{M}))\text{vol}(\text{int}\,\mathscr{H}(N))$. By [**11**, theorem 6·4], $\hat{f} \mid \mathscr{H}(\hat{M})$ is properly homotopic to a $\deg(f)$-fold covering $\hat{g} : \mathscr{H}(\hat{M}) \to \mathscr{H}(N)$. This homotopy can be followed by a whole homotopy of $f : M \to N$. That is, $f : M \to N$ is properly homotopic rel. $M - \mathscr{N}(\mathscr{H}(M), M)$ to a map $g : M \to N$ such that $g \mid \mathscr{H}(M) = \hat{g} \circ q^{-1} \mid \mathscr{H}(M)$. This completes the proof.

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