On the operational solution of linear mixed difference differential equations. By JACOB NEUFELD, Moore School of Electrical Engineering, University of Pennsylvania, Philadelphia, Pa., U.S.A. (Communicated by Mr H. JEFFREYS.)

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L. M. Milne-Thomson has recently* described a method of solving linear finite difference equations by using processes somewhat analogous to those employed by Heaviside.

The following note gives a solution of a more general type of equation, i.e. of mixed difference differential equations in terms of the initial conditions. The method employed, although different from that used by Milne-Thomson, resembles closely that used in operational calculus.

Consider a linear mixed difference differential equation of the type

$$\sum_{r=0}^{r=n} \sum_{k=0}^{k=N_r} \frac{d^r}{dx^r} a_{rk} u(x-k) = F(x), \tag{1}$$

where $F(x)$ is a known function of x, a_{rk} $(r = 0, 1, 2, \ldots, n;$ $k = 0, 1, \ldots, N_r)$ are constants, and at $t = 0$ the values

$$u(0) = u_0, \quad u'(0) = u_0', \quad \ldots, \quad u^{(n-1)}(0) = u_0^{(n-1)} \tag{2}$$

are given. $u(x)$ is the unknown function which, in accordance with the Heaviside method, is subject to the following restrictions:

$$u(x) \equiv 0 \quad \text{for} \quad x < 0, \tag{3}$$

$$\lim_{x \to \infty} u(x-k) e^{-px} = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots, N_r \tag{4}$$

and the infinite integral

$$\int_0^\infty u(x) e^{-px} dx = \frac{f(p)}{p} \tag{5}$$

exists.

The relation (5) is usually denoted symbolically by

$$u(x) = f(p)\, 1,$$

where 1 denotes Heaviside's "unit function"†.

* L. M. Milne-Thomson, "On the operational solution of linear finite difference equations", *Proc. Camb. Phil. Soc.* 27 (1931), 26–36.

† See H. Jeffreys, *Operational Methods in Mathematical Physics* (Cambridge Tracts, 1931); B. van der Pol, "A simple proof and an extension of Heaviside's operational calculus for invariable systems", *Phil. Mag.* (7), 7 (1929), 1153–1162.

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Substituting in (5) $x = z - k$, we have

$$\int_k^\infty u(z-k)\, e^{-pz} e^{pk}\, dz = \frac{f(p)}{p}, \tag{6}$$

and, since

$$u(z-k) \equiv 0 \quad \text{for} \quad z < k,$$

we can change the lower limit in the above integral from $z = k$ to $z = 0$. Thus

$$\int_0^\infty u(z-k)\, e^{-pz}\, dz = \frac{f(p)}{p}\, e^{-pk}. \tag{7}$$

Consider now the expression

$$\int_0^\infty \frac{d}{dx}\, u(x-k)\, e^{-px}\, dx. \tag{8}$$

Integrating by parts we can write it in the form

$$\left[u(x-k)\, e^{-px}\right]_0^\infty + p\int_0^\infty u(x-k) e^{-px}\, dx = -u(-k) + f(p) e^{-pk}. \tag{9}$$

In a similar way it can be shown that

$$\int_0^\infty \frac{d^r}{dx^r}\, u(x-k)\, e^{-px}\, dx = -u^{(r-1)}(-k) - pu^{(r-2)}(-k) - \ldots - p^{r-1} u(-k) + p^{r-1} f(p)\, e^{pk}. \tag{10}$$

For $k = 0$ the expression (10) becomes

$$\int_0^\infty \frac{d^r}{dx^r}\, u(x)\, e^{-px}\, dx = -\sum_{\nu=0}^{\nu=r-1} p^\nu u_0^{(r-1-\nu)} + p^{r-1} f(p). \tag{11}$$

Since $u(x)$ is identically zero for negative values of x, the expressions $u^{(r-1)}(-k)$, $u^{(r-2)}(-k)$, ..., $u(-k)$ vanish for

$$k = 1, 2, 3, \ldots, N_r.$$

Consequently

$$\int_0^\infty \frac{d^r}{dx^r}\, u(x-k)\, e^{-px}\, dx = p^{r-1} f(p)\, e^{pk}. \tag{12}$$

Multiplying both sides of the equation (1) by $e^{-pt}\, dt$ and integrating the result from $t = 0$ to $t = \infty$ we obtain

$$\begin{aligned}\int_0^\infty \left[\sum_{r=0}^{r=n} \sum_{k=0}^{k=N} \frac{d^r}{dx^r}\, a_{rk}\, u(x-k)\right] e^{-px}\, dx \\ = \sum_{r=0}^{r=n} \sum_{k=0}^{k=N_r} a_{rk} p^{r-1} f(p)\, e^{pk} - \sum_{r=0}^{r=n} \sum_{\nu=0}^{\nu=r-1} a_{r_0} p^\nu u_0^{(r-1-\nu)} \\ = \int_0^\infty e^{-px} F(x)\, dx. \end{aligned} \tag{13}$$

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Put
$$\int_0^\infty e^{-px} F(x)\, dx = \frac{\phi(p)}{p}; \tag{14}$$

we then have

$$f(p) \sum_{r=0}^{r=n} \sum_{k=0}^{k=N_r} a_{rk} p^{r-1} e^{pk} - \sum_{r=0}^{r=n} \sum_{\nu=0}^{\nu=r-1} a_{r_0} p^{\nu} u_0^{(r-1-\nu)} = \frac{\phi(p)}{p}, \tag{15}$$

from which we get

$$f(p) = \frac{\phi(p) + p \sum_{r=0}^{r=n} \sum_{\nu=0}^{\nu=r-1} a_{r_0} p^{\nu} u_0^{(r-1-\nu)}}{p \sum_{r=0}^{r=n} \sum_{k=0}^{k=N_r} a_{rk} p^{r-1} e^{pk}} \tag{16}$$

This value is substituted in the integral equation (5). The solution of this equation is known to be

$$u(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(p)}{p} e^{pt} dp.$$

This expression can be computed in various cases by means of the expansion theorem or by development in convergent or asymptotic series.

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