All parabolas through three non-collinear points

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If no two of three non-collinear points share the same x-coordinate, then the parabola $y = a_2x^2 + a_1x + a_0$ through the points is easily found by solving a system of linear equations. That is but one of an infinite number of parabolas through the three points. How does one find the other parabolas? In this note, we find all parabolas through any three non-collinear points by reducing the problem to finding the equation of a parabola by using rotations.

The parabola $y = a_2x^2 + a_1x + a_0$ has an axis of symmetry parallel to the y-axis. Other parabolas have an axis of symmetry that is parallel to some line $y = mx$. We focus on the angle θ that the axis of symmetry makes with the y-axis, as in Figure 1, so that $\tan\theta = 1/m$. To find the parabola associated with θ that goes through three non-collinear points, we rotate the three points counterclockwise by θ, find the equation of the parabola, and then rotate the parabola (and the three points) counterclockwise back by $-\theta$ so that the parabola goes through the original points. As we explain below, this approach shows that there are three angles for which there are no parabolas through the three points and allows us to find the parabola with the widest opening, or widest parabola, through the points.

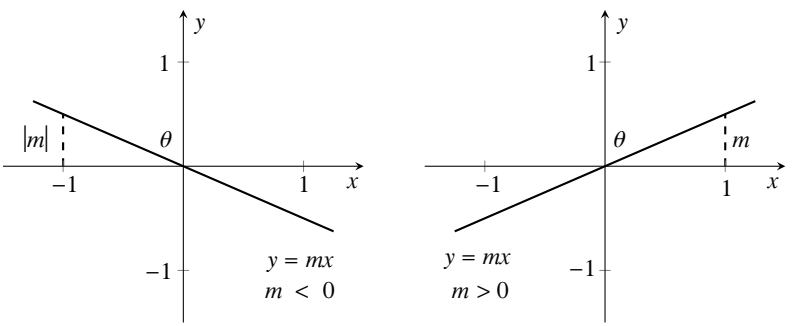


FIGURE 1: $\theta \in (-\pi/2,\ 0)$ (Left); $\theta \in (0,\ \pi/2)$ (Right)

1. *A motivating example*

To find the parabola through the points $(2, 3)$, $(-1,\ 1)$ and $(0,\ 0)$ and with axis of symmetry parallel to $y = x$, we rotate the points around the origin counterclockwise by $\theta = \pi/4$ because $m = 1$ and $\tan\theta = 1/m$. The points are rotated by the rotation matrix $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ for $\theta = \pi/4$. The origin $(0, 0)$ remains at the origin after rotation, but the points $(2, 3)$ and $(-1,\ 1)$ become, respectively,

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$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{5\sqrt{2}}{2} \end{bmatrix} \text{ and } \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} \\ 0 \end{bmatrix}.$$

Now, we solve for the equation of the parabola $y = a_2x^2 + a_1x + a_0$ through the rotated points by solving the system of equations in the a_i when the rotated points are substituted into the equation of the parabola. Because the parabola goes through the point $(0, 0)$, it follows that $a_0 = 0$. The other two points result in two linear equations and two unknowns:

$$\left(\frac{-\sqrt{2}}{2}\right)^2 a_2 + \left(\frac{-\sqrt{2}}{2}\right)a_1 = \frac{5\sqrt{2}}{2} \text{ and}$$

$$\left(-\sqrt{2}\right)^2 a_2 + \left(-\sqrt{2}\right)a_1 = 0.$$

Solving these equations simultaneously gives $a_1 = -10$ and $a_2 = -5\sqrt{2}$. The resulting parabola $y = -5\sqrt{2}x^2 - 10x$ goes through the rotated points, as shown in Figure 2 (Left). For the next step, we will refer to this parabola as P and by the equation $f(x, y) = -5\sqrt{2}x^2 - 10x - y = 0$.

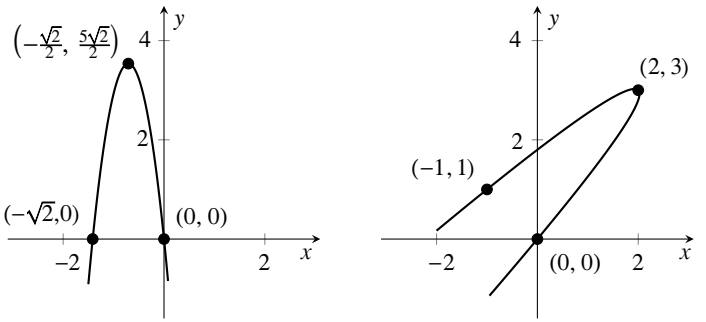


FIGURE 2: The parabola $y = -5\sqrt{2}x^2 - 10x$ through the points $\left(-\sqrt{2}/2,\ 5\sqrt{2}/2\right)$, $\left(-\sqrt{2},\ 0\right)$ and $(0, 0)$ (Left); The parabola $5y^2 - 10xy + 5x^2 + 11x - 9y = 0$ through the points $(2, 3)$, $(-1,\ 1)$ and $(0, 0)$ (Right)

Finally, the parabola P given by $f(x, y) = 0$ is rotated back by $-\theta$, using the rotation matrix $R_{-\theta}$. This rotates the points back to their initial positions and rotates the parabola as well. To generate the parabola through the original points, P is rotated counterclockwise by $-\theta$ (i.e. clockwise by θ) which yields the desired parabola P'. A point (x, y) lies on the parabola P' if, and only if, $R_\theta(x, y)$ lies on P. (This follows because $R_{-\theta}(P) = P'$, so that $P = R_\theta(P')$.) Hence the points

$$R_\theta(x, y) = \left(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta\right)$$

must satisfy $f(x, y) = 0$, or, equivalently, the parabola P' is the set of (x, y) for which $f\left(x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta\right) = 0$.

For the example where $-\theta = -\pi/4$, then the parabola P' is the set of (x, y) for which $f\left((\sqrt{2}/2)x - (\sqrt{2}/2)y,\ (\sqrt{2}/2)x + (\sqrt{2}/2)y\right) = 0$, which is equivalent to

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$$-5\sqrt{2}\left[(\sqrt{2}/2)x - (\sqrt{2}/2)y\right]^2 - 10\left[(\sqrt{2}/2)x - (\sqrt{2}/2)y\right]$$
$$-(\sqrt{2}/2)x - (\sqrt{2}/2)y = 0.$$

This simplifies to $5y^2 - 10xy + 5x^2 + 11x - 9y = 0$. This parabola P' has axis of symmetry parallel to $y = x$ and goes through the original three points; see Figure 2 (Right).

2. *The general problem*

Suppose we want to find a parabola with an axis of symmetry parallel to $y = mx$ and through the non-collinear points (x_1, y_1), (x_2, y_2) and (x_3, y_3). We may assume that $(x_3, y_3) = (0, 0)$; if not, then we can solve the problem for $(x_1 - x_3, y_1 - y_3)$, $(x_2 - x_3, y_2 - y_3)$ and $(0, 0)$ and then translate the resulting parabola by (x_3, y_3). The advantage of using $(0, 0)$ for the third point is that $(0, 0)$ remains fixed upon rotation; this ensures that the constant term a_0 of the parabolic equation $y = a_2x^2 + a_1x + a_0$ is 0.

In the introduction, we noted that if any of the three non-collinear points shared an x-value, then no parabola goes through the three points. From this observation, it is clear that there are three axes of symmetry for which there is no parabola through the three points. These axes have slopes parallel to the sides of the triangle made from the three points, as rotating the points by the associated angles results in two of the three points sharing x-values. For the initial points (x_1, y_1), (x_2, y_2) and $(0, 0)$, a rotation by $\theta_i = \tan^{-1}(1/m_i)$ when the axis of symmetry has a slope of

$$m_1 = \frac{y_1 - y_2}{x_1 - x_2}, \; m_2 = \frac{y_1 - 0}{x_1 - 0}, \text{ or } m_3 = \frac{y_2 - 0}{x_2 - 0}$$

results in two of the rotated points having the same x-values. The angles θ_i for $i = 1$ to 3 are pictured in Figure 3.

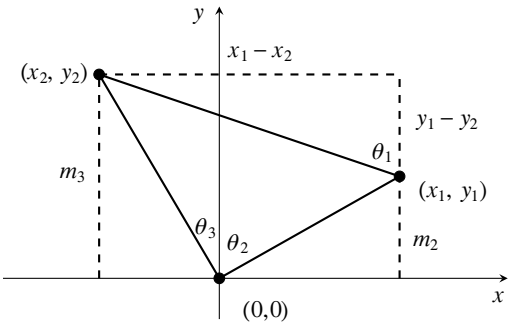


FIGURE 3: The angles θ_i for which there are no parabolas

We now consider the general problem for non-collinear points (x_1, y_1), (x_2, y_2) and $(0, 0)$ and a desired axis of symmetry with slope m where $m \neq m_i$ for the m_i as defined above for $i = 1$ to 3. Let $\theta = \tan^{-1}(1/m)$. First, we rotate the points using the rotation matrix R_θ. Recall that $(0, 0)$ is

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fixed under rotation. Rotating (x_1, y_1) and (x_2, y_2) gives

$$R_\theta \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} \cos\theta\, x_i - \sin\theta\, y_i \\ \sin\theta\, x_i + \cos\theta\, y_i \end{bmatrix}$$

The rotated points $(\cos\theta x_i - \sin\theta y_i, \sin\theta x_i + \cos\theta y_i)$, for $i = 1, 2$, are substituted into the equation $y = a_2x^2 + a_1x$ (because $a_0 = 0$) to yield two equations and two unknowns:

$$\sin\theta x_i + \cos\theta y_i = a_2[\cos\theta x_i - \sin\theta y_i]^2 + a_1[\cos\theta x_i - \sin\theta y_i], \text{ for } i = 1, 2.$$

The two equations can be written as $t_1 = a_2s_1^2 + a_1s_1$ and $t_2 = a_2s_2^2 + a_1s_2$, where

$$t_1 = \sin\theta x_1 + \cos\theta y_1 \qquad t_2 = \sin\theta x_2 + \cos\theta y_2 \qquad \text{and}$$
$$s_1 = \cos\theta x_1 - \sin\theta y_1 \qquad s_2 = \cos\theta x_2 - \sin\theta y_2.$$

Solving this system of equations for a_1 and a_2 yields

$$a_1 = \frac{s_1^2t_2 - s_2^2t_1}{s_1s_2(s_1 - s_2)} \qquad \text{and} \qquad a_2 = \frac{s_2t_1 - s_1t_2}{s_1s_2(s_1 - s_2)}.$$

It follows that the parabola through the rotated points is

$$f(x, y) = a_2x^2 + a_1x - y$$
$$= \left(\frac{s_2t_1 - s_1t_2}{s_1s_2(s_1 - s_2)}\right)x^2 + \left(\frac{s_1^2t_2 - s_2^2t_1}{s_1s_2(s_1 - s_2)}\right)x - y = 0. \qquad (1)$$

Notice that the coefficients a_1 and a_2 have the same denominator. This denominator is 0 when $s_1 = 0$, $s_2 = 0$, or $s_1 = s_2$. In these cases, there is no parabola through the rotated points. The following lemma shows that these conditions match our intuition about how rotations by θ_1, θ_2 and θ_3, as pictured in Figure 3, do not yield parabolas.

Lemma: There is no parabola through the points (x_1, y_1), (x_2, y_2) and $(0, 0)$ with axis of symmetry parallel to $y = m_ix$ for $i = 1$ to 3 where

$$m_1 = \frac{y_1 - y_2}{x_1 - x_2}, \; m_2 = \frac{y_1}{x_1}, \text{ or } m_3 = \frac{y_2}{x_2}. \qquad (2)$$

Proof: The equation of the parabola in (1) is undefined when $s_1s_2(s_1 - s_2) = 0$, which is equivalent to $s_1 = 0$, $s_2 = 0$ or $s_1 = s_2$. These conditions reduce to $\tan\theta = x_1/y_1$, $\tan\theta = x_2/y_2$ or

$$(y_1 - y_2)\sin\theta = (x_1 - x_2)\cos\theta,$$

respectively. The last one can be rewritten as

$$\tan\theta = \frac{x_1 - x_2}{y_1 - y_2}.$$

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Because $\tan\theta = 1/m$, the slopes for which a parabola does not exist are those in (2) with θ_i as in Figure 3.

To get the final parabola, as in the description of the example, the parabola from (1) is rotated counterclockwise by $-\theta$, using the rotation matrix $R_{-\theta}$. As before, the parabola P' is the set of (x, y) for which

$$f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) = 0.$$

Hence, the parabola through the three initial points (x_1, y_1), (x_2, y_2) and $(0, 0)$ is given by the equation

$$a_2(\cos\theta x - \sin\theta y)^2 + a_1(\cos\theta x - \sin\theta y) - \cos\theta y - \sin\theta x = 0.$$

By substituting in the values for a_1 and a_2, writing the parabola in standard form, and using a considerable amount of routine algebra and the identity $\cos^2\theta + \sin^2\theta = 1$, we obtain the following theorem.

Theorem: The parabola through (x_1, y_1), (x_2, y_2) and $(0, 0)$ with axis of symmetry parallel to $y = mx$ has equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{3}$$

where $\theta = \tan^{-1}(1/m)$ and

$$\begin{aligned}
A &= (x_2y_1 - x_1y_2)\cos^2\theta \\
B &= (x_2y_1 - x_1y_2)(-2\cos\theta\sin\theta) \\
C &= (x_2y_1 - x_1y_2)\sin^2\theta \\
D &= (x_1^2y_2 - x_2^2y_1)\cos^2\theta + (x_1y_1y_2 - x_2y_1y_2)(-2\cos\theta\sin\theta) \\
&\quad + (y_1^2y_2 - y_1y_2^2)\sin^2\theta \\
E &= (x_1x_2^2 - x_1^2x_2)\cos^2\theta + (x_1x_2y_2 - x_1x_2y_1)(-2\cos\theta\sin\theta) \\
&\quad + (x_1y_2^2 - x_2y_1^2)\sin^2\theta \\
F &= 0.
\end{aligned}$$

3. *Focus, directrix and vertex*

Because a parabola is the locus of points equidistant from a point (the focus) and a line (the directrix), we determine the focus, directrix, and vertex of the parabola with axis of symmetry of angle θ through (x_1, y_1), (x_2, y_2) and $(0, 0)$. First we find the focus, directrix, and vertex of the equation of the parabola $y = a_2x^2 + a_1x$ from (1) and then rotate the results by $-\theta$. The equation of a parabola may be represented as $4p(y - k) = (x - h)^2$, which has focus $(h, k + p)$, directrix $y = k - p$, and vertex (h, k). Completing the square in the equation $y = a_2x^2 + a_1x$ yields

$$y + \left(\frac{a_1^2}{4a_2}\right) = a_2\left(x + \frac{a_1}{2a_2}\right)^2.$$

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This implies that $4p = 1/a_2$. Because it is cumbersome to give the focus, directrix, and vertex for the parabolic function in terms of θ, we leave them in terms of the a_i in Table 1.

Focus	Directrix	Vertex
$\left(\frac{-a_1}{2a_2}, \frac{1 - a_1^2}{4a_2}\right)$	$y = -\frac{a_1^2 + 1}{4a_2}$	$\left(\frac{-a_1}{2a_2}, \frac{-a_1^2}{4a_2}\right)$

TABLE 1: The focus, directrix, and vertex for the equation of the parabola $y = a_2x^2 + a_1x$

To get the focus, directrix, and vertex of the final parabola, we need only rotate by the focus, directrix, and vertex from Table 1 counterclockwise by $-\theta$. The results appear in Table 2.

Focus	$\left(\frac{-\cos\theta a_1}{2a_2} + \frac{\sin\theta(1 - a_1^2)}{4a_2}, \frac{\sin\theta a_1}{2a_2} + \frac{\cos\theta(1 - a_1^2)}{4a_2}\right)$
Directrix	$\cos\theta y + \sin\theta x = -\frac{a_1^2 + 1}{4a_2}$
Vertex	$\left(\frac{-\cos\theta a_1}{2a_2} - \frac{\sin\theta_1^2}{4a_2}, \frac{\sin\theta a_1}{2a_2} - \frac{\cos\theta a_1^2}{4a_2}\right)$

TABLE 2: The focus, directrix, and vertex for the parabola through (x_1, y_1), (x_2, y_2) and $(0, 0)$

4. *An application: the widest parabola*

Our presentation of the infinitely many parabolas through (x_1, y_1), (x_2, y_2) and $(0, 0)$ makes it straightforward to find the widest parabola through the points. Because the final parabola is just a rotation of the parabola $y = a_2x^2 + a_1x$, the widest parabola is the rotation of the parabola from (1) with the minimal absolute value of a_2 (the coefficient of x^2). Recall that $a_2 = \frac{s_2t_1 - s_1t_2}{s_1s_2(s_1 - s_2)}$. With a little algebra, one notices that the numerator of a_2 is a constant in θ. In particular, $s_2t_1 - s_1t_2 = x_2y_1 - x_1y_2$. Taking the derivative of a_2 with respect to θ gives

$$a_2'(\theta) = \frac{\left[-\left(2s_1s_2 - s_2^2\right)s_1' + \left(2s_1s_2 - s_1^2\right)s_2'\right](x_2y_1 - x_1y_2)}{s_1^2s_2^2(s_1 - s_2)^2},$$

where s_i and t_j are functions of θ. This is cumbersome to evaluate in general, but is possible for specific cases. Because a_2 may be negative or positive, the minimal absolute value may be a local minimum when a_2 is positive or a local maximum when a_2 is negative.

Returning to our motivating example where $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (-1, 1)$, then $a_2'(\theta) = 0$, when

$$\left[-\left(2s_1s_2 - s_2^2\right)s_1' + \left(2s_1s_2 - s_1^2\right)s_2'\right](x_2y_1 - x_1y_2) = 0,$$

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which reduces to

$$\frac{-5}{4}\left[11\,(\cos\theta\ -\ \sin\theta)\ -\ 39\,(\cos 3\theta\ +\ \sin 3\theta\right]\ =\ 0.$$

From this, a_2 takes on a local minimum at $\theta \approx -0.1836$ and this is the global minimum of $|a_2|$. There are two other critical points that are local maxima that occur when a_2 is negative. Each of these local maxima is a candidate for the widest parabola as $|a_2|$ is at a local minimum. Substituting $\theta \approx -0.1836$ into (3), the widest parabola through our three initial points is approximately $4.83x^2 + 1.79xy + 0.17y^2 - 4.4x - 7.6y = 0$, as shown in Figure 4.

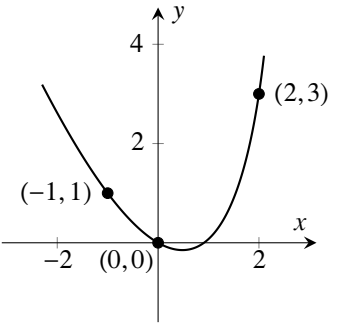


FIGURE 4: The widest parabola through the points (2, 3), (−1, 1) and (0, 0) is approximately $4.83x^2 + 1.79xy + 0.17y^2 - 4.4x - 7.6y = 0$

In contrast, there is no narrowest parabola through the points (x_1, y_1), (x_2, y_2) and (0, 0). This follows because a_2 approaches positive or negative infinity as θ approaches θ_i for $i = 1$ to 3 (because the denominator of a_2 approaches 0) and the greater $|a_2|$, the narrower the parabola. The relationship between a_2 and θ also explains why there are three values of θ to consider for the widest parabola. The idea is that the θ_i divide the interval $(-\pi/2, \pi/2]$ (where $-\pi/2$ and $\pi/2$ are identified, so that the interval is really a circle) into three regions and $|a_2|$ takes on a local minimum on each region.

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