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Coherence for monoidal endofunctors[†]

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The goal of this paper is to prove coherence results with respect to relational graphs for monoidal endofunctors, that is, endofunctors of a monoidal category that preserve the monoidal structure up to a natural transformation that need not be an isomorphism. These results are proved first in the absence of symmetry in the monoidal structure, and then with this symmetry. In the later parts of the paper, the coherence results are extended to monoidal endofunctors in monoidal categories that have diagonal or codiagonal natural transformations, or where the monoidal structure is given by finite products or coproducts. Monoidal endofunctors are interesting because they stand behind monoidal monads and comonads, for which coherence will be proved in a sequel to this paper.

1. Introduction

A monoidal functor is a functor between monoidal categories that preserves the monoidal structure up to a natural transformation that need not be an isomorphism (see Section 3 below; this notion originated in Eilenberg and Kelly (1966, Section II.1)). Coherence results for monoidal functors were obtained long ago in Epstein (1966) and Lewis (1972). Epstein (1966) presented a result for such functors between symmetric monoidal categories in the absence of unit objects, while Lewis (1972) allowed unit objects, and also considered non-symmetric monoidal categories.

To get coherence with the unit objects, Lewis (1972) implicitly introduces graphs that connect occurrences of the generating functor (see the beginning of the next section). The standard graphs, which go back to Kelly and Mac Lane (1971), as well as earlier work of Mac Lane and Kelly, connect occurrences of generating objects.

Our goal in this paper is first to extend these old coherence results to the situation where instead of a monoidal functor between two categories, we have an endofunctor of a single monoidal category. This involves matters that go beyond Lewis (1972), where application of functors cannot be iterated. Monoidal endofunctors are interesting because they lie behind *monoidal* monads and comonads, and the present paper lays the foundations for a study of coherence in these monads and comonads.

Monoidal monads were first introduced in Kock (1970; 1972); more recent papers on monoidal comonads are Moerdijk (2002), Bruguières and Virelizier (2007) and Pastro and Street (2009). We will prove coherence results for monoidal monads and comonads in a sequel to this paper (Došen and Petrić 2010). In both the present paper and the sequel, we

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understand coherence with respect to graphs like those of Lewis (1972). Coherence states that there is a faithful functor from a freely generated categorial structure, for which we prove coherence, into the category whose arrows are such graphs. We obtain thereby a characterisation of the freely generated categorial structure in terms of graphs. Such coherence results give very useful procedures for deciding whether a diagram of canonical arrows commutes. (A general treatment of coherence in this spirit may be found in Došen and Petrić (2004).)

Moggi (1991) used a notion of monad inspired by Kock (1970; 1972) and based on the notions of left and right monoidal endofunctors, for which we are also going to prove coherence. We prove our results first in the absence of symmetry, and then with symmetry. In the later part of the paper we extend our coherence results to monoidal endofunctors in monoidal categories that have diagonal or codiagonal natural transformations (we call these monoidal categories *relevant* categories), or where the monoidal structure is given by finite products or coproducts.

Some of our coherence results may be understood as basic coherence results for equations between deductions in modal logic. In this paper we find systems that may be understood as fragments of K with the necessity operator primitive; in the sequel, which considers comonads, we will find fragments of $S4$ with the necessity operator primitive.

2. Endofunctors in monoidal categories

In this section we deal with coherence for monoidal categories with endofunctors that are not necessarily monoidal. This is a basic auxiliary result, which we will need later.

A monoidal category is, as usual, a category with a biendofunctor $\otimes$, a special unit object I and the natural isomorphisms whose components are the arrows

$$a_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$$
$$l_A : I \otimes A \to A$$
$$r_A : A \otimes I \to A,$$

which satisfy Mac Lane's coherence equations (see Mac Lane (1998, Section VII.1); our notation follows Eilenberg and Kelly (1966, Section II.1).

Let $\mathscr{E}$ be the free monoidal category with a family of endofunctors; both here, and later, we use freedom to mean free generation by two arbitrary sets, one of which is considered to be the set of *generating objects*, and the other as the set of *generating functors* $\{E^i \mid i \in \mathscr{I}\}$. Collectively, we call the generating objects and the generating functors *generators*. If $\mathscr{I}$ is empty, then $\mathscr{E}$ is just the free monoidal category generated by a set of generating objects.

The category $\mathscr{E}$ is made of syntactical material. Its objects are *propositional formulae* built using the binary connective $\otimes$, the unary connectives E^i and the nullary connective I out of the generating objects, which we take to be the propositional letters $p, q, r, \ldots$ An object of $\mathscr{E}$ is *atomic* when it is a generating object or is of the form E^iA. An object of $\mathscr{E}$ is *diversified on generating objects* when every generating object occurs in it at most once. We define diversification on generating functors analogously, and say that an object is

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diversified when it is diversified both on generating objects and generating functors. For E^iA a subformula of an object B of $\mathscr{E}$, the *scope in* B of the outermost occurrence of E^i in E^iA is the set of all the generators in A.

The arrows of $\mathscr{E}$ are equivalence classes of arrow terms made out of the primitive arrow terms $\mathbf{1}_A$, $a_{A,B,C}$, l_A, r_A, with the operations $\circ$, $\otimes$ and E^i so that the equations assumed for defining a monoidal category together with the functorial equations for E^i, for each $i \in \mathscr{I}$, are satisfied (*cf.* Došen and Petrić (2004, Chapter 2)). We will omit the superscripts i of E^i except when they are essential (we will do the same later with ψ, ψ_0, ψ^L and ψ^R). The existence of free structures like $\mathscr{E}$ is guaranteed by the purely equational definition of these structures.

Every arrow term f of $\mathscr{E}$ is equal to an arrow term $f_n \circ \cdots \circ f_1$, which is said to be *developed*, that:

— if $n = 0$, is $\mathbf{1}_A$;
— if $n \geqslant 1$, is such that for every $j \in \{1, \ldots, n\}$, there is in f_j exactly one occurrence of a, l or r and no occurrence of $\circ$; such an f_i is called a *factor*.

The subterm $a_{A,B,C}$, or l_A, or r_A, of a factor is its *head* (Došen and Petrić 2004, Section 2.7). A factor with head $a_{A,B,C}$ is an *a-factor*, and analogously in other cases. We are going to prove the following theorem.

Theorem 1 ($\mathscr{E}$-Coherence). The category $\mathscr{E}$ is a preorder.

Proof. Suppose we have two arrow terms $f, g : A \to B$ of $\mathscr{E}$. To show that $f = g$, we proceed by induction on the number n of occurrences of E in A, which is equal to this number in B. In the basis, when $n = 0$, we have Mac Lane's coherence result for monoidal categories (see Mac Lane (1998, Section VII.2) or Došen and Petrić (2004, Chapter 4)).

When $n > 0$, we take a single arbitrary occurrence of E^i in A, and replace it by E^j such that j is not an index of any E in A (if all the generating functors occur in A, we can enlarge the set of generating functors with a new functor E^j, which just acts as a placeholder). We make this replacement at the appropriate place in B and in the arrow terms f and g to obtain the arrow terms $f', g' : A' \to B'$ of $\mathscr{E}$. By naturality and the functorial equations, f' is equal to a developed arrow term $f_2 \circ f_1$ such that no head of a factor of $f_1 : A' \to C$ is in the scope of E^j and all the heads of the factors of $f_2 : C \to B'$ are in the scope of E^j. The object C is completely determined by A' and B'. Analogously, we have $g' = g_2 \circ g_1$, with $g_1 : A' \to C$ and $g_2 : C \to B'$. Since E^jD in f_1 and g_1 amounts to a generating object (it is a parameter), and since in f_2 and g_2 only what is within the scope of E^j counts, by the induction hypothesis, we have $f_1 = g_1$ and $f_2 = g_2$, so $f' = g'$, from which $f = g$ follows by substitution. □

3. Monoidal and locally monoidal endofunctors

An endofunctor may preserve the monoidal structure of a category up to a natural transformation either globally or locally. We have *global* preservation when our endofunctor E is a *monoidal functor* in the sense of Eilenberg and Kelly (1966, Section II.1) – see also Mac Lane (1998, Section XI.2). This means that we have a natural transformation

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in our monoidal category whose components are the arrows

$$\psi_{A,B} : EA \otimes EB \to E(A \otimes B),$$

and we also have the arrow $\psi_0 : I \to EI$; the monoidal structure is preserved up to ψ and ψ_0, which means that the following equations hold:

$$Ea_{A,B,C} \circ \psi_{A\otimes B,C} \circ (\psi_{A,B} \otimes \mathbf{1}_{EC}) = \psi_{A,B\otimes C} \circ (\mathbf{1}_{EA} \otimes \psi_{B,C}) \circ a_{EA,EB,EC} \qquad (\psi a)$$

$$El_A \circ \psi_{I,A} \circ (\psi_0 \otimes \mathbf{1}_{EA}) = l_{EA} \qquad (\psi l)$$

$$Er_A \circ \psi_{A,I} \circ (\mathbf{1}_{EA} \otimes \psi_0) = r_{EA}. \qquad (\psi r)$$

The global character of the preservation is manifested in (ψa) by E from the left-hand side falling on every index of a on the right-hand side. (The notation using ψ is taken from Kock (1970; 1972).)

We have *local* preservation using the following three endofunctor notions suggested by Kock (1970; 1972). Monoidal functors need not be endofunctors, but the notions we are going to consider now are tied to endofunctors only.

We say that an endofunctor E of a monoidal category is *left monoidal* when we have a natural transformation whose components are the arrows

$$\psi^L_{A,B} : EA \otimes B \to E(A \otimes B),$$

and the monoidal structure is preserved up to ψ^L, which means that the following equations hold:

$$Ea_{A,B,C} \circ \psi^L_{A\otimes B,C} \circ (\psi^L_{A,B} \otimes \mathbf{1}_C) = \psi^L_{A,B\otimes C} \circ a_{EA,B,C} \qquad (\psi^L a)$$

$$Er_A \circ \psi^L_{A,I} = r_{EA}. \qquad (\psi^L r)$$

The local character of the preservation is manifested in $(\psi^L a)$ by E from the left-hand side falling on a single index of a on the right-hand side.

We say, analogously, that E is *right monoidal* when we have a natural transformation whose components are the arrows

$$\psi^R_{A,B} : A \otimes EB \to E(A \otimes B),$$

and the monoidal structure is preserved up to ψ^R, which means that the following equations hold:

$$Ea_{A,B,C} \circ \psi^R_{A\otimes B,C} = \psi^R_{A,B\otimes C} \circ (\mathbf{1}_A \otimes \psi^R_{B,C}) \circ a_{A,B,EC} \qquad (\psi^R a)$$

$$El_A \circ \psi^R_{I,A} = l_{EA}. \qquad (\psi^R l)$$

We say that E is *locally* monoidal when it is both left and right monoidal, and, moreover, we have the equation

$$Ea_{A,B,C} \circ \psi^L_{A\otimes B,C} \circ (\psi^R_{A,B} \otimes \mathbf{1}_C) = \psi^R_{A,B\otimes C} \circ (\mathbf{1}_A \otimes \psi^L_{B,C}) \circ a_{A,EB,C}. \qquad (\psi^L \psi^R)$$

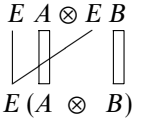
We define, in the same way as we defined the category $\mathscr{E}$ in the preceding section, the free monoidal categories with a family of monoidal endofunctors, a family of left

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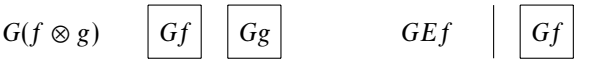
monoidal endofunctors, a family of right monoidal endofunctors or a family of locally monoidal endofunctors, which we call $\mathscr{M}$, $\mathscr{L}^L$, $\mathscr{L}^R$ and $\mathscr{L}$, respectively. All these categories have the same propositional formulae as objects (provided the sets of generators are the same). Then it is easy to see that the category $\mathscr{L}^R$ is an isomorphic mirror image of $\mathscr{L}^L$. For the categories $\mathscr{M}$, $\mathscr{L}^L$, $\mathscr{L}^R$ and $\mathscr{L}$, we define the notions of developed arrow term, factor and head of a factor analogously to the way we did for $\mathscr{E}$ in the preceding section.

We define a functor G from $\mathscr{M}$ to the category *Fun* of functions between finite ordinals by stipulating that GA, for A a propositional formula, is the number of occurrences of E in A (that is, the number of all E^i's in A, for every i), while Gf for an arrow term f of $\mathscr{M}$ is defined inductively on the complexity of f. We have that $G\mathbf{1}_A$, $Ga_{A,B,C}$, Gl_A and Gr_A are identity functions, while for the remaining primitive arrow terms, we have the clause corresponding to the picture

$G\psi_{A,B}$

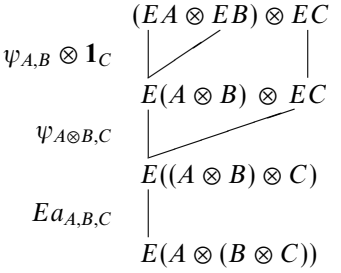


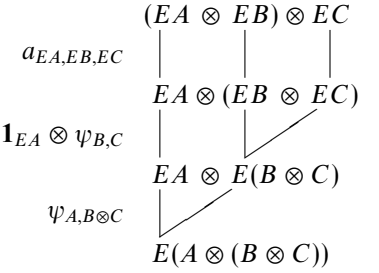
and $G\psi_0$ is the empty function from Ø, which is GI, to $\{\emptyset\}$, which is GEI (in our picture we obtain a crossing if there is an E in A, which may or may not be different from the E in the picture). We also have clauses corresponding to the pictures



and $G(g \circ f)$ is the composition of the functions Gf and Gg. It is easy to verify by induction on the length of derivation that G so defined on the arrow terms of $\mathscr{M}$ induces a functor from $\mathscr{M}$ to *Fun*.

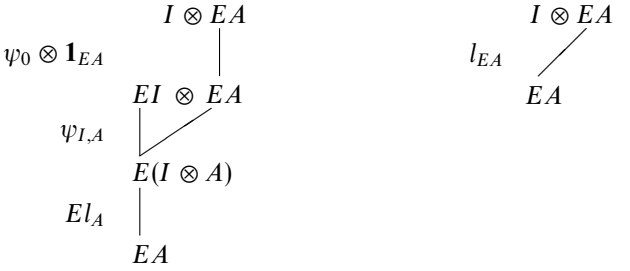
Intuitively, $Gf : GA \to GB$ shows from which occurrences of E in A the occurrences of E in B originate. We call Gf a *graph*. For example, for the two sides of the equation (ψa) we have the pictures





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and for the two sides of (ψl) we have the pictures



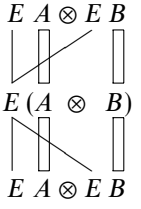
The functors from $\mathscr{L}^L$, $\mathscr{L}^R$ and $\mathscr{L}$ to *Fun* analogous to G, which we shall call G, are defined in the same way as G apart from the clauses corresponding to the pictures



(this means that $G\psi^L_{A,B}$ is an identity arrow). The target category of G for $\mathscr{L}^R$ and $\mathscr{L}$ is the subcategory of *Fun* of bijections between finite ordinals, and for $\mathscr{L}^L$ this is the discrete subcategory of *Fun*, with all arrows just identity arrows.

If $\mathscr{K}$ is a category like $\mathscr{M}$, $\mathscr{L}^L$, $\mathscr{L}^R$ or $\mathscr{L}$, then we use the name $\mathscr{K}$*-Coherence* for the proposition that G from $\mathscr{K}$ to *Fun*, or a category like *Fun*, is a faithful functor. Since the image of $\mathscr{L}^L$ under G is a discrete category, $\mathscr{L}^L$-Coherence amounts to the proposition that $\mathscr{L}^L$ is a preorder, and since $\mathscr{L}^L$ and $\mathscr{L}^R$ are isomorphic, $\mathscr{L}^R$-Coherence also amounts to the proposition that $\mathscr{L}^R$ is a preorder. (Our notion of $\mathscr{K}$-coherence is a standard notion of coherence, which follows from Mac Lane's coherence results for monoidal and symmetric monoidal categories; see Mac Lane (1998) and Došen and Petrić (2004), and references therein.)

Note that with this understanding of coherence, we cannot expect $\psi_{A,B}$ and ψ_0 to be isomorphisms. With the natural G image (now not in *Fun*) of the inverses of $\psi_{A,B}$ and ψ_0, we have



EI

I

EI

neither of which corresponds to an identity arrow. The reasons for this failure of isomorphism are similar to the reasons for the failure of the isomorphism of distribution investigated in Došen and Petrić (2004).

Instead of formulating our coherence results in terms of G and graphs, we could have produced formulations based on diversified objects (see the preceding section). For example, $\mathscr{M}$-Coherence, which we are going to prove in the next section, is equivalent to

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the proposition that for all arrow terms $f, g : A \to B$ of $\mathscr{M}$ with B diversified, we have $f = g$ in $\mathscr{M}$.

4. $\mathscr{M}$-Coherence

The category $\mathscr{M}$ is equivalent to its strictification $\mathscr{M}^{str}$, where

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \qquad a_{A,B,C} = \mathbf{1}_{A \otimes B \otimes C}$$

$$I \otimes A = A = A \otimes I \qquad l_A = \mathbf{1}_A = r_A.$$

The preordered groupoid subcategory of $\mathscr{M}$ over which we make the strictification is the category $\mathscr{E}$ of Section 2 (see Došen and Petrić (2004, Section 3.2), which, together with Došen and Petrić (2004, Section 3.1), provides a general treatment of strictification along with references to earlier approaches).

Let H be the functor from $\mathscr{M}^{str}$ to $\mathscr{M}$, and H' be the functor in the opposite direction, by which $\mathscr{M}^{str}$ and $\mathscr{M}$ are equivalent categories. We will show that the composite functor GH from $\mathscr{M}^{str}$ to *Fun* is faithful. This implies, as follows, that G from $\mathscr{M}$ to *Fun* is faithful, that is, we have $\mathscr{M}$-Coherence. Suppose $Gf = Gg$. Then, since $GHH'h = Gh$ for every arrow h of $\mathscr{M}$, we get $GHH'f = GHH'g$, and, by the faithfulness of GH, we have $H'f = H'g$, from which we get $HH'f = HH'g$, so $f = g$ in $\mathscr{M}$.

Proposition 1. The functor GH from $\mathscr{M}^{str}$ to *Fun* is faithful.

Proof. Every arrow term f of $\mathscr{M}^{str}$ is equal to a developed arrow term $f_n \circ \cdots \circ f_1$ such that each f_j is either a ψ-factor or a ψ_0-factor. By applying naturality and the functorial equations, and the equations

$$\psi_{I,A} \circ (\psi_0 \otimes \mathbf{1}_{EA}) = \mathbf{1}_{EA} = \psi_{A,I} \circ (\mathbf{1}_{EA} \otimes \psi_0),$$

which are (ψl) and (ψr) strictified, we obtain from a developed arrow term an arrow term equal to it of the form $h \circ g_m \circ \cdots \circ g_1$, where in h we have no occurrence of ψ, and $\circ$ may occur only in subterms of h of the form $E \cdots E\psi_0 \circ \cdots \circ \psi_0 : I \to E \cdots EI$; for $g_m \circ \cdots \circ g_1$ we assume that it is developed without ψ_0-factors.

If $m \geqslant 1$, each g_j is a ψ-factor, and we assign to g_j a finite ordinal $\tau(g_j)$ obtained by applying the function $GH(g_m \circ \cdots \circ g_{j+1})$ to the number $\kappa(g_j)$ of occurrences of E in g_j to the left of ψ. Intuitively, this is the place where the contracted E of g_j will end up in the codomain of g_m. For example, with $g_2 \circ g_1$ being

$$E^1\psi^2_{p,E^1(p \otimes q)} \circ E^1(\mathbf{1}_{E^2p} \otimes E^2\psi^1_{p,q}),$$

we have $\kappa(g_1) = 3$ and $\tau(g_1) = 2$, which is clear from the following picture:

$$\begin{array}{ll} & E^1(E^2p \otimes E^2(E^1p \otimes E^1q)) \\ GHg_1 & \;\;|\;\;\;|\;\;\;\;\;|\;\;\diagdown \\ & E^1(E^2p \otimes E^2\;E^1(p \otimes q)) \\ GHg_2 & \;\;|\;\;\diagdown\;\;\;\diagdown \\ & \underset{0}{E^1}\;\underset{1}{E^2}(p \otimes \underset{2}{E^1}(p \otimes q)) \end{array}$$

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It is not difficult to see that for the ψ-factors g_i and g_{i+1} such that $\tau(g_{i+1}) = k < l = \tau(g_i)$ we have, by naturality and the functorial equations, that $g_{i+1} \circ g_i = g'_{i+1} \circ g'_i$ for some ψ-factors g'_i and g'_{i+1} such that $\tau(g'_i) = k$ and $\tau(g'_{i+1}) = l$. So $g_m \circ \cdots \circ g_1$ is equal to $g'_m \circ \cdots \circ g'_1$ such that $\tau(g'_{i+1}) \geqslant \tau(g'_i)$. If $\tau(g'_{i+1}) = \tau(g'_i)$, they can be permuted by the equation (ψa) strictified. (Note that with five applications of that equation we may also permute the two rightmost factors of $\psi_{p\otimes q,r\otimes s} \circ (\mathbf{1}_{E(p\otimes q)} \otimes \psi_{r,s}) \circ (\psi_{p,q} \otimes \mathbf{1}_{Er\otimes Es})$.) We take $h \circ g'_m \circ \cdots \circ g'_1$ to be a normal form of f. As an arrow term, this normal form is not unique, because we may have differences based on the last-mentioned permutations or on equations like $E\mathbf{1}_A = \mathbf{1}_{EA}$.

We may show, however, that if $GHf = GHf'$ and f and f' are in normal form, then f and f' differ from each other only in the way described in the preceding sentence. In f, let a *block* $\vec{f}_i$ be a composition of ψ-factors $f_{i_k} \circ \cdots \circ f_{i_1}$ such that $\tau(f_{i_k}) = \cdots = \tau(f_{i_1}) = l$. We stipulate then that $\tau(\vec{f}_i) = l$. Let f be $h \circ \vec{f}_n \circ \cdots \circ \vec{f}_1$ such that for $i, j \in \{1, \ldots, n\}$, if $i < j$, then $\tau(\vec{f}_i) < \tau(\vec{f}_j)$. The arrangement of these blocks is *strictly increasing*. Analogously, let f' be $h' \circ \vec{f}'_n \circ \cdots \circ \vec{f}'_1$. From $GHf = GHf'$, we can conclude first that $n = n'$, and then proceed by induction on n. If $n = 0$, we conclude easily that $h = h'$. If $n > 0$, we conclude that $\tau(\vec{f}_1) = \tau(\vec{f}'_1)$, that $GH\vec{f}_1 = GH\vec{f}'_1$ and that $GH(h \circ \vec{f}_n \circ \cdots \circ \vec{f}_2) = GH(h' \circ \vec{f}'_n \circ \cdots \circ \vec{f}'_2)$. From this we conclude that $\vec{f}_1 = \vec{f}'_1$, and, by the induction hypothesis, $h \circ \vec{f}_n \circ \cdots \circ \vec{f}_2 = h' \circ \vec{f}'_n \circ \cdots \circ \vec{f}'_2$. So $f = g$ in $\mathscr{M}^{str}$. □

Hence we have $\mathscr{M}$-Coherence.

Note that $\mathscr{M}$ is not a preorder. The two arrows

$$(\psi_0 \otimes \mathbf{1}_{EI}) \circ l^{-1}_{EI}, (\mathbf{1}_{EI} \otimes \psi_0) \circ r^{-1}_{EI} : EI \to EI \otimes EI$$

have different G images, and are different in $\mathscr{M}$; this counterexample to it being a preorder is from Lewis (1972, Section 0). Another counterexample is given by the two arrows

$$(\mathbf{1}_{EA} \otimes (El_B \circ \psi_{I,B})) \circ a_{EA,EI,EB}, (Er_A \circ \psi_{A,I}) \otimes \mathbf{1}_{EB} : (EA \otimes EI) \otimes EB \to EA \otimes EB.$$

This counterexample shows that graphs are essential for coherence even in the absence of ψ_0.

However, let $\mathscr{M}^-$ be the category defined like $\mathscr{M}$ except that we exclude I and everything that involves it – namely, l, r and ψ_0. The category $\mathscr{M}^-$ is a preorder, and graphs are irrelevant for its coherence. When we try to determine whether there is an arrow of $\mathscr{M}^-$ of a given type (that is, with a given source and target), we find that if there is such an arrow, it must be unique.

This will become clear with the following example. Suppose we want to determine whether there is an arrow

$$f : E(Ep \otimes E(Eq \otimes Ep)) \to EE(p \otimes E(q \otimes p)).$$

We first diversify the propositional letters and the occurrences of E in the target, and the question is then whether we have an arrow

$$f' : E(Ep \otimes E(Eq \otimes Er)) \to E^1E^2(p \otimes E^3(q \otimes r))$$

for some superscripts assigned to the occurrences of E in the source. The leftmost E in the source must be E^1. Since this E has $\{p, q, r\}$ in its scope as E^1 in the target, we are

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done with E^1. The leftmost of the remaining Es in the source must be E^2. Since this E has only $\{p\}$ in its scope, while E^2 in the target has $\{p,q,r\}$, we take as E^2 the leftmost of the remaining Es in the source in whose scope we find $\{q,r\}$. By iterating this procedure, we find the arrow

$$E^1E^2(\mathbf{1}_p \otimes \psi^3_{a,r}) \circ E^1\psi^2_{p,E^3q\otimes E^3r} : E^1(E^2p \otimes E^2(E^3q \otimes E^3r)) \to E^1E^2(p \otimes E^3(q \otimes r)).$$

5. $\mathscr{L}^L$, $\mathscr{L}^R$ and $\mathscr{L}$-Coherence

To prove $\mathscr{L}^L$-Coherence, which, as we said towards the end of Section 3, amounts to $\mathscr{L}^L$ being a preorder, we proceed as for $\mathscr{M}$-Coherence. We introduce the strictification of $\mathscr{L}^L$ and in the proof of the faithfulness of GH we have a normal form that is a simplified version of the normal form of the preceding proof for $\mathscr{M}^{str}$. The h part of the normal form with ψ_0-factors does not exist, and instead of the ψ-factors part, we have a ψ^L-factors part. The blocks are now of length 1, that is, single ψ^L-factors, according to the equation $(\psi^L a)$ strictified, and ψ^L-factors with $\psi^L_{A,I}$ are identity arrows by the equation $(\psi^L r)$ strictified. We may prove $\mathscr{L}^R$-Coherence either directly in the same manner, or just appeal to the isomorphism of $\mathscr{L}^L$ and $\mathscr{L}^R$.

To prove $\mathscr{L}$-Coherence, we proceed as before. Again, the h part of the normal form does not exist, and we have ψ^L-factors and ψ^R-factors. A block is a single ψ^L-factor, a single ψ^R-factor or a pair of factors made of one ψ^L-factor and one ψ^R-factor, which may be permuted according to the equation $(\psi^L\psi^R a)$ strictified.

6. Coherence with linear endofunctors

A symmetric monoidal category is, as usual, a monoidal category with the natural isomorphism whose components are the arrows

$$c_{A,B} : A \otimes B \to B \otimes A,$$

which satisfy Mac Lane's coherence conditions (see Mac Lane (1998, Section XI.1)).

A *linear* endofunctor in a symmetric monoidal category is a monoidal endofunctor E that preserves c globally; that is, we have the equation

$$Ec_{A,B} \circ \psi_{A,B} = \psi_{B,A} \circ c_{EA,EB}. \qquad (\psi c)$$

(We use the word *linear* rather than *symmetric monoidal* for brevity, and because of the connection with the structural fragment of *linear* logic, whose name comes from *linear* algebra.)

A *locally linear* endofunctor in a symmetric monoidal category may be defined as a locally monoidal endofunctor E that satisfies

$$Ec_{A,B} \circ \psi^L_{A,B} = \psi^R_{B,A} \circ c_{EA,B}. \qquad (\psi^L\psi^R c)$$

An alternative, simpler, definition is that it is either a left monoidal or a right monoidal endofunctor in a symmetric monoidal category. If it is left monoidal, then from $(\psi^L\psi^R c)$

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we obtain the definition of ψ^R in terms of ψ^L and c, and we derive $(\psi^R a)$, $(\psi^R l)$ and $(\psi^L \psi^R a)$.

Let $\mathscr{M}_c$ and $\mathscr{L}_c$ be the free symmetric monoidal categories with a family of linear or locally linear endofunctors, respectively; these categories are defined analogously to $\mathscr{M}$ and $\mathscr{L}$. We define the functors G from $\mathscr{M}_c$ and $\mathscr{L}_c$ to *Fun* by stipulating first that GA is the number of occurrences of generators in A. Earlier we took GA to be the number of occurrences of generating functors in A, though we could just as well have counted occurrences of generating objects, but it would have been superfluous. The remaining definitions of the new functors G are analogous to the definitions of G from $\mathscr{M}$ and $\mathscr{L}$ to *Fun*, except that we add the clause corresponding to the picture

$$Gc_{A,B} \qquad \begin{matrix} A \otimes B \\ \times \\ B \otimes A \end{matrix}$$

The category $\mathscr{M}_c$ is equivalent to its strictification $\mathscr{M}_c^{str}$, as $\mathscr{M}$ is equivalent to $\mathscr{M}^{str}$ (see Section 4), and, as before, we prove the following proposition, which entails $\mathscr{M}_c$-Coherence.

Proposition 2. The functor GH from $\mathscr{M}_c^{str}$ to *Fun* is faithful.

Proof. We will use the following abbreviation in $\mathscr{M}_c^{str}$:

$$\Psi_{A_1,A_2;B} =_{df} (\psi_{A_1,A_2} \otimes \mathbf{1}_B) \circ (\mathbf{1}_{EA_1} \otimes c_{B,EA_2}) : EA_1 \otimes B \otimes EA_2 \to E(A_1 \otimes A_2) \otimes B.$$

We obtain a Ψ-*developed* arrow term from a developed arrow term of $\mathscr{M}_c^{str}$ by replacing the head ψ_{A_1,A_2} of every ψ-factor by $\Psi_{A_1,A_2;I}$; this replacement is justified by the equation $\psi_{A_1,A_2} = \Psi_{A_1,A_2;I}$ of $\mathscr{M}_c^{str}$. Every Ψ-developed arrow term is equal in $\mathscr{M}_c^{str}$ to an arrow term of the form $h \circ g_m \circ \cdots \circ g_1$, where in h we have no occurrences Ψ and c, while occurrences of $\circ$ are restricted as in the proof of Proposition 1; for $g_m \circ \cdots \circ g_1$, we suppose that it is Ψ-developed without ψ_0-factors; that is, it has only Ψ-factors and c-factors.

If $m \geqslant 1$, and g_i is a Ψ-factor, we assign to g_i a finite ordinal $\tau(g_i)$ exactly as we did in the proof of Proposition 1. We then proceed along the same lines as in that proof to obtain a normal form. When $\tau(g_{i+1}) < \tau(g_i)$, we proceed exactly as before. The new cases we have to consider are as follows.

Suppose we have the Ψ-factors g_i and g_{i+1} such that $\tau(g_{i+1}) = \tau(g_i)$. Then we may have the opportunity to apply the following equations of $\mathscr{M}_c^{str}$ from left to right:

$$\begin{aligned} &(\Psi_{A_1 \otimes A_3, A_2; B_1} \otimes \mathbf{1}_{B_2}) \circ \Psi_{A_1,A_3;B_1 \otimes EA_2 \otimes B_2} \\ &\quad = (E(\mathbf{1}_{A_1} \otimes c_{A_2,A_3}) \otimes \mathbf{1}_{B_1 \otimes B_2}) \circ \Psi_{A_1 \otimes A_2, A_3; B_1 \otimes B_2} \circ (\Psi_{A_1,A_2;B_1} \otimes \mathbf{1}_{B_2 \otimes EA_3}) \end{aligned} \tag{$\Psi\Psi 1$}$$

$$\begin{aligned} &(\Psi_{A_1, A_2 \otimes A_3; B_1} \otimes \mathbf{1}_{B_2}) \circ (\mathbf{1}_{EA_1 \otimes B_1} \otimes \Psi_{A_2,A_3;B_2}) \\ &\quad = \Psi_{A_1 \otimes A_2, A_3; B_1 \otimes B_2} \circ (\Psi_{A_1,A_2;B_1} \otimes \mathbf{1}_{B_2 \otimes EA_3}). \end{aligned} \tag{$\Psi\Psi 2$}$$

We say a c-factor is *atomised* when in its head $c_{A,B}$ the objects A and B are atomic (see Section 2). By the strictified version of Mac Lane's hexagonal coherence condition for symmetric monoidal categories (Mac Lane 1998, Section XI.1), and since $c_{A,I} = l_A^{-1} \circ r_A = \mathbf{1}_A$, we may assume that all our c-factors are atomised. Suppose we have an atomic c-factor

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g_i and a Ψ-factor g_{i+1}. Then we may have the opportunity to apply either the naturality and the functorial equations, or the equation (ψc), or the equations

$$\Psi_{A_1,A_2;B_1\otimes B_2}\circ(c_{B_1,EA_1}\otimes \mathbf{1}_{B_2\otimes EA_2}) = (c_{B_1,E(A_1\otimes A_2)}\otimes \mathbf{1}_{B_2})\circ(\mathbf{1}_{B_1}\otimes \Psi_{A_1,A_2;B_2}) \tag{$\Psi c1$}$$

$$(\mathbf{1}_{B_1}\otimes \Psi_{A_1,A_2;B_2})\circ(c_{EA_1,B_1}\otimes \mathbf{1}_{B_2\otimes EA_2}) = (c_{E(A_1\otimes A_2),B_1}\otimes \mathbf{1}_{B_2})\circ \Psi_{A_1,A_2;B_1\otimes B_2} \tag{$\Psi c2$}$$

of $\mathscr{M}_c^{str}$ in order to obtain $g_{i+1}\circ g_i = g'_{i+1}\circ g'_i$ for a Ψ-factor g'_i and a c-factor g'_{i+1}. Otherwise, we must have the opportunity to apply the equations

$$(\Psi_{A_1,A_2;B_1}\otimes \mathbf{1}_{B_2})\circ(\mathbf{1}_{EA_1\otimes B_1}\otimes c_{B_2,EA_2}) = \Psi_{A_1,A_2;B_1\otimes B_2} \tag{$\Psi c3$}$$

$$\Psi_{A_1,A_2;B_1\otimes B_2}\circ(\mathbf{1}_{EA_1\otimes B_1}\otimes c_{EA_2,B_2}) = \Psi_{A_1,A_2;B_1}\otimes \mathbf{1}_{B_2}, \tag{$\Psi c4$}$$

which follow from the definition of Ψ, in order to obtain $g_{i+1}\circ g_i = g$ for a Ψ-factor g. After applying all these reductions, we reach our normal form, which has the following structure

Let a *block* $\vec{f}_i$ be a composition of Ψ-factors $f_{i_k}\circ\cdots\circ f_{i_1}$, all with the same τ value and such that $f_{i_{j+1}}\circ f_{i_j}$ is never of the form of the left-hand side of ($\Psi\Psi 1$) and ($\Psi\Psi 2$). Our normal form is $h\circ g\circ \vec{f}_n\circ\cdots\circ\vec{f}_1$ such that:

— $n \geqslant 0$ and the arrangement of the blocks is strictly increasing (see the proof of Proposition 1 in Section 4);
— the arrow term g has no occurrence of ψ or ψ_0 (but c may occur), and h has no occurrence of ψ or c (but ψ_0 may occur);
— $\circ$ may occur in h only as specified in the proof of Proposition 1.

The last part of the proof is obtained using a slight modification of the last part of the proof of Proposition 1. We have the same kind of induction, but in the basis we do not only have $h = h'$, but also $h\circ g = h'\circ g'$. The fact that $h = h'$ follows as before, while $g = g'$ follows from a coherence result generalising Mac Lane's symmetric monoidal coherence (see Mac Lane (1998, Section XI.1) or Došen and Petrić (2004, Chapter 5)) in the same way as the $\mathscr{E}$-Coherence of Section 2 generalises Mac Lane's monoidal coherence, and which is proved analogously to the result for $\mathscr{E}$-Coherence. □

We can then infer $\mathscr{M}_c$-Coherence from this proposition.

To prove $\mathscr{L}_c$-Coherence, we proceed as for $\mathscr{M}_c$-Coherence. We introduce the strictification $\mathscr{L}_c^{str}$ of $\mathscr{L}_c$ and prove the following proposition, from which we will infer $\mathscr{L}_c$-Coherence.

Proposition 3. The functor GH from $\mathscr{L}_c^{str}$ to *Fun* is faithful.

Proof. We have a normal form for the arrow terms of $\mathscr{L}_c^{str}$, which is a modification of the normal form of the proof of Proposition 2. We use the following abbreviations for

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this normal form in $\mathscr{L}_c^{str}$:

$$\Psi^L_{A_1,A_2;B} =_{df} (\psi^L_{A_1,A_2} \otimes \mathbf{1}_B) \circ (\mathbf{1}_{EA_1} \otimes c_{B,A_2}) : EA_1 \otimes B \otimes A_2 \to E(A_1 \otimes A_2) \otimes B$$

$$\Psi^R_{A_1,A_2;B} =_{df} (\psi^R_{A_1,A_2} \otimes \mathbf{1}_B) \circ (\mathbf{1}_{A_1} \otimes c_{B,EA_2}) : A_1 \otimes B \otimes EA_2 \to E(A_1 \otimes A_2) \otimes B,$$

which are both obtained from the definition of $\Psi_{A_1,A_2;B}$ by adding the superscripts L or R to ψ and deleting some occurrences of E in the subscripted indices.

The h part of the normal form with ψ_0-factors does not exist now, and instead of Ψ-factors, we have Ψ^L-factors and Ψ^R-factors, which, collectively, we call Ψ-factors. For a Ψ-factor g_j, we define $\tau(g_j)$ as before, and proceed as before when $\tau(g_{i+1}) < \tau(g_i)$ for the Ψ-factors g_i and g_{i+1}.

When $\tau(g_{i+1}) = \tau(g_i)$, we may apply one of the following equations of $\mathscr{L}_c^{str}$ from left to right:

$$(\Psi^L_{A_1\otimes A_3,A_2;B_1} \otimes \mathbf{1}_{B_2}) \circ \Psi^L_{A_1,A_3;B_1\otimes A_2\otimes B_2} \qquad (\Psi^L\Psi^L)$$
$$= (E(\mathbf{1}_{A_1} \otimes c_{A_2,A_3}) \otimes \mathbf{1}_{B_1\otimes B_2}) \circ \Psi^L_{A_1\otimes A_2,A_3;B_1\otimes B_2} \circ (\Psi^L_{A_1,A_2;B_1} \otimes \mathbf{1}_{B_2\otimes A_3})$$

$$\Psi^L_{A_1\otimes A_2,A_3;B_1\otimes B_2} \circ (\Psi^R_{A_1,A_2;B_1} \otimes \mathbf{1}_{B_2\otimes A_3}) \qquad (\Psi^L\Psi^R1)$$
$$= (\Psi^R_{A_1,A_2\otimes A_3;B_1} \otimes \mathbf{1}_{B_2}) \circ (\mathbf{1}_{A_1\otimes B_1} \otimes \Psi^L_{A_2,A_3;B_2})$$

$$(\Psi^L_{A_1\otimes A_3,A_2;B_1} \otimes \mathbf{1}_{B_2}) \circ \Psi^R_{A_1,A_3;B_1\otimes A_2\otimes B_2} \qquad (\Psi^L\Psi^R2)$$
$$= (E(\mathbf{1}_{A_1} \otimes c_{A_2,A_3}) \otimes \mathbf{1}_{B_1\otimes B_2}) \circ (\Psi^R_{A_1,A_2\otimes A_3;B_1} \otimes \mathbf{1}_{B_2}) \circ (\mathbf{1}_{A_1\otimes B_1} \otimes \Psi^R_{A_2,A_3;B_2}).$$

The equation $(\Psi^L\Psi^L)$ is obtained from the equation $(\Psi\Psi1)$ in the proof of Proposition 2 above by adding the superscripts L to Ψ and deleting both occurrences of E in the subscripted indices. The equation $(\Psi^L\Psi^R1)$ is obtained in a similar manner from $(\Psi\Psi2)$ read from right to left. The equation $(\Psi^L\Psi^R2)$, which is analogous to $(\Psi^L\Psi^L)$, could be obtained similarly from an equation of $\mathscr{M}_c^{str}$, which we did not need, and did not mention before.

We have, moreover, eight equations obtained from the equations $(\Psi c1)$-$(\Psi c4)$ by adding the superscripts L or R uniformly to Ψ and deleting some occurrences of E in the subscripted indices. These equations enable us to obtain a normal form with the following structure.

A *block* $\vec{f}_i$ is a composition of Ψ-factors $f_{i_k} \circ \cdots \circ f_{i_1}$ all with the same τ value such that $f_{i_{j+1}} \circ f_{i_j}$ is never of the form of the left-hand side of $(\Psi^L\Psi^L)$, and it is never the case that f_{i_j} is a Ψ^R-factor while $f_{i_{j+1}}$ is a Ψ^L-factor. Our normal form is $g \circ \vec{f}_n \circ \cdots \circ \vec{f}_1$ such that, as before:

— $n \geqslant 0$ and the arrangement of the blocks is strictly increasing (see the proof of Proposition 1);
— the arrow term g has no occurrence of ψ^L and ψ^R (but c may occur).

With this normal form, we then proceed as in the proofs of Propositions 1 and 2. □

If we had defined $\Psi^R_{A_1,A_2;B}$ as

$$(\mathbf{1}_B \otimes \psi^R_{A_1,A_2}) \circ (c_{A_1,B} \otimes \mathbf{1}_{EA_2}) : A_1 \otimes B \otimes EA_2 \to B \otimes E(A_1 \otimes A_2),$$

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Ψ^L and Ψ^R would have been more symmetric, and we could have used a modification of our normal form that would not favour pushing Ψ^L to the right as in $(\Psi^L\Psi^R 1)$ and $(\Psi^L\Psi^R 2)$. However, this approach would have made our exposition rather more involved.

7. Coherence with conjunctive relevant endofunctors

A *conjunctive relevant* category is a symmetric monoidal category with a diagonal natural transformation, whose components are the arrows

$$\Delta_A : A \to A \otimes A.$$

We assume the following coherence equations for Δ:

$$a_{A,A,A} \circ (\Delta_A \otimes \mathbf{1}_A) \circ \Delta_A = (\mathbf{1}_A \otimes \Delta_A) \circ \Delta_A \qquad (\Delta a)$$

$$l_I \circ \Delta_I = \mathbf{1}_I \qquad (\Delta l)$$

$$c_{A,A} \circ \Delta_A = \Delta_A \qquad (\Delta c)$$

$$\Delta_{A\otimes B} = c^m_{A,A,B,B} \circ (\Delta_A \otimes \Delta_B), \qquad (\Delta ac)$$

where

$$c^m_{A,B,C,D} =_{df} a^{-1}_{A,C,B\otimes D} \circ (\mathbf{1}_A \otimes (a_{C,B,D} \circ (c_{B,C} \otimes \mathbf{1}_D) \circ a^{-1}_{B,C,D})) \circ a_{A,B,C\otimes D} :$$
$$(A \otimes B) \otimes (C \otimes D) \to (A \otimes C) \otimes (B \otimes D),$$

(Došen and Petrić 1996, Section 2; Petrić 2002, Section 1; Došen and Petrić 2004, Sections 9.1 and 9.2); the word *relevant* is used here because of the connection with the structural fragment of *relevant* logic).

A *conjunctive relevant* endofunctor in a conjunctive relevant category is a linear endofunctor E in this category that preserves Δ globally; that is, we have the equation

$$E\Delta_A = \psi_{A,A} \circ \Delta_{EA}. \qquad (\psi\Delta)$$

Let $\mathscr{R}$ be the free conjunctive relevant category with a family of conjunctive relevant endofunctors. Let *Rel* be the category whose arrows are relations between finite ordinals (and not between any sets, which one might have expected from what is perhaps the more common usage of the denotation *Rel*; our category *Rel* is the skeleton of the category of relations between finite sets, but a notation like $Sk(Rel_{fin})$ would be too cumbersome). We define the functor G from $\mathscr{R}$ to *Rel* as the functor G from $\mathscr{M}_c$ to *Fun* with an additional clause corresponding to the following picture:

$G\Delta_A$ 

Let $\mathscr{R}^-$ be the free conjunctive relevant category, and let G from $\mathscr{R}^-$ to *Rel* (as a matter of fact, Fun^{op}) be defined by restricting G from $\mathscr{R}$ to *Rel*. A proof of $\mathscr{R}^-$-Coherence can be found in Petrić (2002, Section 5).

The category $\mathscr{R}$ is equivalent to its strictification $\mathscr{R}^{str}$, since $\mathscr{M}$ is equivalent to $\mathscr{M}^{str}$ (see Section 4), and, as before, our goal is to prove the following proposition, which entails $\mathscr{R}$-Coherence.

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Proposition 4. The functor GH from $\mathscr{R}^{str}$ to *Rel* is faithful.

We will first prove the following auxiliary lemma for $\mathscr{M}_c$ (see Section 2 for the notions of diversification and scope).

Lemma 1 ($\mathscr{M}_c$-Theoremhood Lemma). For A diversified on generating objects and B diversified, there is an arrow $f : A \to B$ of $\mathscr{M}_c$ if and only if the generators of A and B coincide, and for every generating functor E^i of B the union of the scopes of the occurrences of E^i in A is equal to the scope of E^i in B.

Proof. The proof in the left to right direction is by a trivial induction on the length of f in developed form.

For the other direction, we suppose $\{E^1, \ldots, E^n\}$ is the set of generating functors of B, and proceed by induction on n.

If $n = 0$, this set is empty, and we trivially have an arrow from A to B of symmetric monoidal categories.

For the induction step, let E^1 in E^1B_1 be the leftmost E of B. Since E^1 is not in the scope of any other E in B, by the assumptions of the lemma, it is not in the scope of any other E in A either. So we may assume that A is of the form

$$D_1 \otimes E^1A_1 \otimes \cdots \otimes D_n \otimes E^1A_n \otimes D_{n+1},$$

with parentheses associated arbitrarily, and D_i being E^1-free. It is clear that we have an arrow of $\mathscr{M}_c$ from A to $E^1(A_1 \otimes \cdots \otimes A_n) \otimes D_1 \otimes \cdots \otimes D_{n+1}$. By the induction hypothesis, there is an arrow of $\mathscr{M}_c$ from $A_1 \otimes \cdots \otimes A_n$ to B_1, and hence an arrow of $\mathscr{M}_c$ from $E^1(A_1 \otimes \cdots \otimes A_n)$ to E^1B_1. By appealing again to the induction hypothesis, we have an arrow of $\mathscr{M}_c$ from $p \otimes D_1 \otimes \cdots \otimes D_n$ to B in which E^1B_1 is replaced by p. From all this we obtain an arrow of $\mathscr{M}_c$ from A to B. □

Let B be a part (proper or not) of an object of $\mathscr{M}_c^{str}$, denoted by $A[B]$, and let $A[B']$ be obtained from this object by replacing B by B' (we replace a single part B by a single B'). For $f : B \to B'$, let $A[f] : A[B] \to A[B']$ be constructed from f with identity arrows, $\otimes$ and E in the obvious way. We can then prove the following lemma.

Lemma 2. For the arrow term

$$f : A[EA_1 \otimes D \otimes EA_2] \to B$$

of $\mathscr{M}_c^{str}$ and with $g : B \to C$ a ψ-factor such that the ordinals corresponding to the outermost occurrences of E in EA_1 and EA_2 are i and j, respectively, and $(GHf)(i) \neq (GHf)(j)$, but $(GH(g \circ f))(i) = (GH(g \circ f))(j)$, there exists an arrow term

$$f' : A[E(A_1 \otimes A_2) \otimes D] \to C$$

of $\mathscr{M}_c^{str}$ such that $g \circ f = f' \circ A[\Psi_{A_1,A_2;D}]$.

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Proof. Note first that every arrow term of $\mathscr{M}_c^{str}$ is a substitution instance of an arrow term of $\mathscr{M}_c^{str}$ with a diversified target. So we may assume that C in the lemma is diversified. The fact that f' exists follows from the assumption that we have $g \circ f$ and from the $\mathscr{M}_c$-Theoremhood Lemma (Lemma 1). The fact that $g \circ f = f' \circ A[\Psi_{A_1,A_2;D}]$ then follows from $\mathscr{M}_c$-Coherence. □

Remark 1. Consider the arrow term $f : A[D \otimes D] \to B$ of $\mathscr{M}_c^{str}$ with D atomic, and let i and j, with $i < j$, be the ordinals corresponding either to the outermost occurrences of E in D when D is of the form ED', or, otherwise, to the two occurrences of D when D is a propositional letter. If $(GHf)(j) < (GHf)(i)$, then for h being $f \circ A[c_{D,D}]$ we have $(GHh)(i) < (GHh)(j)$ and $f = h \circ A[c_{D,D}]$.

A developed arrow term made only of Δ-factors is called a Δ-*term*. If h is a Δ-term, then for $GHh : GHA \to GHB$, the converse relation $(GHh)^{-1} : GHB \to GHA$ is an onto function. A Δ-term is *atomised* when for every Δ-factor in it, in the head Δ_A of this Δ-factor, A is atomic.

Let h be an atomised Δ-term such that E^j occurs exactly once in its source. By naturality and the functorial equations, h is equal to an atomised Δ-term $h_3 \circ h_2 \circ h_1$ such that, for every factor of h_1:

— its head is neither in the scope of E^j nor is it of the form $\Delta_{E^jA'}$;
— all the heads of the factors of h_2 are of the form $\Delta_{E^jA'}$; and
— all the heads of the factors of h_3 are in the scope of E^j (*cf.* the proof of $\mathscr{E}$-Coherence in Section 2).

Analogously, we can transform every atomised Δ-term into the *normal form* $h_3 \circ h_2 \circ h_1$ *relative to an occurrence* E^i in its source (we have to replace this particular occurrence of E^i by a genuinely new E^j, then factor the newly obtained arrow term as above, and, finally, we substitute E^i for E^j everywhere in the term).

A Δ-*capped* arrow term is an arrow term $f \circ h$ of $\mathscr{R}^{str}$ such that $h : D \to A$ is a Δ-term and $f : A \to C$ is an arrow term of $\mathscr{M}_c^{str}$. A Δ-capped arrow term $f \circ h$ is *atomised* when h is an atomised Δ-term.

A *short circuit* in an arrow term $f \circ h$, with $h : D \to A$, is a pair of ordinals (i, j) such that $i, j \in GHA$, $i < j$, $(GHh)^{-1}(i) = (GHh)^{-1}(j)$ and $(GHf)(j) = (GHf)(i)$.

A *useless crossing* is defined analogously to a short circuit, except that here we have $(GHf)(j) < (GHf)(i)$. For an example of a short circuit and a useless crossing see the picture after Lemma 6.

Lemma 3. Every arrow term of $\mathscr{R}^{str}$ is equal to an atomised Δ-capped arrow term.

To prove this lemma we just apply naturality and the functorial equations together with $\Delta_I = \mathbf{1}_I$, which is (Δl) strictified, and (Δac).

Lemma 4. Every arrow term of $\mathscr{R}^{str}$ is equal to an atomised Δ-capped arrow term without short circuits.

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Proof. We first apply Lemma 3 and then proceed by induction on the number n of short circuits in an atomised Δ-capped arrow term. If $n = 0$, we are done. If $n > 0$, our arrow term is of the form $k \circ g \circ f \circ h$, for $f \circ h$ an atomised Δ-capped arrow term without short circuits, $g \circ f \circ h$ an atomised Δ-capped arrow term with a single short circuit (i, j) and g a ψ-factor. Let $h_3 \circ h_2 \circ h_1$ be the normal form of the Δ-term h relative to the occurrence E^i in the source of h that corresponds to the ordinal $(GHh)^{-1}(i)$, which is equal to $(GHh)^{-1}(j)$, and let $g \circ f$ be transformed according to Lemma 2; here h_2 is not an identity arrow. Now we can apply naturality and the functorial equations to 'permute' h_3 with $A[\Psi_{A_1,A_2;D}]$, and then the equations (Δa) strictified, (Δc) and $(\psi\Delta)$ in order to decrease n. After applying $(\psi\Delta)$, we may have to apply $\Delta_I = \mathbf{1}_I$ and (Δac) again to atomise the resulting Δ-capped arrow term, to which we then apply the induction hypothesis. □

Lemma 5. If for the Δ-terms $f, g : A \to B$ we have $GHf = GHg$, then $f = g$.

Proof. We proceed essentially as in the proof of $\mathscr{E}$-Coherence in Section 2, except that:

— When E does not occur in A, we rely on $\mathscr{R}^-$-Coherence instead of Mac Lane's monoidal coherence.

— When E occurs in A, the interpolant C is determined not only by A' and B', but we must also take into account Gf, which is equal to Gg. From $Gf = Gg$, we may also infer $Gf_1 = Gg_1$ and $Gf_2 = Gg_2$.

Another difference when compared with the proof of $\mathscr{E}$-Coherence is that the number of occurrences of E in A is not equal to the number in B, and in C we may have more than one occurrence of E^j. However, there is no essential difference compared with the previous proof because we do not have to deal with Δ-terms like $E^j\Delta_{E^jA}$ that have Δ in the scope of E^j and E^j in the index of Δ. Between two occurrences of E^j in C, there will always be a $\otimes$ in whose scope they are. Hence in f_1 and g_1 the subformula E^jD will again amount to a generating object, and in f_2 and g_2 only what is within the scope of E^j counts. However, since there may be more than one occurrence of E^j in C, we may need to apply the induction hypothesis more than once to establish that $f_2 = g_2$. □

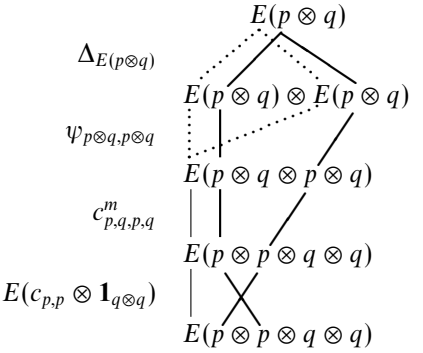
Note that the normal form of f relative to an occurrence of E^i in A is just a refinement of the factorisation $f_2 \circ f_1$ used in the proof of $\mathscr{E}$-Coherence in Section 2, and in the proof of Lemma 5. For $\mathscr{E}$-Coherence, we could take the factorisation $f_2 \circ f_1$ to be such that all the heads of the factors of f_1 are in the scope of E^j, and no head of the factors of f_2 is in the scope of E^j, while in the case of Δ-terms, switching the roles of f_1 and f_2 is impossible.

An arrow term of $\mathscr{R}^{str}$ is in *normal form* when it is an atomised Δ-capped arrow term without short circuits and without useless crossings. We can now prove the following lemma.

Lemma 6. Every arrow term of $\mathscr{R}^{str}$ is equal to an arrow term in normal form.

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To prove this, we just apply Lemma 4, Remark 1 and the equation (Δc). Note that we could not apply Remark 1 without previously applying Lemma 4. For example, we could have



where dotted lines are tied to a short circuit and bold lines to a useless crossing.

We can now prove Proposition 4, which says that the functor GH from $\mathscr{R}^{str}$ to *Rel* is faithful.

Proof of Proposition 4. Note that for an arrow term $f \circ h$ of $\mathscr{R}^{str}$ in normal form, where $h : A \to B$ is a Δ-term and $f : B \to C$ is an arrow term of $\mathscr{M}_c^{str}$, we have that $G(f \circ h)$ uniquely determines Gh, Gf and B. This is not entirely trivial, and is established along the same lines as the more general Decomposition Proposition of Došen and Petrić (2010, Section 8). We then conclude the proof of the proposition using Lemma 5 and Proposition 2, that is, $\mathscr{M}_c$-Coherence. □

8. Coherence in cartesian categories with relevant endofunctors

A *cartesian* category is a conjunctive relevant category with the monoidal unit object I being a terminal object. The unique arrow from A to I is $¡_A : A \to I$. This notion of a cartesian category is equivalent to the usual notion, where a cartesian category is a category with all finite products (see Došen and Petrić (2004, Sections 9.1 and 9.2); some authors use *cartesian* to denote categories with finite limits other than just finite products – see Johnstone (2002, Volume I, Section A1.2) and Leinster (2003, Section 4.1)).

In accordance with what we had before, a cartesian endofunctor in a cartesian category should preserve ¡, which would yield the equation

$$E¡_A = \psi_0 \circ ¡_{EA} \qquad (\psi¡)$$

analogous to ($\psi\Delta$). However, since $GI = \emptyset$, we see easily that no definition of $G¡_A$ would enable us to obtain coherence with ($\psi¡$); even the functoriality of G, that is, $G(g \circ f) = Gg \circ Gf$, would fail (*cf.* the last picture in Section 3, and the beginning of Section 5 in Došen and Petrić (2010)).

However, we still get coherence for conjunctive relevant endofunctors in cartesian categories, which we shall prove now. Let $\mathscr{C}$ be the free cartesian category with a family

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of conjunctive relevant functors. We define the functor G from $\mathscr{C}$ to *Rel* as the functor G from $\mathscr{R}$ to *Rel* with an additional clause that says that $G¡$ is the empty relation between GA and $\emptyset$, which is GI.

The category $\mathscr{C}$ is equivalent to its strictification $\mathscr{C}^{str}$ since $\mathscr{M}$ is equivalent to $\mathscr{M}^{str}$ (see Section 4), and, as before, our goal is to prove the following proposition, which entails $\mathscr{C}$-Coherence.

Proposition 5. The functor GH from $\mathscr{C}^{str}$ to *Rel* is faithful.

Proof. We proceed analogously to the proof of Proposition 4, but modify the lemmas and terminology given there to take account of the presence of ¡ in $\mathscr{C}^{str}$.

A developed arrow term made only of Δ-factors and ¡-factors is called a $\Delta¡$-*term*. If h is a $\Delta¡$-term, then for $GHh : GHA \to GHB$, the converse relation $(GHh)^{-1} : GHB \to GHA$ is a function. A $\Delta¡$-term is *atomised* when for every Δ-factor and every ¡-factor in it, in the heads Δ_A or ¡$_A$ of this Δ-factor or ¡-factor, A is atomic.

For every atomised $\Delta¡$-term h, and every occurrence E^i in its source, we define the *normal form* $h_3 \circ h_2 \circ h_1$ of h *relative to this occurrence of* E^i in exactly the same way as we defined it for atomised Δ-terms in Section 7. A $\Delta¡$-*capped* arrow term is an arrow term $f \circ h$ of $\mathscr{C}^{str}$ such that h is a $\Delta¡$-term and f is an arrow term of $\mathscr{M}_c^{str}$. A $\Delta¡$-capped arrow term $f \circ h$ is *atomised* when h is an atomised $\Delta¡$-term. The short circuit and useless crossing notions are defined exactly as in Section 7.

An arrow term of $\mathscr{C}^{str}$ is in *normal form* when it is an atomised $\Delta¡$-capped arrow term without short circuits and without useless crossings. Lemmas 3–6 with $\mathscr{R}^{str}$ replaced by $\mathscr{C}^{str}$ and Δ replaced by $\Delta¡$ can be proved along the same lines as in the proofs in Section 7. For the proof of the modification of Lemma 5, where we relied on $\mathscr{R}^{-}$-Coherence, we now rely on cartesian coherence, that is, the faithfulness of G from the free cartesian category into *Rel* (see Došen and Petrić (2004, Section 9.2), and references therein).

Unlike in the proof of Proposition 4, here there is no difficulty in showing that $G(f \circ h)$ uniquely determines Gh, Gf and B because we may assume that the target of $f \circ h$ is $\otimes$-free, which we can show by composing with

$$E^n(r_a \circ (\mathbf{1}_a \otimes ¡_B)) : E^n(A \otimes B) \to E^n A$$

and

$$E^n(l_B \circ (¡_A \otimes \mathbf{1}_B)) : E^n(A \otimes B) \to E^n B,$$

where E^n is the sequence of n occurrences of E, and then using the following 'extensionality' equation of $\mathscr{C}$:

$$E^{n-1}\psi_{A,B} \circ \cdots \circ \psi_{E^{n-1}A,E^{n-1}B} \circ (E^n(r_a \circ (\mathbf{1}_a \otimes ¡_B)) \otimes E^n(l_B \circ (¡_A \otimes \mathbf{1}_B))) \circ \Delta_{E^n(A\otimes B)} = \mathbf{1}_{E^n(A\otimes B)}.$$

Since we may assume that the target of $f \circ h$ is $\otimes$-free, we may assume that h is Δ-free, and Gh, Gf and B are then uniquely determined from $G(f \circ h)$ in a straightforward manner. □

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9. Coherence in cocartesian categories with endofunctors

A *disjunctive relevant* category is a symmetric monoidal category with a codiagonal natural transformation whose components are the arrows

$$\nabla_A : A \otimes A \to A.$$

For ∇, we assume coherence equations dual to (Δa), (Δl), (Δc) and (Δac).

A *disjunctive relevant* endofunctor in a disjunctive relevant category is a linear endofunctor E in this category that satisfies the equation

$$E\nabla_A \circ \psi_{A,A} = \nabla_{EA}. \qquad (\psi\nabla)$$

This equation, together with others, enables us to reduce to a propositional letter all the indices of ∇, and that, together with the naturality of ∇, is, essentially, all we need to push every occurrence of ∇ to the left. This enables us to prove coherence for disjunctive relevant categories.

A *cocartesian* category is a disjunctive relevant category with the monoidal unit object I being an initial object. The unique arrow from I to A is $!_A : I \to A$. Equivalently, cocartesian categories are defined as categories with all finite coproducts.

It will follow from the coherence result below that any endofunctor E in a cocartesian category is a disjunctive relevant endofunctor with the definitions

$$\begin{aligned} \psi_{A,B} &=_{df} \nabla_{E(A\otimes B)} \circ (E((\mathbf{1}_A \otimes\ !_B) \circ r_A^{-1}) \otimes E((!_A \otimes \mathbf{1}_B) \circ l_B^{-1})) \\ \psi_0 &=_{df}\ !_{EI}. \end{aligned}$$

Moreover, this endofunctor preserves !, in the sense that it satisfies the equation

$$E!_A \circ \psi_0 = !_{EA}. \qquad (\psi!)$$

Unlike $(\psi\mathrm{i})$, this equation is in accordance with coherence. Note that the equations defining ψ_0 and ψ above follow from the initiality of I, the requirement that $\otimes$ is a coproduct and the equations (ψl) and (ψr).

Let $\mathscr{D}$ be the free cocartesian category with a family of endofunctors. We define the functor G from $\mathscr{D}$ to *Fun* as the functor G from $\mathscr{M}_c$ to *Fun* with the additional clause that corresponds to the picture

$$G\nabla_A \qquad \begin{array}{c} A \otimes A \\ \bigvee \\ A \end{array}$$

and the clause that says that $G!_A$ is the empty function from $\emptyset$, which is GI, to GA. Then we can prove $\mathscr{D}$*-Coherence*.

We proceed essentially as in the proof of $\mathscr{E}$-Coherence in Section 2, and as in the proofs of Lemma 5 and its modifications in the two preceding sections, but now rely on cocartesian coherence, instead of monoidal coherence, $\mathscr{R}^{-}$-Coherence and cartesian coherence, respectively (the only difference is that now we work relative to an occurrence E^j in the target B, and the factorisation $f_2 \circ f_1$ is such that all the heads of the factors of f_1 are in the scope of E^j and no head of a factor of f_2 is in the scope of E^j).

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An alternative way to prove $\mathscr{D}$-Coherence is to use the factorisation $f_2 \circ f_1$ such that f_1 is an arrow term of $\mathscr{L}_c$, and in the developed arrow term f_2 every factor is either a ∇-factor or a !-factor such that the index of its head is a propositional letter. To do this, we use the equations $(\psi\nabla)$ and $(\psi!)$ above.

Cocartesian coherence, that is, the faithfulness of G from the free cocartesian category into *Fun*, follows from cartesian coherence (see Došen and Petrić (2004, Section 9.2), and references therein). Cocartesian coherence may be proved using a normal form inspired by Gentzen's cut elimination (see Došen and Petrić (2004, Sections 9.1 and 9.2)), but we could alternatively use a developed strictified normal form $f_3 \circ f_2 \circ f_1$, where f_1 has atomised c-factors only, f_2 has atomised ∇-factors only, and f_3 has atomised !-factors only.

The coherence results of this paper yield coherence results for categories with arrows oriented in the opposite direction. In these categories, we do not have monoidal functors with ψ and ψ_0, but *comonoidal* functors with arrows oriented oppositely to ψ and ψ_0.

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