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Feynman's kernel and the classical path

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Abstract. The quantum mechanical kernel $K(q'',t'';\,q',t')$ in Feynman's form of a sum over paths is evaluated by means of the canonical transformation to the system Q, P for which the Hamiltonian vanishes identically. The kernel is found to be completely determined by the classical paths, and is given by

$$K(q'',t'';\,q',t') = A\sum_c \frac{\partial(Q,P)}{\partial(q',q'')}\exp\left[\frac{i}{\hbar}S(q'',P,t'';\,q',P,t')\right].$$

The sum is over all classical paths leading from q' at t' to q'' at t''. $S(q'',P,t'';\,q',P,t')$ is the classical action along the classical path characterized by the constant momentum P.

1. *Introduction.* The kernel $K(q'',t'';q',t')$ transforms a wave function $\psi(q',t')$ into a wave function $\psi(q'',t'')$:

$$\psi(q'',t'') = \int dq' K(q'',t'';\,q',t')\,\psi(q',t').$$

Feynman (2) has shown that the kernel can be expressed as a sum over all paths leading from coordinates q' at time t' to coordinates q'' at time t'':

$$K(q'',t'';\,q',t') = A\sum_q \exp\left[\frac{i}{\hbar}\int_{t'}^{t''} L(q,\dot{q},t)\,dt\right],$$

where $q(t') = q'$ and $q(t'') = q''$ in $\sum_q$. A is a normalizing constant, which ensures that the kernel is a unitary transformation, and $L(q,\dot{q},t)$ is the classical Lagrangian.

Davies (1) has shown that the kernel can be expressed in terms of the Hamiltonian $H(p,q)$, the sum being taken over all paths in momentum as well as in coordinate space:

$$K(q'',t'';\,q',t') = A\sum_{p,q} \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\{p\dot{q}-H(p,q)\}\,dt\right],$$

where $q(t') = q', q(t'') = q''$, and p is unrestricted in $\sum_{p,q}$.

It is always possible in principle to find a canonical transformation to new coordinates Q and momenta P for which the new Hamiltonian vanishes identically. The classical equations of motion are then $\dot{P} = 0$, $\dot{Q} = 0$, or $P'' = P'$, $Q'' = Q'$.

In terms of the new coordinates, Davies has shown that the kernel becomes simply

$$K(Q'',t'';\,Q',t') = \delta(Q''-Q').$$

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In this special coordinate system, it seems that only the classical path ($Q'' = Q'$) contributes to the quantum mechanical kernel: where there is no classical path, the kernel is zero.

The purpose of this paper is to show that this remains true in an arbitrary coordinate system. We shall use the canonical transformation to the system in which the Hamiltonian vanishes identically in order to evaluate the kernel expressed in terms of the old coordinates. We shall then see that the kernel is entirely determined by the classical paths, and is given by

$$K(q'',t'';\,q',t') = A\sum_c \frac{\partial(Q,P)}{\partial(q',q'')}\exp\left[\frac{i}{\hbar}S\,(q'',P,t'';q',P,t')\right],$$

where the summation is over all classical paths leading from q' at t' to q'' at t''.

$S(q'',P,t'';\,q',P,t')$ is the classical action along the classical path characterized by the constant momentum P. The Jacobian $\partial(Q,P)/\partial(q',q'')$ is proportional to the density of paths leading from the interval dq' to the interval dq''.

This implies that the quantum mechanical solution to a dynamical problem follows immediately from a complete classical solution. Crudely speaking, we may say that electrons travel along classical paths, carrying with them a phase proportional to the classical action.

This does not contradict the possibility of such non-classical phenomena as 'tunnelling'. Classically, a particle inside a finite potential well is permanently imprisoned only if it is constrained to have energy and momentum below some upper bound. However, the quantum mechanical kernel imposes no restraint on the momenta, and a classical path across a finite potential barrier can always be found provided the particle is given a sufficiently high momentum.

2. *Application of the canonical transformation.* For p and q to be canonical coordinates and momenta, the variation of the action integral must vanish

$$\delta\int_{t'}^{t''}\{p\dot{q}-H(p,q)\}\,dt = 0.$$

This must remain true after a canonical transformation to the coordinates Q and the momenta P for which the Hamiltonian vanishes identically

$$\delta\int_{t'}^{t''}P\dot{Q}\,dt = 0.$$

The difference between the two integrals must then be a pure boundary term, independent of the intermediate path

$$\int_{t'}^{t''}\{p\dot{q}-H(p,q)\}\,dt = \int_{t'}^{t''}P\dot{Q}\,dt + S(q'',P'',t'';\,q',P',t').$$

The exponential of the action integral is transformed into

$$\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}\{p\dot{q}-H(p,q)\}\,dt\right] = \exp\left[\frac{i}{\hbar}\int_{t'}^{t''}P\dot{Q}\,dt\right]\exp\left[\frac{i}{\hbar}S(q'',P'',t'';\,q',P',t')\right].$$

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The summation over all paths can now be carried out in two stages: first, sum the first exponential over all paths with fixed end-points P', Q', and P'', Q'' (for which the second exponential remains a constant); second, integrate over all P', Q' compatible with q', and over all P'', Q'' compatible with q''.

3. *Evaluation of the sum over paths with fixed end-points.* To evaluate the sum over paths of the exponential

$$\exp\left[\frac{i}{\hbar}\int_{t'}^{t''} P\dot{Q}\,dt\right]$$

with fixed P', Q', P'', Q'', we approximate the integral by a sum, and integrate over all intermediate values of P and Q. It is legitimate to replace integration over p, q by integration over P, Q, because the volume element, $dp\,dq$, remains invariant under a canonical transformation.

It is convenient to use the trapezium rule with an even number of intervals, so that $t'' = t' + 2n\delta t$. We write

$$P(t' + k\delta t) = P_k \quad \text{and} \quad Q(t' + k\delta t) = Q_k.$$

The integral becomes

$$\begin{aligned}
&\int_{t'}^{t''} P\dot{Q}\,dt \\
&\quad \doteq \delta t\,[\tfrac{1}{2}P'\dot{Q}' + P_1\dot{Q}_1 + \ldots + P_{2n-1}\dot{Q}_{2n-1} + \tfrac{1}{2}P''\dot{Q}''] \\
&\quad \doteq \delta t\left[\tfrac{1}{2}P'\frac{Q_1 - Q'}{\delta t} + P_1\frac{Q_2 - Q'}{2\delta t} + P_2\frac{Q_3 - Q_1}{2\delta t} + \ldots + P_{2n-1}\frac{Q'' - Q_{2n-2}}{2\delta t} + \tfrac{1}{2}P''\frac{Q'' - Q_{2n-1}}{\delta t}\right] \\
&\quad = \tfrac{1}{2}[Q_1(P' - P_2) + Q_3(P_2 - P_4) + \ldots + Q_{2n-1}(P_{2n-2} - P'') + P_1(Q_2 - Q') + P_3(Q_4 - Q_2) + \ldots \\
&\qquad + P_{2n-1}(Q'' - Q_{2n-2})] + \tfrac{1}{2}(P''Q'' - P'Q').
\end{aligned}$$

The exponential now becomes

$$\begin{aligned}
&\exp\left[\frac{i}{2\hbar}Q_1(P' - P_2)\right]\exp\left[\frac{i}{2\hbar}Q_3(P_2 - P_4)\right]\ldots\exp\left[\frac{i}{2\hbar}Q_{2n-1}(P_{2n-2} - P'')\right] \\
&\quad \times \exp\left[\frac{i}{2\hbar}P_1(Q_2 - Q')\right]\exp\left[\frac{i}{2\hbar}P_3(Q_4 - Q_2)\right]\ldots\exp\left[\frac{i}{2\hbar}P_{2n-1}(Q'' - Q_{2n-2})\right] \\
&\quad \times \exp\left[\frac{i}{2\hbar}(P''Q'' - P'Q')\right].
\end{aligned}$$

On integrating over Q_{2k+1}, we have

$$\int \exp\left[\frac{i}{2\hbar}Q_{2k+1}(P_{2k} - P_{2k+2})\right]dQ_{2k+1} \propto \delta(P_{2k} - P_{2k+2}).$$

The proportionality constant may be ignored, since it is taken care of by the final normalization. On integrating over $Q_1, Q_3, \ldots, Q_{2n-1}, P_1, P_3, \ldots, P_{2n-1}$, we obtain

$$\begin{aligned}
&\delta(P' - P_2)\,\delta(P_2 - P_4)\ldots\delta(P_{2n-2} - P'') \times \delta(Q' - Q_2)\,\delta(Q_2 - Q_4)\ldots\delta(Q_{2n-2} - Q'') \\
&\qquad\qquad \times \exp\left[\frac{i}{2\hbar}(P''Q'' - P'Q')\right].
\end{aligned}$$

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On integrating over $Q_2, Q_4, \ldots, Q_{2n-2}, P_2, P_4, \ldots, P_{2n-2}$, this becomes

$$\delta(P'-P'')\,\delta(Q'-Q'')\exp\left[\frac{i}{2\hbar}(P''Q''-P'Q')\right] = \delta(P'-P'')\,\delta(Q'-Q'').$$

We see from this that only the classical paths contribute to the kernel.

4. *Integration over the end-points.* We must now find the appropriate range over which to integrate P', Q', P'', Q''. Now, $\psi(q',t')\,dq'$ is the probability amplitude for $q = q(Q',P',t')$ to lie between q' and $q'+dq'$, and similarly for $\psi(q'',t'')$. Hence the kernel in the form $dq'\,dq''K(q'',t'';\,q',t')$ is a sum $\sum_{dq}$ over all paths such that

$$\left.\begin{aligned} q' \leqslant q(Q',P',t') \leqslant q'+dq', \\ q'' \leqslant q(Q'',P'',t'') \leqslant q''+dq''. \end{aligned}\right\}$$

The kernel now becomes

$$\begin{aligned} &dq'\,dq''K(q'',t'';\,q',t') \\ &\quad = A\sum_{dq}\exp\left[\frac{i}{\hbar}\int_{t'}^{t''}P\dot{Q}\,dt\right]\exp\left[\frac{i}{\hbar}S(q'',P'',t'';\,q',P',t'\right] \\ &\quad = A\int dP'\,dQ'\,dP''dQ''\,\delta(P'-P'')\,\delta(Q'-Q'')\exp\left[\frac{i}{\hbar}S(q'',P'',t'';\,q',P',t')\right] \\ &\quad = A\int dP\,dQ\exp\left[\frac{i}{\hbar}S(q'',P,t'';\,q',P,t')\right], \end{aligned}$$

where the range of integration is now given by

$$\left.\begin{aligned} q' \leqslant q(Q,P,t') \leqslant q'+dq', \\ q'' \leqslant q(Q,P,t'') \leqslant q''+dq''. \end{aligned}\right\}$$

In general, there will be a number of different classical paths from q' at t' to q'' at t''. For example, a particle in a potential well can either proceed directly from q' to q'', or it can bounce off one of the walls first.

Each classical path will have a different momentum P and a different classical action $S(q'',P,t'';\,q',P,t')$. For any one classical path $S(q'',P,t'';\,q',P,t')$ is approximately constant over the small range of integration and

$$\int dP\,dQ = \frac{\partial(Q,P)}{\partial(q',q'')}\,dq'\,dq''.$$

The kernel is therefore given by

$$K(q'',t'';\,q',t') = A\sum_{c}\frac{\partial(Q,P)}{\partial(q',q'')}\exp\left[\frac{i}{\hbar}S(q'',P,t'';\,q',P,t')\right],$$

where the summation $\sum_{c}$ is over all classical paths from q',t' to q'',t''.

It is worth noticing that the Jacobian $\partial(Q,P)/\partial(q',q'')$ can be expressed entirely in terms of the old coordinates and momenta

$$\frac{\partial(Q,P)}{\partial(q',q'')} = \frac{\partial(Q,P)}{\partial(q',p')}\frac{\partial(q',p')}{\partial(q',q'')} = \frac{\partial(q',p')}{\partial(q',q'')} = \left\|\frac{\partial q''}{\partial p'}\right\|^{-1}.$$

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5. *The quadratic Lagrangian.* Kilmister (3) has shown that when the Lagrangian is quadratic in the coordinates and velocities, then the kernel becomes

$$K(q'',t'';\,q',t') = A\exp\left[\frac{i}{\hbar}S(q'',t'';\,q',t')\right].$$

This implies that, if our form for the kernel is correct, then the Jacobian $\|\partial q''/\partial p'\|^{-1}$ is a constant, independent of q' and q'', whenever the Lagrangian is quadratic. It is of interest to see that this is, in fact, so.

When the Lagrangian is quadratic, then the differential equations of motion are linear. The coordinates at any time t are then linear combinations of the initial coordinates and momenta. In particular,

$$q'' = q(t'') = F(t'')\,q' + G(t'')\,p',$$

where $F(t)$ and $G(t)$ are matrix functions.

The Jacobian is now given by

$$\left\|\frac{\partial q''}{\partial p'}\right\|^{-1} = \|G(t'')\|^{-1},$$

which is indeed independent of q' and q''.

6. *Conclusions and discussion.* We have seen that only the classical paths contribute to the quantum mechanical kernel, which is given by

$$K(q'',t'';\,q',t') = A\sum_c \frac{\partial(Q,P)}{\partial(q',q'')}\exp\left[\frac{i}{\hbar}S(q'',P,t'';\,q',P,t')\right].$$

The summation $\sum_c$ is over all classical paths, and S is the classical action.

This result is an expression of Huygen's principle for matter waves. Each portion dq' of an advancing wave-front makes a contribution

$$K(q'',t'';\,q',t')\,\psi(q',t')\,dq'$$

to the amplitude $\psi(q'',t'')$ of the wave-front at time t''. The amplitude of the contribution is proportional to the density of classical paths leading from the interval dq' at t' to the interval dq'' at t'', which in turn is proportional to the Jacobian

$$\partial(Q,P)/\partial(q',q'').$$

The phase of the contribution is proportional to the classical action. Where there are several classical paths, the total contribution is the sum of the partial contributions corresponding to each classical path.

REFERENCES

(1) Davies, H. *Proc. Cambridge Philos. Soc.* **59** (1963), 147–155.
(2) Feynman, R. P. *Rev. Mod. Phys.* **20** (1948), 367–387.
(3) Kilmister, C. W. *Proc. Cambridge Philos. Soc.* **54** (1958), 302–304.

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