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Recurrence, rigidity, and popular differences

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Abstract. We construct a set of integers S such that every translate of S is a set of recurrence and a set of rigidity for a weak mixing measure preserving system. Here 'set of rigidity' means that enumerating S as $(s_n)_{n\in\mathbb{N}}$ produces a rigidity sequence. This construction generalizes or strengthens results of Katznelson, Saeki (on equidistribution and the Bohr topology), Forrest (on sets of recurrence and strong recurrence), and Fayad and Kanigowski (on rigidity sequences). The construction also provides a density analogue of Julia Wolf's results on popular differences in finite abelian groups.

1. *Motivation*

Given a set of integers A, we write $d^*(A)$ for its upper Banach density; see §2 for definitions. The *difference set* $A - A$ is $\{a - b : a, b \in A\}$. If $n \in \mathbb{Z}$, write $n + A$ for the *translate* $\{n + a : a \in A\}$. A *Bohr neighborhood* in the integers is a non-empty set of the form $\{n : p(n) > 0\}$, where $p(n) = \sum_{j=0}^{m} c_j \sin(\lambda_j n) + d_j \cos(\lambda_j n)$ is a trigonometric polynomial ($c_j, d_j, \lambda_j \in \mathbb{R}$). Ruzsa asked the following question in [**14**, Part II, §5.8].

Question 1.1. If $d^*(A) > 0$, must $A - A$ contain a Bohr neighborhood?

Question 1.1 was also asked in [**5**, **20**], and in a slightly different form in [**17**]. While we do not answer Question 1.1 in the present paper, we answer the closely related Questions 1.2 and 1.3, providing support for a negative answer to Question 1.1. Most techniques for analyzing the difference set $A - A$ of a set A having positive upper Banach density actually provide information about the *c-popular differences* $P_c(A) := \{n : d^*(A \cap (A - n)) > c\}$, where $c > 0$ is some fixed number. Note that $P_c(A) \subseteq A - A$ for each $c \geq 0$. The following questions now present themselves.

Question 1.2. If $d^*(A) > 0$, is there necessarily a $c > 0$ such that $P_c(A)$ contains a Bohr neighborhood of some $n_0 \in \mathbb{Z}$?

Question 1.3. Given $A \subseteq \mathbb{Z}$ having $d^*(A) > 0$, is there a $c > 0$, an $n_0 \in \mathbb{Z}$, and a set $B \subseteq \mathbb{Z}$ having $d^*(B) > 0$ such that $P_c(A)$ contains $n_0 + B - B$?

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Our Corollary 2.6 answers Questions 1.2 and 1.3 in the negative. Corollary 2.6 follows from our main result, Theorem 2.1, an ergodic theoretic construction generalizing some classical results in harmonic analysis and recent results in ergodic theory.

The next section summarizes some concepts from measure preserving dynamics and states our main results.

2. *Recurrence and rigidity*

A *measure preserving system* (or just 'system') (X, μ, T) is a probability measure space (X, μ), together with a map $T : X \to X$ preserving μ: $\mu(T^{-1}D) = \mu(D)$ for every measurable $D \subseteq X$. In this paper, T will always be invertible. We do not mention the σ-algebra of measurable sets on X, as we only ever consider measurable subsets of X. See any of [**8**, **9**, **13**, **15**, **28**, **32**] for standard definitions regarding measure preserving systems, including *ergodicity*, *weak mixing*, and the definition of isomorphism for measure preserving systems.

2.1. *Definitions.*

A set $S \subseteq \mathbb{Z}$ is a:

- *set of recurrence* if for every system (X, μ, T) and every $D \subseteq X$ having $\mu(D) > 0$, there exists $n \in S$ such that $\mu(D \cap T^n D) > 0$;
- *set of strong recurrence* if for every system (X, μ, T) and every $D \subseteq X$ with $\mu(D) > 0$, there is a $c > 0$ such that $\mu(D \cap T^n D) > c$ for infinitely many $n \in S$;
- *set of optimal recurrence*† if for every system (X, μ, T), all $\varepsilon > 0$, and every $D \subseteq X$ having $\mu(D) > 0$, there exists $n \in \mathbb{Z}$ such that $\mu(D \cap T^n D) > \mu(D)^2 - \varepsilon$;
- *set of rigidity* if S is infinite and there is a non-trivial ergodic system $\mathbf{X} = (X, \mu, T)$ such that for all measurable $D \subseteq X$ and all $\varepsilon > 0$, the set $\{n \in S : \mu(D \triangle T^n D) > \varepsilon\}$ is finite. For such $\mathbf{X}$, we say that S is *set of rigidity for* $\mathbf{X}$. If S is a set of rigidity for a non-trivial weak mixing system $\mathbf{X}$, we say S is a *WM set of rigidity*.

Forrest [**12**] constructed an example of a set of recurrence which is not a set of strong recurrence. See [**25**] for another proof of Forrest's result.

For technical reasons, some variations on the above definitions are in order. Let $\mathbf{X} = (X, \mu, T)$ be a measure preserving system.

A set $S \subseteq \mathbb{Z}$ is a *set of strong recurrence for* $\mathbf{X}$ if for every set $D \subseteq X$ having $\mu(D) > 0$, there is a $c > 0$ such that $\{n \in S : \mu(D \cap T^n D) > c\}$ is infinite. The set S is a *WM set of strong recurrence* if it is a set of strong recurrence for every weak mixing system.

A sequence of integers $(s_n)_{n\in\mathbb{N}}$ is a *rigidity sequence for* $\mathbf{X}$ if for every set $D \subseteq X$, $\lim_{n\to\infty} \mu(D \triangle T^{s_n} D) = 0$. The set S is a set of rigidity for $\mathbf{X}$ if and only if $(s_n)_{n\in\mathbb{N}}$ is a rigidity sequence for $\mathbf{X}$ whenever $(s_n)_{n\in\mathbb{N}}$ is an enumeration of S.

A system $\mathbf{X}$ is a *rigid-recurrence system* if there is a set of rigidity S for $\mathbf{X}$ such that every translate of S is a set of recurrence, meaning $S + n$ is a set of recurrence for every $n \in \mathbb{Z}$.

2.2. *Results.*

Our main result is Theorem 2.1; its proof uses measure concentration results similar to the use of Kleitman's theorem in [**12**] and [**25**], and to the measure

† Also known as 'nice recurrence'.

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concentration arguments of [**33**]. See §§5 and 8 for further discussion of Theorem 2.1 and its relation to other work.

THEOREM 2.1. *There is a set $S \subseteq \mathbb{Z}$ such that every translate of S is both a set of recurrence and a WM set of rigidity.*

We prove Theorem 2.1 in §5. An immediate consequence of the theorem is the following proposition.

PROPOSITION 2.2. *There is a weak mixing rigid-recurrence system.*

Our first corollary generalizes the main result of [**12**].

COROLLARY 2.3. *There is a set $S \subseteq \mathbb{Z}$ such that every translate of S is a set of recurrence, and no translate of S is a WM set of strong recurrence. Consequently, no translate of S is a set of strong recurrence.*

Corollary 2.3 is proved in §7.

2.3. *Popular differences.* If G is an abelian group, $t \in G$, and $A, B \subseteq G$, we write $A + t$ for $\{a + t : a \in A\}$, and $A - B$ for $\{a - b : a \in A, b \in B\}$. The difference set $A - B$ is exactly the set $\{g \in G : B \cap (A - g) \neq \varnothing\}$.

If G is a finite abelian group, $A \subseteq G$, and $c \geq 0$, the *c-popular difference set* $P_c(A)$ is the set of elements $g \in G$ such that there are more than $c|G|$ pairs $(a, b) \in A \times A$ satisfying $a - b = g$. In other words, $P_c(A) := \{g : |A \cap (A - g)| > c|G|\}$. Wolf constructed dense sets $A \subseteq \mathbb{Z}/N\mathbb{Z}$ where the popular difference set of A does not contain any difference set $B - B$, where B is not small. To be more precise, we have the following theorem.

THEOREM 2.4. [**33**, Theorem 1.1] *There is a function $c : \mathbb{N} \to \mathbb{R}$ having $\lim_{n\to\infty} c(n) = 0$, and there are sets $A_N \subseteq \mathbb{Z}/N\mathbb{Z}$ having $|A_N| > N/3$ such that $P_{c(N)}(A_N)$ does not contain any difference set $B - B$, where $B \subseteq \mathbb{Z}/N\mathbb{Z}$ has $|B| > c(N) \cdot N$.*

Corollary 2.6 provides an analogue of Theorem 2.4 in terms of upper Banach density, while Corollary 2.5 is an analogue of Theorem 2.4 for measure preserving systems.

2.4. *Upper Banach density and Følner sequences.* If $A \subseteq \mathbb{Z}$, the *upper Banach density* of A is $d^*(A) := \lim_{N\to\infty} \sup_{M\in\mathbb{Z}} (|A \cap [M, M + N - 1]|/N)$.

A *Følner sequence* for $\mathbb{Z}$ is a sequence $(\Phi_j)_{j\in\mathbb{N}}$ of finite subsets of $\mathbb{Z}$ satisfying $\lim_{j\to\infty} (|\Phi_j \triangle (\Phi_j + n)|/|\Phi_j|) = 0$ for every $n \in \mathbb{Z}$. A sequence of intervals $I_j = \{M_j, M_j + 1, \ldots, N_j\}$ is a Følner sequence if and only if $\lim_{j\to\infty} |I_j| = \infty$. If $A \subseteq \mathbb{Z}$, and $\mathbf{\Phi} = (\Phi_j)_{j\in\mathbb{N}}$ is a Følner sequence, we write $d_{\mathbf{\Phi}}(A)$ for $\lim_{j\to\infty} (|A \cap \Phi_j|/|\Phi_j|)$. Observe that $d^*(A)$ is the maximum of all values of $d_{\mathbf{\Phi}}(A)$ where $\mathbf{\Phi}$ is a Følner sequence.

For a set of integers $A \subseteq \mathbb{Z}$, an analogue of the c-popular difference set is $P_c(A) := \{n \in \mathbb{Z} : d^*(A \cap (A - n)) > c\}$. For a measure preserving system (X, μ, T) and a set $D \subseteq X$, the analogue of the c-popular difference set is the *c-popular recurrence set* $R_c(D) := \{n : \mu(D \cap T^n D) > c\}$. From Corollary 2.3 we will deduce the following analogues of Theorem 2.4.

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COROLLARY 2.5. *There is a weak mixing system (X, μ, T) such that for all $D \subseteq X$ having $\mu(D) > 0$ and $D \cap TD = \varnothing$, and all $c > 0$, the set $R_c(D)$ does not contain a set of the form $n_0 + B - B$, where $d^*(B) > 0$ and $n_0 \in \mathbb{Z}$. Consequently, $R_c(D)$ does not contain a Bohr neighborhood.*

See §8 for discussion of the Bohr topology; Proposition 8.2 explains why the second assertion of Corollary 2.5 follows from the first.

Corollary 2.5 should be contrasted with the classical result that for every system $\mathbf{X} = (X, \mu, T)$, every $D \subseteq X$, and all $c < \mu(D)^2$, $R_c(D)$ contains a set of the form $U \setminus C$, where $U \subseteq \mathbb{Z}$ is a Bohr neighborhood of 0 and $d^*(C) = 0$. Furthermore, if $\mathbf{X}$ is weak mixing, the complement $E := \mathbb{Z} \setminus R_c(D)$ has $d^*(E) = 0$. These results can be understood in terms of Fourier transforms of measures; see [**3**, §1.3] for a brief exposition.

COROLLARY 2.6. *For all $\varepsilon > 0$, there is a set $A \subseteq \mathbb{Z}$ having $d^*(A) > \frac{1}{2} - \varepsilon$ such that for all $c > 0$, the set $P_c(A)$ does not contain a set of the form $n_0 + B - B$, where $d^*(B) > 0$ and $n_0 \in \mathbb{Z}$. Consequently, $P_c(A)$ does not contain a Bohr neighborhood.*

In fact, the examples A in Corollary 2.6 can be taken to have *uniform density*, meaning $\lim_{n\to\infty} (|A \cap \Phi_n|/|\Phi_n|) = d^*(A)$ for every Følner sequence $(\Phi_n)_{n\in\mathbb{N}}$; this is proved in Proposition 7.3. In particular, there are syndetic† sets A none of whose popular difference sets $P_c(A)$, $c > 0$, contain a translate of a difference set $n_0 + B - B$, where $d^*(B) > 0$.

Remark 2.7. We state the conclusion of Corollary 2.5 in terms of difference sets $B - B$, where $d^*(B) > 0$, rather than recurrence sets $R_0(E)$. Lemma 3.3 shows that it makes no difference whether we use $R_0(E)$ or $B - B$ in the conclusion.

2.5. *Outline of the paper.* Section 3 presents a version of Furstenberg's correspondence principle needed in the proofs of Theorem 2.1 and its corollaries. Section 4 provides one of the key lemmas, showing that certain subsets of finite groups exhibit a recurrence property. Theorem 2.1 is proved in §5. Section 6 contains lemmas needed for Corollaries 2.3, 2.5, and 2.6, which are proved in §7. Section 8 discusses the Bohr topology on $\mathbb{Z}$ and how our results are related to the existing literature. Section 9 contains a characterization of upper Banach density needed in the proof of Theorem 2.1.

3. *The correspondence principle*

Sets of recurrence may be characterized as those sets which have non-empty intersection with every difference set $A - A$, where $d^*(A) > 0$. Corollary 3.4 formalizes this statement, which plays a key role in the proof of Theorem 2.1.

The following version of Furstenberg's correspondence principle is a special case in [**3**, Proposition 3.1].

LEMMA 3.1. *Let $A \subseteq \mathbb{Z}$. Then there is an ergodic measure preserving system (X, μ, T) and a set $D \subseteq X$ having $\mu(D) = d^*(A)$, and*

$$d^*(A \cap (A - m)) \geq \mu(D \cap T^{-m} D)$$

for all $m \in \mathbb{Z}$.

† A set $S \subseteq \mathbb{Z}$ is *syndetic* if there is a finite set F such that $F + S = \mathbb{Z}$.

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Earlier versions of the correspondence principle, as in [**1**, **13**], did not assert the ergodicity of (X, μ, T), as ergodic decomposition was often employed in the analysis of (X, μ, T).

The next lemma is a partial converse of Lemma 3.1. It is a special case in [**26**, Proposition 3.2.7].

LEMMA 3.2. *Let (X, μ, T) be a measure preserving system and let $D \subseteq X$ have $\mu(D) > 0$. Then there is a set $A \subseteq \mathbb{Z}$ having $d^*(A) \geq \mu(D)$ such that*

$$A - A \subseteq \{m : \mu(D \cap T^{-m} D) > 0\}.$$

Combining Lemmas 3.1 and 3.2 yields the following result. We use the notation $R_c(D)$ defined in §2.4.

LEMMA 3.3. *Let $\delta > 0$ and let $E \subseteq \mathbb{Z}$. The following statements are equivalent.*

(i) *There exists $A \subseteq \mathbb{Z}$ having $d^*(A) \geq \delta$ such that $A - A \subseteq E$.*

(ii) *There exist a measure preserving system (X, μ, T) and a set $D \subseteq X$ having $\mu(D) \geq \delta$ such that $R_0(D) \subseteq E$.*

(iii) *There exist an* ergodic *measure preserving system (X, μ, T) and a set $D \subseteq X$ having $\mu(D) \geq \delta$ such that $R_0(D) \subseteq E$.*

The following well-known corollary (see [**4**] for another proof) allows us to treat sets of recurrence from a combinatorial perspective.

COROLLARY 3.4. *Let $S \subseteq \mathbb{Z}$. The following statements are equivalent.*

(i) *S is a set of recurrence.*

(ii) *If $A \subseteq \mathbb{Z}$ has $d^*(A) > 0$, then $(A - A) \cap S \neq \varnothing$.*

(iii) *For every* ergodic *measure preserving system (X, μ, T) and every $D \subseteq X$ having $\mu(D) > 0$, there exists $n \in S$ such that $\mu(D \cap T^n D) > 0$.*

The corollary is an immediate consequence of Lemma 3.3, so we omit the proof.

The connection between popular differences $P_c(A)$ and popular recurrence sets $R_c(D)$ for $c > 0$ is not as straightforward as the connection between difference sets $A - A$ and recurrence sets $R_0(D)$. Rather than attempt to prove an analogue of Lemma 3.3 for popular differences and popular recurrence, in §7 we resort to *ad hoc* constructions using uniquely ergodic systems.

4. *Quantitative analogues and concentration of measure*

We consider a quantitative version of recurrence in some finite abelian groups. This will be used in the proof of Theorem 2.1 to show that the set S there constructed is indeed a set of recurrence. The main result of this section is Lemma 4.3, the key lemma in proving Proposition 5.2. Our approach is virtually identical to the proof of [**33**, Lemma 2.4], and we require some results from [**27**]. To state those, we introduce some terminology.

Let $k, r \in \mathbb{N}$, and let $\Lambda_k := \{\lambda \in \mathbb{C} : \lambda^k = 1\}$, the group of kth roots of unity, where the group operation is ordinary multiplication. Consider the group $\Lambda_k^r := (\Lambda_k)^r$, the rth cartesian power of Λ_k. We write the group operation on Λ_k^r using multiplicative notation.

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If $A, B \subseteq \Lambda_k^r$, we write $A \cdot B$ for the product set $\{ab : a \in A, b \in B\}$, and $A^{-1}A$ for the difference set† $\{a^{-1}b : a, b \in A\}$.

Write an element of Λ_k^r as $x = (x_1, \ldots, x_r)$. Let d_0 denote one half of euclidean distance in $\mathbb{C}$, so that $d(x, y) := \sum_{j=1}^r d_0(x_j, y_j)$ defines a translation-invariant metric on Λ_k^r with diameter less than or equal to r. For $t \geq 0$ and $x \in \Lambda_k^r$, let $U_t(x)$ denote the d-ball of radius t around the element $x = (x_1, \ldots, x_r)$, meaning $U_t(x) = \{y \in \Lambda_k^r : \sum_{j=1}^r d_0(x_j, y_j) \leq t\}$. We refer to $U_t(x)$ as the *Hamming ball* of radius t around x.

The following lemma may be proved in exactly the same manner‡ as [**27**, Proposition 7.7].

LEMMA 4.1. *Let $A \subseteq \Lambda_k^r$ be non-empty, $t \geq 0$, and let $\alpha = |A|/|\Lambda_k^r|$. Then*

$$|A \cdot U_t(0)|/|\Lambda_k^r| \geq 1 - \alpha^{-1} \exp(-t^2/2r).$$

Consequently, for every $x \in \Lambda_k^r$,

$$|A \cdot U_t(x)|/|\Lambda_k^r| \geq 1 - \alpha^{-1} \exp(-t^2/2r).$$

An immediate consequence of Lemma 4.1 is that Hamming balls of moderate radius can be seen as sets of recurrence.

LEMMA 4.2. *Let $A \subseteq \Lambda_k^r$, $x \in \Lambda_k^r$, and $t > 0$. If $(A^{-1}A) \cap U_t(x) = \varnothing$, then $|A|/|\Lambda_k^r| \leq \exp(-t^2/4r)$.*

Proof. Assume $A^{-1}A \cap U_t(x) = \varnothing$. Write α for $|A|/|\Lambda_k^r|$. Note that

$$A^{-1}A \cap U_t(x) = \varnothing \iff A \cap (A \cdot U_t(x)) = \varnothing,$$

which implies $|A| + |A \cdot U_t(x)| \leq |\Lambda_k^r|$, or

$$\frac{|A|}{|\Lambda_k^r|} + \frac{|A \cdot U_t(x)|}{|\Lambda_k^r|} \leq 1. \tag{4.1}$$

Lemma 4.1 implies $|A \cdot U_t(x)|/|\Lambda_k^r| \geq 1 - \alpha^{-1} \exp(-t^2/2r)$; substituting into inequality (4.1), we get

$$\frac{|A|}{|\Lambda_k^r|} + 1 - \alpha^{-1} \exp(-t^2/2r) \leq 1,$$

meaning $\alpha + 1 - \alpha^{-1} \exp(-t^2/2r) \leq 1$. Solving for α, we find $\alpha^2 \leq \exp(-t^2/2r)$, or $\alpha \leq \exp(-t^2/4r)$, which is the desired conclusion. □

The next lemma is a qualitative restatement of Lemma 4.2.

LEMMA 4.3. *For all ε, $\delta > 0$ and all $k \in \mathbb{N}$, there exists $N = N(\delta, \varepsilon, k)$ such that: if $r > N$ and $A \subseteq \Lambda_k^r$ has $|A| \geq \delta|\Lambda_k^r|$, then for all $x \in \Lambda_k^r$, there are $a_1, a_2 \in A$ such that $a_1^{-1}a_2 \in U_{r \cdot \varepsilon}(x)$.*

† Perhaps we should write 'quotient set', but we abuse terminology instead.

‡ We use the parameter r where [**27**] uses the parameter n. Take $\gamma = 1$ in the conclusion of [**27**, Proposition 7.7] to get the expression $-t^2/2r$ in the exponent here.

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Proof. If $(A^{-1}A) \cap U_{r\cdot\varepsilon}(x) = \varnothing$, Lemma 4.2 implies

$$|A|/|\Lambda_k^r| \le \exp(-\varepsilon^2 r/4).$$

For r sufficiently large, this implies $|A|/|\Lambda_k^r| < \delta$. □

Remark 4.4. Consider Λ_k^r as the set of functions $f : \{1, \dots, r\} \to \Lambda_k$. If m is normalized counting measure on $\{1, \dots, r\}$ and $x \in \Lambda_k^r$, then $\{y \in \Lambda_k^r : \|y - x\|_{L^1(m)} < \varepsilon\}$ is the set of y satisfying $(1/r)\sum_{j=1}^r |y_j - x_j| < \varepsilon$, or

$$\sum_{j=1}^{r} \frac{1}{2}|y_j - x_j| < r \cdot \varepsilon/2.$$

So the $L^1(m)$-ball of radius ε around x is the Hamming ball $U_{r\cdot\varepsilon/2}(x)$.

5. *Fourier transforms of measures; proof of Theorem 2.1*

As we construct the sets S of Theorem 2.1 we view $\mathbb{Z}$ as the group of exponential functions $\{e_n : n \in \mathbb{Z}\}$, where $e_n : [0, 1) \to \mathbb{C}$ is given by $e_n(x) := \exp(2\pi i n x)$. In §5.2 we find collections of such functions imitating Hamming balls in the groups Λ_k^r considered in §4; this will be made precise by the proof of Proposition 5.2. Section 5.3 provides lemmas for assembling finite pieces of the sets found in Proposition 5.2 into the set S described in Theorem 2.1, and the assembly is carried out in §5.4.

5.1. *Fourier transforms of measures.*

Proposition 5.2, the main technical fact underlying the proof of Theorem 2.1, is of independent interest. Before stating the proposition, we summarize some background on Fourier transforms of measures and Kronecker sets.

Here $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ is the torus with the usual topology. We identify $\mathbb{T}$ with the interval $[0, 1)$, so the functions e_n correspond to the characters of the group $\mathbb{T}$.

A Borel measure μ on a topological space X is *continuous* if $\mu(\{x\}) = 0$ for every $x \in X$.

If σ is a measure on $\mathbb{T}$, its Fourier transform is the function $\hat{\sigma} : \mathbb{Z} \to \mathbb{C}$ given by $\hat{\sigma}(n) = \int e_n(\theta)\, d\sigma(\theta)$. For a Borel probability measure σ on $\mathbb{T}$ and a sequence of integers $(s_n)_{n\in\mathbb{N}}$, the condition $\lim_{n\to\infty} \hat{\sigma}(s_n) = 1$ is equivalent to $\lim_{n\to\infty} \int |\exp(2\pi i s_n\theta) - 1|\, d\sigma(\theta) = 0$. Under the additional assumption that σ is continuous, these conditions imply that the sequence $(s_n)_{n\in\mathbb{N}}$ is a rigidity sequence for a weak mixing system, as the following lemma states.

LEMMA 5.1.

(i) *A sequence $(s_n)_{n\in\mathbb{N}}$ is a rigidity sequence for a weak mixing system if and only if there is a continuous measure σ on $\mathbb{T}$ such that $\lim_{n\to\infty} \|e_{s_n} - 1\|_{L^1(\sigma)} = 0$. Consequently:*

(ii) *A translated sequence $(s_n - m)_{n\in\mathbb{N}}$ is a rigidity sequence for a weak mixing system if and only if there is a continuous probability measure σ on $\mathbb{T}$ such that $\lim_{n\to\infty} \|e_{s_n} - e_m\|_{L^1(\sigma)} = 0$.*

See [**2**, Proposition 2.10] for a proof of Lemma 5.1(i) and further exposition. Part (ii) follows from part (i) and multiplying the integrand in $\|e_{s_n-m} - 1\|_{L^1(\sigma)}$ by e_m.

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5.2. *Kronecker sets.* The unit circle $\{z \in \mathbb{C} : |z| = 1\}$ is denoted by $\mathcal{S}^1$.

A set $K \subseteq \mathbb{T}$ is a *Kronecker set* if, for every continuous function $f : K \to \mathcal{S}^1$ and all $\varepsilon > 0$, there is an exponential function e_n such that $\sup_{x \in K} |f(x) - e_n(x)| < \varepsilon$. The group $\mathbb{T}$ contains a non-empty compact Kronecker set with no isolated points ([**21**, §5]; see also [**16**, **29**]). In other words, $\mathbb{T}$ contains a non-empty perfect Kronecker set. Consequently, there are continuous probability measures supported on Kronecker sets.

PROPOSITION 5.2. *Let σ be a continuous probability measure supported on a Kronecker set $K \subseteq \mathbb{T}$.*

(a) *For all $\varepsilon > 0$, every translate of the set*

$$Q_\varepsilon := \left\{ n : \int |e_n(x) - 1| \, d\sigma(x) < \varepsilon \right\}$$

is a set of recurrence.

(b) *Let $f : K \to \mathcal{S}^1$ be a measurable function and $\varepsilon > 0$. Every translate of the set*

$$Q_{\varepsilon, f} := \left\{ n : \int |e_n(x) - f(x)| \, d\sigma(x) < \varepsilon \right\}$$

is a set of recurrence.

Proposition 5.2 strengthens [**23**, Theorem 2.2], which says† that there is a continuous probability measure σ on $\mathbb{T}$ such that for each $\varepsilon > 0$, $\{n : |\hat{\sigma}(n)| > 1 - \varepsilon\}$ is dense in the Bohr topology of $\mathbb{Z}$. The proposition also improves a special case of the main result of [**30**], which says (in the special case) that if σ is a continuous probability measure supported on a Kronecker set and $U \subseteq \mathbb{C}$ is an open subset of the closed unit disk, then $\{n : \hat{\sigma}(n) \in U\}$ is dense in the Bohr topology. Proposition 8.2 below explains why Proposition 5.2 implies the earlier results. It is currently unknown whether every set dense in the Bohr topology is also a set of recurrence; this is equivalent to Question 1.1. If so, Proposition 5.2 merely recovers [**23**, Theorem 2.2] and the aforementioned special case of [**30**].

Proof of Proposition 5.2. Fix a non-empty perfect Kronecker set $K \subseteq \mathbb{T}$ and a continuous probability measure σ supported on K. We will use the following property of probability measures supported on Kronecker sets: for all Borel measurable functions $f : \mathbb{T} \to \mathcal{S}^1$ and all $\varepsilon > 0$, there is an $n \in \mathbb{Z}$ such that $e_n(x) := \exp(2\pi i n x)$ satisfies $\|f - e_n\|_{L^1(\sigma)} < \varepsilon$.

Our main task is proving part (a). Fixing $\varepsilon > 0$, we aim to prove that every translate of Q_ε is a set of recurrence. We write a translate $Q_\varepsilon + m$ as

$$\begin{aligned} Q_\varepsilon + m &= \left\{ n + m : \int |e_n - 1| \, d\sigma < \varepsilon \right\} \\ &= \left\{ n : \int |e_{n-m} - 1| \, d\sigma < \varepsilon \right\} \\ &= \left\{ n : \int |e_n - e_m| \, d\sigma < \varepsilon \right\}; \end{aligned}$$

† Theorem 2.2 of [**23**] has a slightly weaker statement than that given here, but the proof is easily adapted to prove the stronger statement.

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multiply the integrand $|e_{n-m} - 1|$ by $|e_m|$ in the second line to get the third line. For the remainder of the proof, we fix a set of integers B having $d^*(B) > 0$, we fix an $m \in \mathbb{Z}$, and we aim to prove that $(B - B) \cap (Q_\varepsilon + m) \neq \varnothing$. In other words, we make the following claim.

Claim 1. There are $a, b \in B$ such that $\int |e_{a-b}(x) - e_m(x)|\, d\sigma(x) < \varepsilon$.

Since B is an arbitrary set having $d^*(B) > 0$, Proposition 5.2 follows from Claim 1 (by way of Corollary 3.4).

We define some auxiliary sets. As in §4, Λ_k is the multiplicative group of kth roots of unity. Choose $k \in \mathbb{N}$ large enough that there is a function $g : K \to \Lambda_k$ satisfying

$$\|e_m - g\|_{L^1(\sigma)} < \varepsilon/3. \tag{5.1}$$

We choose $r > N(d^*(B), \varepsilon/6, k)$ in Lemma 4.3. We also choose r large enough that there is a partition $\mathcal{P} = (P_1, \ldots, P_r)$ of K into r sets of equal σ-measure such that there is a $g : K \to \Lambda_k$ satisfying inequality (5.1) which is constant on the elements of $\mathcal{P}$. For the remainder of the proof, we fix such g and $\mathcal{P}$.

Let $H_{r,k}$ be the set of functions $\psi : K \to \Lambda_k$ which are constant on elements of $\mathcal{P}$. Note that $H_{r,k}$ is a group under pointwise multiplication, isomorphic to Λ_k^r; the isomorphism is given by $h \leftrightarrow y \in \Lambda_k^r$ if and only if $h|_{P_i} \equiv y_i$ for $i = 1, \ldots, r$. Furthermore, the $L^1(\sigma)$-ball of radius ε around g in $H_{r,k}$ may be identified with a Hamming ball $U_{r\cdot\varepsilon/2}(x)$ in Λ_k^r, as mentioned in Remark 4.4.

Claim 2. For $z \in \mathbb{Z}$, let $\tilde{B}_z \subseteq H_{r,k}$ be the set of $\psi \in H_{r,k}$ satisfying

$$\text{there exists } n \in B \text{ such that } \|e_{n+z} - \psi\|_{L^1(\sigma)} < \varepsilon/6. \tag{5.2}$$

Then $|\tilde{B}_z| \geq d^*(B)|H_{r,k}|$ for some z.

Proof. For each $\psi \in H_{r,k}$, let $n(\psi) \in \mathbb{Z}$ satisfy $\|e_{n(\psi)} - \psi\|_{L^1(\sigma)} < \varepsilon/3$. Such an $n(\psi)$ exists, since K is a Kronecker set. Furthermore, choose $n(\psi)$ so that $n(\psi) \neq n(\psi')$ if $\psi \neq \psi'$. Let $E_{r,k} = \{n(\psi) : \psi \in H_{r,k}\}$, so that $|E_{r,k}| = |H_{r,k}|$, and $E_{r,k} \subseteq \mathbb{Z}$. Apply Lemma 9.1 to find a translate $B + z$ of B such that $|(B + z) \cap E_{r,k}| \geq d^*(B)|E_{r,k}|$. The definition of $n(\psi)$ implies $|\tilde{B}_z| \geq |(B + z) \cap E_{r,k}|$, so we have the desired conclusion. □

In light of the isomorphism between Λ_k^r and H_k^r, Lemma 4.3 and Remark 4.4 guarantee that whenever $A \subseteq H_k^r$ has $|A| \geq d^*(B)|H_k^r|$, there are $\psi_1, \psi_2 \in A$ such that $\psi_1\overline{\psi_2}$ lies in the $L^1(\sigma)$ ball of radius $\varepsilon/3$ around g. Fixing z as in the conclusion of Claim 2, we may choose $\psi_1, \psi_2 \in \tilde{B}_z$ satisfying simultaneously:

(i) $\|\psi_1\overline{\psi_2} - g\|_{L^1(\sigma)} < \varepsilon/3$; and

(ii) there are $n_j \in B$ and $z \in \mathbb{Z}$ such that $\|e_{n_j+z} - \psi_j\|_{L^1(\sigma)} < \varepsilon/6$, $j = 1, 2$.

Simplifying $e_{n_1+z}\overline{e_{n_2+z}}$ as $e_{n_1}\overline{e_{n_2}} = e_{n_1-n_2}$ and writing $e_{n_1-n_2} - g$ as

$$(e_{n_1+z}\overline{e_{n_2+z}} - e_{n_1+z}\overline{\psi_2}) + (e_{n_1+z}\overline{\psi_2} - \psi_1\overline{\psi_2}) + (\psi_1\overline{\psi_2} - g),$$

we see that conditions (i), (ii), and the triangle inequality imply

$$\|e_{n_1-n_2} - g\|_{L^1(\sigma)} < 2\varepsilon/3. \tag{5.3}$$

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Setting $a = n_1$, $b = n_2$, inequalities (5.1), (5.3), and the triangle inequality imply $\|e_{a-b} - e_m\|_{L^1(\sigma)} < \varepsilon$, thereby establishing Claim 1. This completes the proof of Proposition 5.2, part (a).

Part (b) follows immediately from part (a): since σ is supported on a Kronecker set, there is an $m \in \mathbb{Z}$ such that $\|f - e_m\|_{L^1(\sigma)} < \varepsilon/2$, so $Q_{\varepsilon,f}$ contains $Q' := \{n : \|e_n - e_m\|_{L^1(\sigma)} < \varepsilon/2\}$. Observe that $Q' - m = Q_{\varepsilon/2}$ (as defined in part (a) of the proposition), so every translate of Q' is a set of recurrence. Consequently, every translate of $Q_{\varepsilon,f}$ is a set of recurrence. □

5.3. *Recurrence is witnessed by finite approximations.* We will prove Theorem 2.1 by piecing together finite subsets of the sets $Q_{\varepsilon,f}$ from Proposition 5.2; for this we need Lemma 5.4.

We say that $S \subseteq \mathbb{Z}$ is a *set of δ-recurrence* if for every measure preserving system (X, μ, T) and every $D \subseteq X$ such that $\mu(D) > \delta$, there exists $n \in S$ such that $\mu(D \cap T^n D) > 0$.

LEMMA 5.3. [**11**, Theorem 2.1] *A set $S \subseteq \mathbb{Z}$ is a set of recurrence if and only if for all $\delta > 0$, there is a finite subset S_δ of S such that S_δ is a set of δ-recurrence.*

LEMMA 5.4. *Let $S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots$ be a descending chain of subsets of $\mathbb{Z}$ such that every translate of S_j is a set of recurrence for each j. Then there is a set S such that $S \setminus S_j$ is finite for each j and every translate of S is a set of recurrence.*

Proof. For each $j \in \mathbb{N}$, let $S'_j \subseteq S_j$ be a finite set such that $S'_j + m$ is a set of $1/j$-recurrence for every $|m| \leq j$; such finite sets exist by Lemma 5.3. Then $S = \bigcup_{j\in\mathbb{N}} S'_j$ is the desired set. □

5.4. *Proof of Theorem 2.1.* Let $K \subseteq \mathbb{T}$ be a non-empty perfect Kronecker set, and let $(K_m)_{m\in\mathbb{Z}}$ be a collection of mutually disjoint non-empty perfect subsets of K. Observe that each K_m is a Kronecker set, for every subset of a Kronecker set is itself a Kronecker set. For each m, let σ_m be a continuous probability measure supported on K_m, and define $\sigma := \frac{1}{3}\sum_{m\in\mathbb{Z}} 2^{-|m|}\sigma_m$, so that σ is a continuous probability measure on K. Let $f : \mathbb{T} \to \mathcal{S}^1$ satisfy $f|_{K_m} = e_m$ for each m, and for each $j \in \mathbb{N}$ let

$$S_j = \left\{ n \in \mathbb{Z} : \int |e_n - f| \, d\sigma < \frac{1}{j} \right\}.$$

By Proposition 5.2(b), every translate of S_j is a set of recurrence. By Lemma 5.4, there is a set S such that every translate of S is a set of recurrence, and $S \setminus S_j$ is finite for every n. Enumerating the elements of S as $(s_n)_{n\in\mathbb{N}}$, we have $e_{s_n} \to f$ in $L^1(\sigma)$, and it follows that $e_{s_n} \to f$ in $L^1(\sigma_m)$ for every m. Consequently, $e_{s_n} \to e_m$ in $L^1(\sigma_m)$, by the definition of f, and we apply Lemma 5.1 to see that $S - m$ is a set of rigidity for every m.

6. *Sets of recurrence*

The main result of this section is Lemma 6.2, which we need to prove Corollaries 2.5 and 2.6. Lemma 6.1 provides a useful characterization of sets S such that every translate of S is a set of recurrence.

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The lemmas in this section use the following property of ergodic systems, which is actually equivalent to ergodicity.

(E) If (X, μ, T) is an ergodic system and $C, D \subseteq X$ have positive measure, then there is an $n \in \mathbb{Z}$ such that $\mu(C \cap T^n D) > 0$.

LEMMA 6.1. *For a set of integers S, the following conditions are equivalent.*

(i) *Every translate of S is a set of recurrence.*

(ii) *For every measure preserving system (X, μ, T) and every pair of sets $C, D \subseteq X$: if there is an $n \in \mathbb{Z}$ such that $\mu(C \cap T^n D) > 0$, then there is an $m \in S$ such that $\mu(C \cap T^m D) > 0$.*

(iii) *If (X, μ, T) is an ergodic measure preserving system and $\mu(D) > 0$, then $\mu\big(\bigcup_{n\in S} T^n D\big) = 1$.*

A set $S \subseteq \mathbb{Z}$ satisfying these conditions is sometimes called *measure expanding* [**19**] or *ergodic* [**6**].

Proof. (i) $\Longrightarrow$ (ii). Suppose every translate of S is a set of recurrence, (X, μ, T) is a measure preserving system, $n \in \mathbb{Z}$, and $\mu(C \cap T^n D) > 0$. Since $S - n$ is a set of recurrence, there is an $m \in S$ such that

$$\mu(C \cap T^n D \cap T^{m-n}(C \cap T^n D)) > 0$$

which implies $\mu(C \cap T^m D) > 0$, since $T^{m-n}(C \cap T^n D) = T^{m-n}C \cap T^m D$.

(ii) $\Longrightarrow$ (iii). Let $\mathbf{X} = (X, \mu, T)$ be an ergodic system, and let $S \subseteq \mathbb{Z}$ satisfy (ii). For $D \subseteq X$, write $S \cdot D$ for $\bigcup_{n\in S} T^n D$. Suppose, to get a contradiction, that $\mu(D) > 0$ and $\mu(S \cdot D) < 1$. Let $E = X \setminus (S \cdot D)$. Then $\mu(E) > 0$ and the ergodicity of $\mathbf{X}$ implies that there is an n such that $\mu(E \cap T^n D) > 0$. Condition (ii) implies that there is an $m \in S$ such that $\mu(E \cap T^m D) > 0$, and this is the desired contradiction.

(iii) $\Longrightarrow$ (i). Suppose S satisfies (iii). Let (X, μ, T) be an ergodic system and $D \subseteq X$ have $\mu(D) > 0$. By Corollary 3.4 it suffices to find $m \in S$ such that $\mu(D \cap T^m D) > 0$. Condition (iii) implies $\mu(S \cdot D) = 1$, so $\mu(D \cap (S \cdot D)) = \mu(D)$, and there must be an $m \in S$ such that $\mu(D \cap T^m D) > 0$. □

LEMMA 6.2. *If every translate of S is a set of recurrence, $d^*(A) > 0$, and $n \in \mathbb{Z}$, then $S \cap (n + A - A)$ is infinite.*

This lemma may seem obvious. To see why we expend effort proving it, observe that there are finite sets of recurrence. In fact, a finite set $F \subseteq \mathbb{Z}$ is a set of recurrence if and only if $0 \in F$.

In the proof we need the following property of weak mixing systems: if $\mathbf{X}$ is ergodic and $\mathbf{Y}$ is weak mixing, then the product system $\mathbf{X} \times \mathbf{Y}$ is ergodic.

Proof of Lemma 6.2. It suffices to show that $S \cap (A - A)$ is infinite whenever $d^*(A) > 0$, since the hypothesis on S is translation invariant. Fix such a set A.

By Lemma 3.1, $A - A$ contains a set of the form $R_0(D) := \{n \in \mathbb{Z} : \mu(D \cap T^n D) > 0\}$, where (X, μ, T) is an ergodic measure preserving system and $D \subseteq X$ has $\mu(D) \geq d^*(A)$. So it suffices to show that $S \cap R_0(D)$ is infinite.

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Let $F = \{m_1, \ldots, m_r\} \subseteq \mathbb{Z}$ be a finite set; we will find an $m \in S \setminus F$ such that $\mu(D \cap T^m D) > 0$. Let $n_0 \in \mathbb{Z} \setminus F$. Let $\mathbf{Y} = (Y, \nu, R)$ be a non-trivial weak mixing system, and let $E \subseteq Y$ have positive measure and satisfy $E \cap R^{m-n_0} E = \varnothing$ for all $m \in F$; this is possible because $\mathbf{Y}$ is totally ergodic† and $m - n_0 \neq 0$ for $m \in F$.

Consider the product system $\mathbf{Z} = (X \times Y, \mu \times \nu, T \times R) = \mathbf{X} \times \mathbf{Y}$, and let C_1, $C_2 \subseteq X \times Y$ be

$$C_1 := D \times E, \quad C_2 := D \times R^{-n_0} E.$$

Write η for $\mu \times \nu$ and $\tilde{T}$ for $T \times R$.

Then for all $m \in F$, we have $C_1 \cap \tilde{T}^m C_2 = \varnothing$, since $E \cap R^{m-n_0} E = \varnothing$. Since $\mathbf{Y}$ is weak mixing and $\mathbf{X}$ is ergodic, the system $\mathbf{Z}$ is ergodic. We conclude that there is an n such that $\eta(C_1 \cap \tilde{T}^n C_2) > 0$. Every translate of S is a set of recurrence, so Lemma 6.1 implies $\eta(C_1 \cap \tilde{T}^m C_2) > 0$ for some $m \in S$. Since $C_1 \cap \tilde{T}^m C_2 = \varnothing$ for all $m \in F$, there is an $m \in S \setminus F$ such that $\eta(C_1 \cap \tilde{T}^m C_2) > 0$, and $\mu(D \cap T^m D) > 0$ for such m. Since F was an arbitrary finite set, we have shown that $S \cap R_0(D)$ is infinite. □

7. *Proofs of corollaries*

In this section we prove the corollaries stated in §2. Corollaries 2.3 and 2.5 will be proved with the aid of Lemma 7.1. Our approach to Corollary 2.6 requires some additional machinery in the form of the Jewett–Krieger theorem, which we review after the proofs of Corollaries 2.3 and 2.5.

7.1. *Corollaries 2.3 and 2.5.*

LEMMA 7.1. *If $\mathbf{X} = (X, \mu, T)$ is a non-trivial weak mixing measure preserving system, S is a set of rigidity for $\mathbf{X}$, and $m \in \mathbb{Z} \setminus \{0\}$, then $S + m$ is not a set of strong recurrence for $\mathbf{X}$.*

Proof. Let S be a set of rigidity for $\mathbf{X}$. Let $D \subseteq X$ have $\mu(D) > 0$ and $D \cap T^m D = \varnothing$. Such a D exists since $\mathbf{X}$ is weak mixing, and therefore totally ergodic. Enumerate the elements of S as $\{s_n : n \in \mathbb{N}\}$, so that $\mu(E \triangle T^{s_n} E) \to 0$ as $n \to \infty$ for every measurable set $E \subseteq X$. Then $\mu(T^m D \triangle T^{s_n+m} D) \to 0$, so $\mu(D \cap T^{s_n+m} D) \to \mu(D \cap T^m D) = 0$ as $n \to \infty$. This shows that $S + m$ is not a set of strong recurrence for $\mathbf{X}$. □

Proof of Corollary 2.3. Lemma 7.1 and Theorem 2.1 together imply Corollary 2.3. □

PROPOSITION 7.2. *If $\mathbf{X}$ is a non-trivial rigid-recurrence system, then for every set $D \subseteq X$ such that $D \cap TD = \varnothing$ and every $c > 0$, $R_c(D)$ does not contain a set of the form $n_0 + B - B$, where $d^*(B) > 0$.*

Proof. Let $S \subseteq \mathbb{Z}$ be a set of rigidity for $\mathbf{X}$ such that every translate of S is a set of recurrence. Assume, to get a contradiction, that there is a set $D \subseteq X$ with $\mu(D) > 0$ such that $D \cap TD = \varnothing$, and for some $c > 0$, the set $R_c(D) = \{n : \mu(D \cap T^n D) > c\}$ contains a set of the form $n_0 + B - B$, where $d^*(B) > 0$ and $n_0 \in \mathbb{Z}$.

† Meaning the system (Y, ν, R^n) is ergodic for each $n \neq 0$, a consequence of weak mixing.

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Every translate of S is a set of recurrence, so every translate of $S+1$ is a set of recurrence. Lemma 6.2 implies $(S+1)\cap(n_0+B-B)$ is infinite, and consequently $(S+1)\cap R_c(D)$ is infinite. On the other hand, if we enumerate S as $(s_n)_{n\in\mathbb{N}}$ we have $\lim_{n\to\infty}\mu(D\cap T^{s_n+1}D)=0$ by an argument similar to the proof of Lemma 7.1. The last limit implies the set $(S+1)\cap R_c(D)$ is finite, contradicting the earlier conclusion that it is infinite. □

Proof of Corollary 2.5. Theorem 2.1 implies the existence of non-trivial weak mixing rigid-recurrence systems, so take $\mathbf{X}$ to be any weak mixing rigid-recurrence system and apply Proposition 7.2. □

7.2. *Topological systems and unique ergodicity.* A *topological system* (X, T) is a compact metric space X together with a self homeomorphism $T: X\to X$. We say that (X, T) is *uniquely ergodic* if there is only one T-invariant Borel probability measure on X. See any of [**8**, **9**, **13**, **15**, **28**, **32**] for background on topological systems.

In the proof of Corollary 2.6, we will need minimal uniquely ergodic models of measure preserving systems. Such models are provided by the Jewett–Krieger theorem [**22**, **24**], which says that if (X, μ, T) is an ergodic measure preserving system, then there is a minimal uniquely ergodic topological system $(\tilde{X}, \tilde{T})$ with invariant probability measure $\tilde{\mu}$ such that (X, μ, T) is isomorphic to $(\tilde{X}, \tilde{\mu}, \tilde{T})$. We can take the space $\tilde{X}$ to be totally disconnected.

We now list some well-known properties of uniquely ergodic topological systems. Let (X, T) be a uniquely ergodic topological system with invariant probability measure μ on a totally disconnected space X. Assume μ is continuous, so that $\mu(\{x\})=0$ for all $x\in X$.

(U1) If $K\subseteq X$ is a clopen set, then for every Følner sequence $(\Phi_j)_{j\in\mathbb{N}}$, we have $\lim_{j\to\infty}(1/|\Phi_j|)\sum_{n\in\Phi_j}1_K(T^nx)=\mu(K)$, for every $x\in X$.

(U2) If $K\subseteq X$, $n\in\mathbb{Z}$, and $x\in X$, let $A:=\{m: T^mx\in K\}$. Then $A\cap(A-n)=\{m: T^mx\in K\cap T^{-n}K\}$. Consequently, if K is clopen, then for every Følner sequence $\mathbf{\Phi}$, $d_{\mathbf{\Phi}}(A\cap(A-n))=\mu(K\cap T^{-n}K)$.

(U3) For all $\varepsilon>0$, there is a clopen set $D\subseteq X$ having $\mu(D)>\frac{1}{2}-\varepsilon$ such that $D\cap TD=\varnothing$.

To prove statement (U3), take N to be sufficiently large and construct a Rohlin tower $\{C, TC, \ldots, T^{2N-1}C\}$ for (X, μ, T) where the T^iC are mutually disjoint clopen sets such that $\mu(X\setminus\bigcup_{i=0}^{2N-1}T^iC)>1-\varepsilon$. Let $D=\bigcup_{i=0}^{N-1}T^{2i}C$. Then $\mu(C)\geq(1-\varepsilon)/2N$, $\mu(D)\geq N\cdot\mu(C)\geq N((1-\varepsilon)/2N)=(1-\varepsilon)/2$, and $D\cap TD=\varnothing$. Such a Rohlin tower may be constructed by following the proof of [**9**, Lemma 2.45], starting with a clopen set A in the proof.

We say a set $A\subseteq\mathbb{Z}$ has *uniform density* α if $d_{\mathbf{\Phi}}(A)=\alpha$ for every Følner sequence $\mathbf{\Phi}$; in this case we write $d_U(A)=\alpha$. The following proposition immediately implies Corollary 2.6.

PROPOSITION 7.3. *For all $\varepsilon>0$, there is a set $A\subseteq\mathbb{Z}$ having uniform density equal to or greater than $\frac{1}{2}-\varepsilon$, such that for all $c>0$, the set $P_c(A)$ does not contain a translate of a difference set n_0+B-B, where $d^*(B)>0$.*

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Proof. Let $\mathbf{X} = (X, \mu, T)$ be a weak mixing system as in the conclusion of Corollary 2.5. By the Jewett–Krieger theorem, there is a minimal uniquely ergodic topological dynamical system $\tilde{\mathbf{X}} = (\tilde{X}, \tilde{T})$ with invariant probability measure $\tilde{\mu}$ such that $(\tilde{X}, \tilde{\mu}, \tilde{T})$ is isomorphic to (X, μ, T), and $\tilde{X}$ is totally disconnected.

Fix $\varepsilon > 0$, and apply property (U3) to find a clopen $D \subseteq \tilde{X}$ such that $D \cap \tilde{T} D = \varnothing$ and $\mu(D) > \frac{1}{2} - \varepsilon$. Let $x \in \tilde{X}$ and $A = \{m : \tilde{T}^m x \in D\}$. Then property (U1) implies $d_U(A) = \mu(D) > \frac{1}{2} - \varepsilon$. Property (U2) implies $\mu(D \cap \tilde{T}^{-n} D) = d_U(A \cap (A - n)) = d^*(A \cap (A - n))$ for all $n \in \mathbb{N}$. The conclusion now follows from Corollary 2.5. □

8. *The Bohr topology and recurrence*

In this section we discuss the relations between our results and other work. As in §5, we write $\mathbb{T}$ for $\mathbb{R}/\mathbb{Z}$ with the usual topology.

8.1. *Bohr topology.*

The *Bohr topology* on $\mathbb{Z}$ is the weakest topology on $\mathbb{Z}$ such that every homomorphism $\chi : \mathbb{Z} \to \mathbb{T}$ is continuous. See [**14**, Part II, Ch. 5] for more on the Bohr topology and difference sets.

Remark 8.1. A subbase for the Bohr topology consists of sets of the form $\mathcal{B}(\alpha, U) := \{n \in \mathbb{Z} : n\alpha \in U\}$, where $U \subseteq \mathbb{T}$ is open and $\alpha \in \mathbb{T}$. In other words, the Bohr topology consists of arbitrary unions of finite intersections of sets of the form $\mathcal{B}(\alpha, U)$.

As mentioned in the introduction, a Bohr neighborhood is a set of the form $\{n \in \mathbb{Z} : p(n) > 0\}$, where p is a trigonometric polynomial, and the Bohr topology on $\mathbb{Z}$ is the weakest topology making all trigonometric polynomials continuous. The next proposition relates the Bohr topology and sets of recurrence.

PROPOSITION 8.2.

(a) *If $U \subseteq \mathbb{Z}$ is a Bohr neighborhood of 0, then U contains a difference set $V - V$, where V is a Bohr neighborhood of 0.*

(b) *If $U \subseteq \mathbb{Z}$ is a Bohr neighborhood of 0, then $d^*(U) > 0$.*

(c) *If $U \subseteq \mathbb{Z}$ is a Bohr neighborhood, then U contains a set of the form $n_0 + B - B$, where $d^*(B) > 0$ and $n_0 \in \mathbb{Z}$.*

Consequently, if every translate of S is a set of recurrence, then S is dense in the Bohr topology of $\mathbb{Z}$.

Proof. These are standard facts, so we only outline their proofs.

We begin with (a) and (b). If U is a Bohr neighborhood of 0 then U contains a set of the form

$$U(\alpha_1, \ldots, \alpha_r; \varepsilon) := \{n \in \mathbb{Z} : n\alpha_i \in (-\varepsilon, \varepsilon),\ 1 \le i \le r\},$$

where $\alpha_i \in \mathbb{T}$ and $\varepsilon > 0$. Setting $V = U(\alpha_1, \ldots, \alpha_r; \varepsilon/2)$, we have that V is a Bohr neighborhood of 0 and $V - V \subseteq U$. The compactness of $\mathbb{T}^r$ implies that finitely many translates of $U(\alpha_1, \ldots, \alpha_r; \varepsilon)$ cover $\mathbb{Z}$, so $d^*(U) > 0$.

Part (c) follows from (a) and (b) and the fact that every Bohr neighborhood is a translate of a Bohr neighborhood of 0.

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Corollary 3.4 and part (c) imply that $S \cap U \neq \varnothing$ whenever U is a Bohr neighborhood and every translate of S is a set of recurrence. This proves the final statement of the proposition. □

Whether every set dense in the Bohr topology of $\mathbb{Z}$ is a set of recurrence is unknown. If the answer to Question 1.1 is 'yes', then every set S dense in the Bohr topology of $\mathbb{Z}$ is a set of recurrence, hence every translate of such a set is a set of recurrence.

8.2. *Rigidity and the Bohr topology.* In answering [**2**, Question 3.5], Fayad and Kanigowski prove the following theorem.

THEOREM 8.3. [**10**, Theorem 2] *There is a set of rigidity $S \subseteq \mathbb{Z}$ such that for all irrational α, $\{n\alpha \bmod 1 : n \in S\}$ is dense in $\mathbb{R}/\mathbb{Z}$, and for all rational α, $\{n\alpha \bmod 1 : n \in S\} = \{n\alpha \bmod 1 : n \in \mathbb{Z}\}$.*

In light of Proposition 8.2 and Remark 8.1, Theorem 2.1 implies Theorem 8.3 and the following strengthening.

THEOREM 8.4. *There is a set of rigidity $S \subseteq \mathbb{Z}$ such that S is dense in the Bohr topology of $\mathbb{Z}$.*

8.3. *Equidistribution.* A set $S \subseteq \mathbb{Z}$ is *equidistributed* if there is a sequence of finite subsets $S_j \subseteq S$ such that, for every $\alpha \in (0, 1)$,

$$\lim_{j\to\infty} \frac{1}{|S_j|} \sum_{n\in S_j} \exp(2\pi i n\alpha) = 0. \tag{8.1}$$

It follows easily from the definition that S is equidistributed if and only if every translate of S is equidistributed. Furthermore, standard arguments show that if $(S_j)_{j\in\mathbb{N}}$ is a sequence of finite sets satisfying equation (8.1), then for every measure preserving system (X, μ, T) and every $D \subseteq X$, we have

$$\lim_{j\to\infty} \frac{1}{|S_j|} \sum_{n\in S_j} \mu(D \cap T^n D) \geq \mu(D)^2;$$

see the second assertion of [**7**, Lemma 6.1] for a proof. Consequently, every translate of every equidistributed set is a set of optimal recurrence.

8.4. *A hierarchy of recurrence properties.* Consider the following properties of a set of integers S:

($R_1^\bullet$) S is equidistributed.
($R_2^\bullet$) Every translate of S is a set of optimal recurrence.
($R_3^\bullet$) Every translate of S is a set of strong recurrence.
($R_4^\bullet$) Every translate of S is a set of recurrence.
($R_5^\bullet$) Every translate of S is a set of topological recurrence†.
($R_6^\bullet$) S is dense in the Bohr topology of $\mathbb{Z}$.

† A set $S \subseteq \mathbb{Z}$ is a set of *topological recurrence* if, for every minimal topological system (X, T) and every non-empty open $U \subseteq X$, there is an n such that $U \cap T^{-n}U \neq \varnothing$.

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For $i = 1, \ldots, 5$ we have $(R_i^\bullet) \implies (R_{i+1}^\bullet)$. The implication $(R_1^\bullet) \implies (R_2^\bullet)$ is discussed above, and the implication $(R_4^\bullet) \implies (R_6^\bullet)$ is Proposition 8.2. Katznelson's and Saeki's examples (Theorem 2.2 of [**23**] and the main theorem of [**30**]) show that $(R_6^\bullet)$ does not imply $(R_1^\bullet)$.

Corollary 2.3 shows that $(R_4^\bullet)$ does not imply $(R_3^\bullet)$, and in fact $(R_4^\bullet)$ does not imply that any translate of S is a set of strong recurrence. Corollary 2.3 provides the first explicit proof that $(R_4^\bullet) \not\implies (R_1^\bullet)$, although it is plausible that the constructions of [**12**] and [**25**] produce sets satisfying $(R_4^\bullet)$ and not $(R_3^\bullet)$.

Does $(R_6^\bullet)$ imply $(R_4^\bullet)$? This is Question 1.1. Does $(R_3^\bullet)$ imply $(R_2^\bullet)$? Does $(R_2^\bullet)$ imply $(R_1^\bullet)$? The answers are likely 'no', but we are unaware of a proof.

The relations between the properties $(R_i^\bullet)$ are discussed in greater generality and detail in [**19**], where it is shown that $(R_5^\bullet) \not\implies (R_4^\bullet)$ for the group $\bigoplus_{n=1}^\infty \mathbb{Z}/2\mathbb{Z}$. A similar construction in [**18**] shows that $(R_6^\bullet) \not\implies (R_4^\bullet)$ for the groups $\bigoplus_{n=1}^\infty \mathbb{Z}/p\mathbb{Z}$ where p is prime.

9. *Characterization of upper Banach density*

LEMMA 9.1. *Let $F \subseteq \mathbb{Z}$ be finite and let $A \subseteq \mathbb{Z}$. There is an $n \in \mathbb{Z}$ such that $|(A - n) \cap F| \geq d^*(A)|F|$.*

Proof. We outline the proof in this paragraph and present an explicit proof in the next. First, find a long interval I having $(1/|I|)|I \cap A| \approx d^*(A)$. We may assume that $I = \{1, \ldots, N\}$ for some large N, $F \subseteq I$, and $(\max F)/N$ is small. Consider the intersections $I \cap (A - n) \cap F$. Summing their cardinalities, we get $\sum_{n=1}^N |I \cap (A - n) \cap F| \approx |F||I \cap A|$, as each element of $I \cap A$ is counted approximately $|F|$ times in the sum. So on average, $|I \cap (A - n) \cap F|$ is approximately $(1/N)|F||I \cap A| \approx |F|d^*(A)$, and there must be an n such that $|(A - n) \cap F| \geq d^*(A)$.

Here is an explicit proof. Let $\mathbf{\Phi} = (\Phi_j)_{j \in \mathbb{N}}$ be a Følner sequence such that $d^*(A) = d_{\mathbf{\Phi}}(A)$. Let $\varepsilon, \varepsilon' > 0$. For sufficiently large j, we have

$$\begin{aligned}
\frac{1}{|\Phi_j|} \sum_{n \in \Phi_j} |(A - n) \cap F| &= \frac{1}{|\Phi_j|} \sum_{n' \in F} \sum_{n \in \Phi_j} 1_{A-n}(n') \\
&= \frac{1}{|\Phi_j|} \sum_{n' \in F} \sum_{n \in \Phi_j} 1_A(n + n') \\
&= \sum_{n' \in F} \frac{1}{|\Phi_j|} |A \cap (\Phi_j - n')| \\
&\geq \sum_{n' \in F} \frac{1}{|\Phi_j|} (|A \cap \Phi_j| - \varepsilon) \\
&= \frac{1}{|\Phi_j|} (|A \cap \Phi_j| - \varepsilon)|F| \\
&\geq (d^*(A) - \varepsilon)|F|.
\end{aligned}$$

From the above inequalities, we see that for ε sufficiently small, the average $(1/|\Phi_j|) \sum_{n \in \Phi_j} |(A - n) \cap F|$ is at least $d^*(A)|F| - \varepsilon'$. The pigeonhole principle then

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implies $|(A-n)\cap F|\geq d^*(A)|F|-\varepsilon'$ for some n. Using the integrality of $|(A-n)\cap F|$, we conclude that $|(A-n)\cap F|\geq d^*(A)|F|$ for some n. □

Deriving the following corollary from Lemma 9.1 is an easy exercise.

COROLLARY 9.2. *For $A\subseteq\mathbb{Z}$, $d^*(A)$ is the largest number δ satisfying the following condition: for all finite sets $F\subseteq\mathbb{Z}$, there exists n such that $|(A-n)\cap F|\geq\delta|F|$.*

This characterization of upper Banach density can be taken as a definition, and in fact [**31**] defines a generalization of upper Banach density to countable semigroups in this way.

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