DEFINITE INTEGRALS INVOLVING G-FUNCTIONS

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1. The main aim of the present note is to evaluate certain definite integrals involving Meijer's G-function. The results proved are of general character and include as particular cases some of the results given earlier by MacRobert ((3), p. 81). For the definitions and properties of the G-functions see (1), §§ 5·3 and 5·3·1.

2. There are two formulae to be established here.

$$\int_0^{\pi} \sin(2n+1)\theta\,(\sin\theta)^{1-2\zeta}\,G_{\gamma,\delta}^{\alpha,\beta}\left(x\sin^{2s}\theta\,\middle|\,\begin{matrix}a_1,\ldots,a_\gamma\\ b_1,\ldots,b_\delta\end{matrix}\right)d\theta$$
$$= \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}\,G_{\gamma+2s,\delta+2s}^{\alpha+s,\beta+s}\left(x\,\middle|\,\begin{matrix}a_1,\ldots,a_\beta,c_1,\ldots,c_{2s},a_{\beta+1},\ldots,a_\gamma\\ b_1,\ldots,b_\alpha,d_1,\ldots,d_{2s},b_{\alpha+1},\ldots,b_\delta\end{matrix}\right), \qquad (1)$$

where $n = 0, 1, 2, \ldots$; $c_{k+1} = (2\zeta - 1 + 2k)/2s$, $c_{s+k+1} = (\zeta + k)/s$,

$d_{k+1} = (\zeta + n + k)/s$, $d_{s+k+1} = (\zeta - n - 1 + k)/s$ for $k = 0, 1, \ldots, (s-1)$

and s is a positive integer.

This result is valid under the following conditions.

(i) $\delta \geqslant 1$, $\gamma < \delta$ (or $\gamma = \delta$ and $|x| < 1$),

$$\Re(3 - 2\zeta + 2sb_j) > 0 \quad \text{for} \quad j = 1, 2, \ldots, \alpha;$$

(ii) $\gamma > \delta$, $\alpha \geqslant 1$, $2(\alpha+\beta) > \gamma+\delta$, $|\arg x| < (\alpha+\beta-\frac{1}{2}\gamma-\frac{1}{2}\delta)\pi$,

$$\Re(3 - 2\zeta + 2sb_j) > 0 \quad \text{for} \quad j = 1, 2, \ldots, \alpha;$$

(iii) $\alpha = 0$, $\gamma > \delta$, $2\beta > \gamma + \delta$ and $|\arg x| < (\beta - \frac{1}{2}\gamma - \frac{1}{2}\delta)\pi$.

$$\int_0^{\pi} \cos n\theta\,(\sin\tfrac{1}{2}\theta)^{-2\zeta}\,G_{\gamma,\delta}^{\alpha,\beta}\left(x\sin^{2s}\tfrac{1}{2}\theta\,\middle|\,\begin{matrix}a_1,\ldots,a_\gamma\\ b_1,\ldots,b_\delta\end{matrix}\right)d\theta$$
$$= \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}}\,G_{\gamma+2s,\delta+2s}^{\alpha+s,\beta+s}\left(x\,\middle|\,\begin{matrix}a_1,\ldots,a_\beta,c_1',\ldots,c_{2s}',a_{\beta+1},\ldots,a_\gamma\\ b_1,\ldots,b_\alpha,d_1',\ldots,d_{2s}',b_{\alpha+1},\ldots,b_\delta\end{matrix}\right), \qquad (2)$$

where $n = 0, 1, 2, \ldots$; $c_{k+1}' = (2\zeta + 1 + 2k)/2s$, $c_{k+s+1}' = (\zeta + k)/s$,

$d_{k+1}' = (\zeta + n + k)/s$, $d_{k+s+1}' = (\zeta - n + k)/s$ for $k = 0, 1, \ldots, (s-1)$

and s is a positive integer.

The conditions of validity of this result are as follows.

(i) $\delta \geqslant 1$, $\gamma < \delta$ (or $\gamma = \delta$ and $|x| < 1$),

$$\Re(1 - 2\zeta + 2sb_j) > 0 \quad \text{for} \quad j = 1, 2, \ldots, \alpha;$$

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(ii) $\gamma > \delta$, $\alpha \geqslant 1$, $\Re(1-2\zeta+2sb_j) > 0$ for $j = 1, 2, \ldots, \alpha$,

$$2(\alpha+\beta) > \gamma+\delta \quad \text{and} \quad |\arg x| < (\alpha+\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta)\pi;$$

(iii) $\gamma > \delta$, $\alpha = 0$, $2\beta > \gamma+\delta$ and $|\arg x| < (\beta-\tfrac{1}{2}\gamma-\tfrac{1}{2}\delta)\pi$.

3. The following results are required in the sequel ((3), p. 80; (2), p. 144; (1), p. 207 and (1), p. 4).

$$\int_0^\pi \sin(2n+1)\theta\,(\sin\theta)^{1-2\zeta}\,d\theta = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2}-\zeta)\,\Gamma(\zeta+n)}{\Gamma(\zeta)\,\Gamma(2+n-\zeta)}, \tag{3}$$

where $\Re(3-2\zeta) > 0$, $n = 0, 1, 2, \ldots$;

$$\int_0^\pi \cos n\theta\,(\sin\tfrac{1}{2}\theta)^{-2\zeta}\,d\theta = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}-\zeta)\,\Gamma(\zeta+n)}{\Gamma(\zeta)\,\Gamma(1+n-\zeta)}, \tag{4}$$

where $\Re(1-2\zeta) > 0$, $n = 0, 1, 2, \ldots$;

$$G^{\alpha,\beta}_{\gamma,\delta}\left(x\,\middle|\,\begin{matrix}a_1,\ldots,a_\gamma\\ b_1,\ldots,b_\delta\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{\alpha}\Gamma(b_j-\xi)\prod_{j=1}^{\beta}\Gamma(1-a_j+\xi)}{\prod_{j=\alpha+1}^{\delta}\Gamma(1-b_j+\xi)\prod_{j=\beta+1}^{\gamma}\Gamma(a_j-\xi)}\,x^\xi\,d\xi, \tag{5}$$

where L is a suitable contour, and the multiplication formula for the Gamma function

$$\prod_{k=0}^{m-1}\Gamma\left(z+\frac{k}{m}\right) = (2\pi)^{\frac{1}{2}m-\frac{1}{2}}\,m^{\frac{1}{2}-mz}\,\Gamma(mz), \tag{6}$$

where m is a positive integer.

4. *Proof of the first integral.* If we substitute the value of the G-function from (5) in the integrand of (1), change the order of integration (which we suppose to be permissible) and evaluate the inner integral with the help of (3), then the value of the integral is

$$\begin{aligned}&\frac{\Gamma(\frac{1}{2})}{2\pi i}\int_L \frac{\prod_{j=1}^{\alpha}\Gamma(b_j-\xi)\prod_{j=1}^{\beta}\Gamma(1-a_j+\xi)\,\Gamma(\frac{3}{2}-\zeta+s\xi)\,\Gamma(n+\zeta-s\xi)}{\prod_{j=\alpha+1}^{\delta}\Gamma(1-b_j+\xi)\prod_{j=\beta+1}^{\gamma}\Gamma(a_j-\xi)\,\Gamma(\zeta-s\xi)\,\Gamma(2+n-\zeta+s\xi)}\,x^\xi\,d\xi\\ &= \frac{\Gamma(\frac{1}{2})}{s^{\frac{1}{2}}\,2\pi i}\int_L \frac{\prod_{j=1}^{\alpha}\Gamma(b_j-\xi)\prod_{j=1}^{\beta}\Gamma(1-a_j+\xi)\left\{\prod_{k=0}^{s-1}\Gamma\left(\frac{3-2\zeta+2k}{2s}+\xi\right)\right\}}{\prod_{j=\alpha+1}^{\delta}\Gamma(1-b_j+\xi)\prod_{j=\beta+1}^{\gamma}\Gamma(a_j-\xi)\left\{\prod_{k=0}^{s-1}\Gamma\left(\frac{\zeta+k}{s}-\xi\right)\right\}}\\ &\qquad\times\frac{\left\{\prod_{k=0}^{s-1}\Gamma\left(\frac{\zeta+n+k}{s}-\xi\right)\right\}}{\left\{\prod_{k=0}^{s-1}\Gamma\left(\frac{2+n-\zeta+k}{s}+\xi\right)\right\}}\,x^\xi\,d\xi,\end{aligned} \tag{7}$$

by virtue of (6).

The contour L is a loop starting and ending at $+\infty$ and encircling all the poles of $\Gamma(b_j-\xi)$ for $j = 1, 2, \ldots, \alpha$ and $\Gamma\{(\zeta+n+k)/s-\xi\}$ for $k = 0, 1, \ldots, (s-1)$, once in the

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negative direction, but none of the poles of $\Gamma(1-a_j+\xi)$ for $j=1,2,\ldots,\beta$ and $\Gamma\{(3-2\zeta+2k)/2s+\xi\}$ for $k=0,1,\ldots,(s-1)$.

On interpreting (7), with the help of (5), we arrive at the result (1) under the conditions (i).

Since ((1), p. 209)

$$G^{\alpha,\beta}_{\gamma,\delta}\left(x^{-1}\left|\begin{matrix}a_1,\ldots,a_\gamma\\ b_1,\ldots,b_\delta\end{matrix}\right.\right)=G^{\beta,\alpha}_{\delta,\gamma}\left(x\left|\begin{matrix}1-b_1,\ldots,1-b_\delta\\ 1-a_1,\ldots,1-a_\gamma\end{matrix}\right.\right), \tag{8}$$

the result (1) also holds when $\gamma>\delta$ under the conditions (ii) and (iii). Then (2) can also be proved in a similar way on using the formula (4).

In (1) and (2) take $\alpha=1$, $\beta=\gamma=l$, $\delta=m+1$, $b_1=0$, write $1-b_{h-1}$ and $1-a_j$ for the parameters b_h and a_j ($h=2,3,\ldots,(m+1)$ and $j=1,2,\ldots,l$), replace x by $1/x$ then by the formula ((1), p. 215),

$$G^{1,l}_{l,m+1}\left(x\left|\begin{matrix}1-a_1,\ldots,1-a_l\\ 0,1-b_1,\ldots,1-b_m\end{matrix}\right.\right)=\mathrm{E}(l;\,a_i:m;\,b_j:1/x), \tag{9}$$

it follows that

$$\int_0^\pi \sin(2n+1)\theta\,(\sin\theta)^{1-2\zeta}\,\mathrm{E}(l;\,a_i:m;\,b_j:x/\sin^{2s}\theta)\,d\theta$$
$$=\frac{\Gamma(\frac12)}{s^{\frac12}}\,G^{1+s,\,l+s}_{l+2s,\,m+1+2s}\left(x^{-1}\left|\begin{matrix}1-a_1,\ldots,1-a_l,c_1,\ldots,c_{2s}\\ 0,1-b_1,\ldots,1-b_m,d_1,\ldots,d_{2s}\end{matrix}\right.\right), \tag{10}$$

where $n=0,1,2,\ldots$ and s is a positive integer, $\Re(3-2\zeta)>0$, $|\arg x|<\frac12(l-m+1)\pi$ if $l>m+1$; $x\neq0$ if $l<m+1$ and $|x|>1$ if $l=m+1$.

$$\int_0^\pi \cos n\theta\,(\sin\tfrac12\theta)^{-2\zeta}\,\mathrm{E}(l;\,a_i:m;\,b_j:x/\sin^{2s}\tfrac12\theta)\,d\theta$$
$$=\frac{\Gamma(\frac12)}{s^{\frac12}}\,G^{1+s,\,l+s}_{l+2s,\,m+1+2s}\left(x^{-1}\left|\begin{matrix}1-a_1,\ldots,1-a_l,c'_1,\ldots,c'_{2s}\\ 0,1-b_1,\ldots,1-b_m,d'_1,\ldots,d'_{2s}\end{matrix}\right.\right), \tag{11}$$

where $\Re(1-2\zeta)>0$, $|\arg x|<\frac12(l-m+1)\pi$ if $l>m+1$; $x\neq0$ if $l<m+1$ and $|x|>1$ if $l=m+1$.

For $s=1$ and $\zeta=0$, (10) and (11) reduce to the results recently given by MacRobert ((3), p. 81).

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