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On coefficient ideals

BY R. CALLEJAS-BEDREGAL
Departamento de Matemática,
CCEN - Centro de Ciências Exatas e da Natureza,
UFPB - Universidade Federal da Paraíba
e-mail: roberto@mat.ufpb.br

V. H. JORGE PÉREZ
Departamento de Matemática,
ICMC - Instituto de Ciências da Computação e Matemática,
USP - Universidade de São Paulo
e-mail: vhjperez@icmc.usp.br

AND M. DUARTE FERRARI
Departamento de Matemática,
CCE - Centro de Ciências Exatas,
UEM - Universidade Estadual de Maringá
e-mail: mdsilva@uem.br

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Abstract

Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an arbitrary ideal of R with analytic spread s. In [**3**] the authors proved the existence of a chain of ideals $I \subseteq I_{[s]} \subseteq \cdots \subseteq I_{[1]}$ such that $\deg\left(P_{I_{[k]}/I}\right) < s - k$. In this article we obtain a structure theorem for this ideals which is similar to that of K. Shah in [**10**] for $\mathfrak{m}$-primary ideals.

1. *Introduction*

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring of Krull dimension d. Let I be a regular ideal in R, that is an ideal which contains a regular element. The ideal

$$\tilde{I} = \bigcup_{n\in\mathbb{N}} \left(I^{n+1} : I^n\right) = \left\{x \in R : xI^n \subseteq I^{n+1} \text{ for some } n\right\}$$

is an interesting ideal first studied by Ratliff and Rush in [**6**]. The ideal $\tilde{I}$ is called the Ratliff–Rush closure of I, and I is called a Ratliff–Rush ideal if $I = \tilde{I}$.

Recall that an ideal $J \subset I$ is called a reduction of I if $JI^n = I^{n+1}$ for some positive integer n.

Ratliff and Rush observed in [**6**, theorem 2·1] that under the above conditions $\tilde{I}$ is the unique largest ideal of R having the same high powers as I, i.e., $(\tilde{I})^n = I^n$ for all large n and if J is an ideal of R such that $J^n = I^n$ for all large n then $J \subseteq \tilde{I}$. In particular, I is a reduction of $\tilde{I}$.

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Hence, every element of $\tilde{I}$ is integral over I, in the sense that there is an equation of the form

$$x^k + b_1 x^{k-1} + \cdots + b_k = 0,$$

where $b_i \in I^i$ for $i = 1, \ldots, k$. Therefore, the ideal $\tilde{I}$ is always between I and the integral closure $\overline{I}$ of I, and hence integrally closed ideals are Ratliff–Rush ideals.

Let us now describe the Ratliff–Rush closure and the integral closure of an $\mathfrak{m}$-primary ideal I of R in terms of its Hilbert–Samuel coefficients. It is well known that the Hilbert–Samuel function $\ell_R(R/I^n)$ is a polynomial $P_I(n)$ in n of degree d, for all large n. This polynomial is called the Hilbert-Samuel polynomial of I, and can be written in the form:

$$P_I(n) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{n+d-i-1}{d-i},$$

where $e_i(I)$ is the ith Hilbert–Samuel coefficient of I, for $i = 0, \ldots, d$. The integer $e_0(I)$ is the classical Hilbert–Samuel multiplicity of I.

The Rees multiplicity theorem [**7**, theorem 3·2] asserts that if R is quasi-unmixed then the integral closure $\overline{I}$ of I is the largest ideal J containing I such that $e_0(I) = e_0(J)$. On the other hand, since the Ratliff–Rush closure $\tilde{I}$ of I has all the same high powers of I, it has the same Hilbert polynomial of I and consequently $\tilde{I}$ is the largest ideal J containing I for which $e_i(I) = e_i(J)$, for all $i = 0, \ldots, d$.

Based on these facts, Shah proved in [**10**], for $0 \leqslant k \leqslant d$, the existence of a unique largest ideals $I_{\{k\}}$, between the ideals $\tilde{I}$ and $\overline{I}$ such that $I_{\{k\}}$ contains I and preserves the first $k+1$ Hilbert–Samuel coefficients of I, i.e., there exists ideals $I_{\{k\}}$ such that $I \subseteq \tilde{I} = I_{\{d\}} \subseteq \ldots \subseteq I_{\{1\}} \subseteq I_{\{0\}} = \overline{I}$ and $e_i(I) = e_i(I_{\{k\}})$, for $i = 0, \ldots, k$. The ideal $I_{\{k\}}$ is called kth coefficient ideal of I. Shah also determined a colon ideal structure for each coefficient ideal $I_{\{k\}}$ of I as follows, $I_{\{k\}} = \left(I^{n_0+1} : x_1, \ldots, x_k\right)$, for $k = 1, \ldots, d$, some fixed $n_0 \geqslant 1$ and some fixed minimal reduction $\{x_1, \ldots, x_d\}$ of I^{n_0}.

Let $J \subset I$ be ideals of R with $\ell(I/J) < \infty$. The work of Amao in [**1**] and Rees in [**8**] shows that the numerical function $h_{I/J}(n) := \ell(I^n/J^n)$ is a polynomial function $P_{I/J}(n)$ of degree at most d. We call $h_{I/J}(n)$ and $P_{I/J}(n)$ the Rees–Amao function and the Rees–Amao polynomial of the pair (J, I) respectively. The Rees multiplicity theorem has been generalised for ideals that are not $\mathfrak{m}$-primary by a number of authors. For instance, in [**8**] Rees showed that if $(R, \mathfrak{m})$ is quasi-unmixed then $J \subset I$ is a reduction of I if and only if $\deg\left(P_{I/J}(n)\right) < d$.

It is natural to ask for the exact degree of $P_{I/J}(n)$ in general. For example, in the case that J, and hence I, is $\mathfrak{m}$-primary it is easy to see that $\deg\left(P_{I/J}(n)\right) < d - k$ if and only if $e_i(I) = e_i(J)$, for $i = 0, \ldots, k$, that is, if and only if $I \subseteq J_{\{k\}}$.

In order to control the degree of $P_{I/J}(n)$ in general, Herzog, Puthenpurakal and Verma, proved in [**3**], that coefficient ideals always exist. Precisely, they proved that if s is the analytic spread of I, then for $1 \leqslant k \leqslant s$, there exist a unique largest ideal $I_{[k]}$ with $\ell(I_{[k]}/I) < \infty$, and $I \subseteq I_{[s]} \subseteq \cdots \subseteq I_{[1]} \subseteq q(I)$, such that $\deg\left(P_{I_{[k]}/I}(n)\right) < s - k$, where $q(I)$ is the relative integral closure of I, defined by $q(I) := \overline{I} \cap I^{sat}$, where $I^{sat} = \bigcup_{n \geqslant 1} (I : \mathfrak{m}^n)$ is the saturation of I.

Computation of coefficient ideals in general seems to be a difficult task at present and in order to have some control of it we extend the result of Shah (*loc.cit.*) proving a fine structure

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theorem for the coefficient ideals of an arbitrary ideal I (see Theorem 3·4). This is the main result of this work.

2. *Some properties of reduction of ideals*

In this section we will prove some properties of reductions of ideals needed for proving the main result of this work.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal of R. The Rees algebra $\mathcal{R}(I)$ of I is defined by $\mathcal{R}(I) := \oplus_{n\in\mathbb{N}} I^n$. The fiber cone $\mathcal{F}(I)$ of I is defined by $\mathcal{F}(I) := \oplus_{n\in\mathbb{N}} I^n/\mathfrak{m}I^n$.

Notice that $\mathcal{F}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. The analytic spread $s(I)$ of I, is the Krull dimension of this fiber cone, i.e., $s(I) := \dim(\mathcal{F}(I))$. It is well known that, if $R/\mathfrak{m}$ is infinit residue field, $s(I)$ is the minimal number of generators of a minimal reduction of I (see for example [**4**, proposition 8·3·7]).

The next result gives a characterisation for a sequence to be a minimal reduction of powers of an ideal.

LEMMA 2·1. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal of R. Let $a_1, \ldots, a_s \in I^{n_0}$ for some fixed integer $n_0 \geqslant 1$, and some $s \geqslant 1$. Denote by $\overline{a}_1, \ldots, \overline{a}_s$ their image in $I^{n_0}/\mathfrak{m}I^{n_0}$. Then,*

$$\ell\left(\frac{\mathcal{F}(I)}{(\overline{a}_1, \ldots, \overline{a}_s)}\right) < \infty \text{ if and only if } (a_1, \ldots, a_s) \text{ is a reduction of } I^{n_0}.$$

Proof. Suppose that $\ell\left(\mathcal{F}(I)/\left(\overline{a}_1, \ldots, \overline{a}_s\right)\right) < \infty$. Then, for some large n,

$$(\overline{a}_1, \ldots, \overline{a}_s)\left(\frac{I^{n_0 n}}{\mathfrak{m}I^{n_0 n}}\right) = \frac{I^{n_0+n_0 n}}{\mathfrak{m}I^{n_0+n_0 n}}.$$

Hence,

$$(a_1, \ldots, a_s)I^{n_0 n} + \mathfrak{m}I^{n_0+n_0 n} = I^{n_0+n_0 n}.$$

Thus, by Nakayama's lemma $(a_1, \ldots, a_s)I^{n_0 n} = I^{n_0+n_0 n}$. Therefore, $(a_1, \ldots, a_s)$ is a reduction of I^{n_0}.

Conversely, suppose that $a_1, \ldots, a_s$ form a reduction of I^{n_0} then, for all large n

$$(a_1, \ldots, a_s)I^{n_0 n} = I^{n_0+n_0 n}.$$

Hence,

$$(\overline{a}_1, \ldots, \overline{a}_s)\left(\frac{I^{n_0 n}}{\mathfrak{m}I^{n_0 n}}\right) = \frac{I^{n_0+n_0 n}}{\mathfrak{m}I^{n_0+n_0 n}}.$$

Then $\ell\left(\mathcal{F}(I)/\left(\overline{a}_1, \ldots, \overline{a}_s\right)\right) < \infty$.

We prove next, for lack of references, some basic results on Rees valuations of ideals which we could not find in the classical literature. Some general references on this subject are [**5**] and [**9**]. These results were proved by Shah in [**10**] for $\mathfrak{m}$-primary ideals and were extended for arbitrary ideals by Callejas-Bedregal, Pérez and Silva in [**2**, theorems 2·5, 2·6]. We include their proof here for completeness.

LEMMA 2·2. *Let R be an universally catenary and analytically unramified local domain and I an ideal of R with $s := s(I)$. Consider $x_1, \ldots, x_s$ a minimal reduction of I. Then $v(x_i) = v(I)$, for all Rees valuation v of I, with $i = 1, \ldots, s$.*

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Proof: Let T be the extended Rees algebra of I, i.e.,

$$T = \bigoplus_{n\in\mathbb{Z}} I^n t^n,$$

where $I^n = R$ for all $n \leqslant 0$ and

$$\overline{T} = \bigoplus_{n\in\mathbb{Z}} \overline{I^n} t^n.$$

Let $u = t^{-1}$ then $G_I(R) = T/uT$ where $G_I(R) = \oplus_{n\in\mathbb{N}} I^n/I^{n+1}$.

The hypothesis about the elements $x_1, \ldots, x_s$ implies that

$$\text{height}\left((x_1t, \ldots, x_st, u)T\right) = s + 1.$$

We claim that

$$\text{height}\left((x_1t, \ldots, x_st, u)\overline{T}\right) = s + 1.$$

To see this, let $\mathfrak{q}$ be a minimal prime ideal over $(x_1t, \ldots, x_st, u)\overline{T}$. As $\overline{T}$ is finitely generated over the universally catenary domain T, we can apply the dimension formula

$$\text{height}(\mathfrak{q}) = \text{height}(\mathfrak{q} \cap T) + \text{tr.deg.}_T(\overline{T}) - \text{tr.deg.}_{\frac{T}{\mathfrak{q}\cap T}}\left(\frac{\overline{T}}{\mathfrak{q}}\right).$$

However we have $\text{tr.deg}_T\left(\overline{T}\right) = 0$, since T and $\overline{T}$ have the same field of fraction and $\text{tr.deg.}_{(T/\mathfrak{q}\cap T)}\left(\frac{\overline{T}}{\mathfrak{q}}\right) = 0$ since $\overline{T}/\mathfrak{q}$ is integral over $T/\mathfrak{q} \cap T$. Hence

$$\text{height}(\mathfrak{q}) = \text{heitht}\left(\mathfrak{q} \cap T\right).$$

Furthermore, since $\eta := \mathfrak{q} \cap T$ is a minimal prime ideal of $(x_1t, \ldots, x_st, u)\, T$, we have that

$$\text{height}\,(\mathfrak{q}) = \dim\left(T_\eta\right) = s + 1$$

as we claimed.

Notice that $\text{height}(x_it, u) = 2$. To prove this, we observe that $x_1t, \ldots, x_st, u$ are part of a system of parameters of the catenary local domain $(\overline{T})_\eta$. Hence, $\text{height}((x_it, u)(\overline{T})_\eta) = 2$. But $(x_it, u)(\overline{T})_\eta$ is an homogenous ideal then we have $\text{height}((x_it, u)\overline{T}) = 2$, as claimed.

Observe that, for $i = 1, \ldots, s$, we have $v(x_i) = v(u)$. In fact, to see this it is sufficient to show that x_it does not belong to any height 1 prime ideal $\mathfrak{p}$ of $\overline{T}$ which contains u. But this follows from the fact that $\text{height}(x_it, u)\overline{T} = 2$. In this way, $v(x_it) = 0$ which implies that $v(x_i) = v(u)$ as claimed.

We lastly claim that $v(u) = v(I)$. Suppose that $\mathfrak{p}$ is a height 1 prime ideal of $\overline{T}$ which contains both u and $(It)\overline{T}$. Then $\mathfrak{p} \supset (x_1t, \ldots, x_st, u)\overline{T}$. So

$$s + 1 = \text{height}(x_1t, \ldots, x_st, u)\overline{T} \leqslant \text{height}(\mathfrak{p}) = 1.$$

Thus $s = 0$, which is a contradiction. Therefore, $v(It) = 0$, given $v(I) = v(u)$.

LEMMA 2·3. *Let R be a quasi-unmixed local ring and $s := s(I)$ the analytic spread of the ideal I. Consider $x_1, \ldots, x_s$ a minimal reduction of I^{n_0}, for some $n_0 \geqslant 1$. Suppose that $yx_{i_0} \in I^{n_0+1}$, for some $y \in R$ and some i_0 with $1 \leqslant i_0 \leqslant s$. Then $y \in \overline{I}$.*

Proof. We may assume that R is complete. In fact, the image $\hat{x}_1, \ldots, \hat{x}_s$ of the elements $x_1, \ldots, x_s$ in the completion $\hat{R}$ form a minimal reduction of $\hat{I}^{n_0}$. Suppose the result is true

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for the completion $\hat{R}$. Then

$$y \in \overline{\hat{I}^{n_0}} \bigcap R = \overline{I}$$

(see [**4**, proposition 1·6·2, p. 15]). Thus, we can assume that R is complete.

We may also assume that R is a domain. For if $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are all the minimal prime ideals of R then, $\dim(R/\mathfrak{p}_i) = d$, for $i = 1, \ldots, t$ since R is a quasi-unmixed ring. This implies that the image of $x_1, \ldots, x_s$ in $R/\mathfrak{p}_i$ form a minimal reduction of $I + \mathfrak{p}_i/\mathfrak{p}_i$. Suppose that the result is true for $R/\mathfrak{p}_i$, for $i = 1, \ldots, t$. Then the image of y in $R/\mathfrak{p}_i$ belongs to $\overline{(I + \mathfrak{p}_i)/\mathfrak{p}_i}$, for $i = 1, \ldots, t$. Then y belongs to $\overline{I}$.

So, we assume that R is a complete domain. Observe that R is an universally catenary, analytically unramified local domain. Then

$$v(I^{n_0}) = v(x_i),$$

for all Rees valuation of I^{n_0}, by Lemma 2·2. We have $yx_i \in I^{n_0+1}$, which gives

$$v(y) + v(x_i) \geqslant v(I^{n_0}) + v(I).$$

Thus

$$v(y) \geqslant v(I).$$

Which implies

$$v(y) \geqslant \frac{1}{n_0} v(I^{n_0}). \tag{1}$$

It is well known (see [**5**] or [**9**]) that if $P_1, \ldots, P_r$ are all the associated primes of $u\overline{T}$ in $\overline{T}$, then they naturally induce discrete valuations $v_1, \ldots, v_r$ on the fraction field of $\overline{T}$, where as before $\overline{T} = \bigoplus_{n\in\mathbb{Z}} \overline{I^n} t^n$ and $u = t^{-1}$.

Now set $v_I(x) = n$ if $x \in I^n - I^{n+1}$, $v_I(0) = \infty$ and $\overline{v}_I(x) = \lim_{n\to\infty} v_I(x^n)/n$. It is also well known (*loc.cit.*) that:

$$\overline{v}_I(x) \geqslant k \quad \text{if, and only if,} \quad x \in \overline{I^k}; \tag{2}$$

$$\overline{v}_I(x) = \min\{v_j(x)/v_j(I) |\ j = 1, \ldots, r\} \tag{3}$$

and

$$\overline{v}_I(x) = n\overline{v}_{I^n}(x), \text{ for all } n \geqslant 1. \tag{4}$$

Hence, by equations (1) and (3) we have $\overline{v}_{I^{n_0}}(y) \geqslant 1/n_0$. So, by equation (4) we have $\overline{v}_{I^{n_0}}(y) \geqslant 1$, which by equation (2) implies that $y \in \overline{I}$.

3. *The structure of the coefficient ideals*

In this section we prove the main result of this work which gives an algebraic description for the coefficient ideals in terms of colon ideals.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and I an ideal of R. The saturation I^{sat} of I is the ideal defined by

$$I^{sat} = \bigcup_{n\in\mathbb{N}} (I : \mathfrak{m}^n) = \{x \in R \mid x\mathfrak{m}^n \subseteq I, \text{ for some } n \in \mathbb{N}\}$$

and the relative integral closure $q(I)$ of I is defined by $q(I) = \overline{I} \bigcap I^{sat}$.

The saturation and the relative integral closure are ideals which contain I.

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Remark 3·1. It is proved in [**3**, Proposition 4.3] that the relative integral closure $q(I)$ of I satisfies the following properties:

(i) I is a reduction of $q(I)$ and $\ell\,(q(I)/I) < \infty$;
(ii) if I is a reduction of K with $\ell\,(K/I) < \infty$ then $K \subseteq q(I)$;
(iii) if $J \subseteq I$ then $q(J) \subseteq q(I)$;
(iv) $q(q(I)) = q(I)$.

The coefficient ideals were introduced by Shah in [**10**] for $\mathfrak{m}$-primary ideals and recently Herzog, Puthenpurakal and Verma in [**3**, theorem 4·4], proved the existence of the coefficient ideals in the arbitrary case. In fact, in case $(R, \mathfrak{m})$ is a quasi-unmixed local ring with infinite residue field $R/\mathfrak{m}$ and I is an ideal of R with analytic spread s, they proved that for each $k = 1, \ldots, s$, there exists a unique largest ideals $I_{[k]}$ with

$$I \subseteq I_{[s]} \subseteq I_{[s-1]} \subseteq \cdots \subseteq I_{[1]} \subseteq q(I)$$

and

(a) $\ell\left(I_{[k]}/I\right) < \infty$, for $k = 1, \ldots, s$;
(b) $\deg\left(P_{I_{[k]}/I}(n)\right) < s - k$.

The next result relates the coefficient ideal $I_{[s]}$ of I with its Ratliff–Rush closure $\tilde{I}$, extending to arbitrary ideals a similar result proved by Shah in [**10**, Corollary 1(E)] for $\mathfrak{m}$-primary ideals.

PROPOSITION 3·2. *Let $(R, \mathfrak{m})$ be a quasi-unmixed local ring with infinite residue field $R/\mathfrak{m}$ and I be a regular ideal of R with analytic spread s. Then*

$$I_{[s]} = \tilde{I} \cap I^{sat}.$$

Proof. Set $J = \tilde{I} \cap I^{sat}$. Since $I \subseteq J \subseteq \tilde{I}$ and $(\tilde{I})^n = I^n$ for all large n, we have that $J^n = I^n$ for all large n. Clearly $J \subseteq q(I)$ and, since $\ell(J^n/I^n) = 0$ for all large n, we have that $J \subseteq I_{[s]}$.

Conversely, since $\deg\left(P_{I_{[s]}/I}(n)\right) < 0$ we have that $\ell\left(I^n_{[s]}/I^n\right) = 0$ for all large n. Thus, $I_{[s]} \subseteq \tilde{I}$ and therefore $I_{[s]} \subseteq \tilde{I} \cap I^{sat}$.

THEOREM 3·3 (Structure theorem for coefficient ideals). *Let $(R, \mathfrak{m})$ be a d-dimensional quasi-unmixed local ring with infinite residue field. Let I be any ideal of R with analytic spread s. Then*

$$I_{[k]} = \bigcup_{n \geqslant 0} \left(I^{n+1} : x_1, \ldots, x_k\right) \cap I^{sat}$$

for all set of elements $\{x_1, \ldots, x_k\}$ extendable to a minimal reduction of I^n.

Proof. Fix $1 \leqslant k \leqslant d$ and let $y \in I_{[k]}$. Then, $I \subseteq (I, y) \subseteq I_{[k]}$, and hence

$$\ell\left(\frac{(I, y)^n}{I^n}\right) \leqslant P_{I_{[k]}/I}(n),$$

where, by definition of coefficient ideal we have $\deg\left(P_{(I_{[k]}/I}(n)\right) < s - k$.

In particular, since $(I, y)I^{n-1} \subseteq (I, y)^n$,

$$\ell\left(\frac{(I, y)I^{n-1}}{I^n}\right) \leqslant \ell\left(\frac{(I, y)^n}{I^n}\right) \leqslant P_{I_{[k]}/I}(n). \tag{5}$$

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Set

$$L = \bigoplus_{n \geqslant 0} L_n \quad \text{where} \quad L_n := \frac{(I, y)I^n}{I^{n+1}}.$$

Clearly L has the structure of an $\mathcal{R}(I)$-module.

By equation (5) we have that $\dim(L) \leqslant s - k$, that is,

$$\dim(L) = \dim\left(\frac{\mathcal{R}(I)}{(0 :_{\mathcal{R}(I)} L)}\right) \leqslant s - k. \tag{6}$$

We can easily prove that

$$(0 :_{\mathcal{R}(I)} L) = \bigoplus_{n \geqslant 0} (I^{n+1} : y) \cap I^n.$$

Consider the natural projection π from $\mathcal{R}(I)$ to $\mathcal{F}(I)$. Then

$$\pi\left(\oplus_{n \geqslant 1}(I^{n+1} : y) \cap I^n\right) = \bigoplus_{n \geqslant 1} \frac{(I^{n+1} : y) \cap I^n + \mathfrak{m}I^n}{\mathfrak{m}I^n} =: H_k. \tag{7}$$

Then we have a surjective R-homomorphism

$$\frac{\mathcal{R}(I)}{(0 : L)} \longrightarrow \frac{\mathcal{F}(I)}{H_k}$$

and hence, by equation (6),

$$\dim\left(\frac{\mathcal{F}(I)}{H_k}\right) \leqslant s - k. \tag{8}$$

But by assumption we have that $\dim(\mathcal{F}(I)) = s$. Hence, by equation (8) and by [**10**, lemma 2(E)], we can obtain elements $\overline{x}_1, \ldots, \overline{x}_s$ in the fiber cone $\mathcal{F}(I)$ such that:

(1) $\overline{x}_1, \ldots, \overline{x}_s$ are homogenous of the same degree $n_0 \geqslant 1$;

(2) $\ell(\frac{\mathcal{F}(I)}{(\overline{x}_1, \ldots, \overline{x}_s)}) < \infty$;

(3) We have $\dim(\frac{\mathcal{F}(I)}{(\overline{x}_1, \ldots, \overline{x}_k)}) = s - k$;

(4) $\overline{x}_1, \ldots, \overline{x}_k$ are in H_k.

Let $x_1, \ldots, x_s$ be any preimages of $\overline{x}_1, \ldots, \overline{x}_s$ in I^{n_0}. By Lemma 2·1 and (2) above, the elements $x_1, \ldots, x_s$ form a minimal reduction of I^{n_0}.

Since $\overline{x}_1, \ldots, \overline{x}_k$ are in H_k we obtain by equation (7) that $y \in (I^{n_0+1} : x_1, \ldots, x_k)$. Therefore, $I_{[k]} \subseteq \bigcup(I^{n+1} : (x_1, \ldots, x_k))$ and hence $I_{[k]} \subseteq \bigcup(I^{n+1} : (x_1, \ldots, x_k)) \cap I^{sat}$.

Conversely, take $y \in (I^{n_0} : x_1, \ldots, x_k) \cap I^{sat}$. In particular $yx_i \in I^{n_0}$. Thus, by Lemma 2·3, $y \in \overline{I}$. Therefore I is a reduction of (I, y), i.e., $(I, y)^{s+n} = (I, y)^s I^n$, for some $s \geqslant 0$

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and all $n \geqslant 0$. Then

$$\begin{aligned}\ell\left(\frac{(I,y)^{s+n}}{I^{n+s}}\right) &= \ell\left(\frac{(I,y)^s I^n}{I^{n+s}}\right)\\ &= \sum_{i=1}^{s} \ell\left(\frac{(I,y)^i I^{n+s-i}}{(I,y)^{i-1} I^{n+s-i-1}}\right)\\ &= \sum_{i=1}^{s} \ell\left(\frac{(I,y)^{i-1} I^{s-i}(I,y) I^n}{(I,y)^{i-1} I^{s-i} I^{n+1}}\right)\\ &\leqslant \sum_{i=1}^{s} c_i \ell\left(\frac{(I,y)\, I^n}{I^{n+1}}\right)\end{aligned}$$

for some positive constant c_i. The last inequality can be seen by canonically mapping for each i

$$\bigoplus_{n=1}^{c_i} L_n \longrightarrow \frac{(I,y)^{i-1} I^{s-i}(I,y) I^n}{(I,y)^{i-1} I^{s-i} I^{n+1}} \longrightarrow 0,$$

where $L_n = (I,y)I^n/I^{n+1}$. Choose c_i as the minimal number of generators of $(I,y)^{i-1} I^{s-i}$.

Set $c = \sum c_i$. Thus

$$\ell\left(\frac{(I,y)^{s+n}}{I^{n+s}}\right) \leqslant c \cdot \ell(L_n). \tag{9}$$

Since $y \in I^{sat}$ there exist a positive integer p such that $y\mathfrak{m}^p \subseteq I$. Set $L = \bigoplus_{n\geqslant 0} L_n$ and $S = \mathcal{R}(I)/\mathfrak{m}^p\mathcal{R}(I) = \bigoplus_{n\geqslant 0} I^n/\mathfrak{m}^p I^n$. Then, since $y\mathfrak{m}^p \subseteq I$, we have that $\mathfrak{m}^p\mathcal{R}(I) \subseteq \mathrm{Ann}_{\mathcal{R}(I)}(L)$. Hence L, which is an $\mathcal{R}(I)$-module has a natural structure of an S-graded module. Thus $\ell(L_n)$ is, for large n, a polynomial in n of degree at most $\dim(L) - 1$. But $\dim(L) = \dim(S/\mathrm{Ann}_S(L))$. Notice that

$$\mathrm{Ann}_S(L) = \bigoplus_{n\geqslant 1} \frac{(I^{n+1} : y) \bigcap I^n}{\mathfrak{m}^p I^n} =: A_k.$$

Since $y \in (I^{n_0} : x_1, \ldots, x_k)$ it follows that $x_1', \ldots, x_k' \in A_k$, where x_i' is the image of x_i in S. Thus, since $\dim S = \dim \mathcal{F}(I) = s$,

$$\dim(S/\mathrm{Ann}_S(L)) \leqslant \dim(S/(x_1', \ldots, x_k')) = s - k.$$

Therefore, by (9), $\ell\left((I,y)^{s+n}/I^{s+n}\right)$ is bounded by a polynomial in n of degree $s-(k+1)$, for all large n. Thus

$$\deg\left(P_{\frac{(I,y)}{I}}(n)\right) < s - k,$$

which implies that $I \subset (I,y) \subset I_{[k]}$.

Therefore $y \in I_{[k]}$.

THEOREM 3·4 (Finer structure theorem for coefficient ideals). *Let $(R,\mathfrak{m})$ be a d-dimensional quasi-unmixed local ring with infinite residue field. Let I be any ideal of R with analytic spread s. Then, there exists $n_0 \geqslant 0$ such that for all $1 \leqslant k \leqslant s$,*

$$I_{[k]} = \left(I^{n_0+1} : (x_1, \ldots, x_k)\right) \bigcap I^{sat}$$

for some fixed minimal reduction $x_1, \ldots, x_s$ of I^{n_0}.

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Proof: We know, by definition of $I_{[k]}$, that for all large n

$$\ell\left(\frac{I^n_{[k]}}{I^n}\right) = P_{\frac{I_{[k]}}{I}}(n) \text{ and that } \deg\left(P_{\frac{I_{[k]}}{I}}(n)\right) < s-k.$$

In particular, since, $I \subseteq I_{[k]}$, we also have

$$\ell\left(\frac{I_{[k]}I^{n-1}}{I^n}\right) \leqslant P_{\frac{I_{[k]}}{I}}(n). \tag{10}$$

Set $L^{(k)} := \bigoplus_{n\geqslant 0}\frac{I_{[k]}I^n}{I^{n+1}}$. Clearly $L^{(k)}$ has the structure of an $\mathcal{R}(I)$-module and, by (10), we have

$$\dim\left(L^{(k)}\right) = \dim\left(\frac{\mathcal{R}(I)}{\left(0 :_{\mathcal{R}(I)} L^{(k)}\right)}\right) \leqslant s-k, \tag{11}$$

We can easily prove that

$$(0 :_{\mathcal{R}(I)} L^{(k)}) = \bigoplus_{n\geqslant 0}(I^{n+1} : I_{[k]}) \bigcap I^n.$$

Consider the natural projection π from $\mathcal{R}(I)$ to $\mathcal{F}(I)$. Then

$$\pi\left(\oplus_{n\geqslant 0}(I^{n+1} : I_{[k]}) \bigcap I^n\right) = \bigoplus_{n\geqslant 1}\frac{(I^{n+1} : I_{[k]}) \bigcap I^n + \mathfrak{m}I^n}{\mathfrak{m}I^n} =: H_k.$$

Then we have a surjective R-homomorphism

$$\frac{\mathcal{R}(I)}{(0 :_{\mathcal{R}(I)} L^{(k)})} \longrightarrow \frac{\mathcal{F}(I)}{H_k}$$

and hence, by equation (11), $\dim\left(\mathcal{F}(I)/H_k\right) \leqslant s-k$. This holds for all $k = 1, \ldots, s$. Further $H_1 \subseteq \cdots \subseteq H_s$ and each H_k is a homogeneous ideal of $\mathcal{F}(I)$. Thus, by [**10**, lemma 2(F)], we can choose homogeneous elements $\overline{x}_1, \ldots, \overline{x}_s$ in $\mathcal{F}(I)$ of some fixed degree $n_0 \geqslant 1$ such that

$$\ell\left(\frac{\mathcal{F}(I)}{(\overline{x}_1, \ldots, \overline{x}_s)}\right) < \infty$$

and $\overline{x}_1, \ldots, \overline{x}_k$ are in H_k, for all $1 \leqslant k \leqslant s$. Let $x_1, \ldots, x_s$ be any preimages of $\overline{x}_1, \ldots, \overline{x}_s$ in I^{n_0}. Then, by Lemma 2·1, $x_1, \ldots, x_s$ form a minimal reduction of I^{n_0}. Now, since $\overline{x}_1, \ldots, \overline{x}_k$ are in H_k we have that $I_{[k]} \subseteq \left(I^{n_0+1} : x_1, \ldots, x_k\right)$. Therefore, $I_{[k]} \subseteq \left(I^{n_0+1} : x_1, \ldots, x_k\right) \bigcap I^{sat}$, for $1 \leqslant k \leqslant s$.

The inclusion $\left(I^{n_0+1} : x_1, \ldots, x_k\right) \bigcap I^{sat} \subseteq I_{[k]}$ follows straight from Theorem 3·3.

REFERENCES

[**1**] J. O. Amao. On a certain Hilbert polynomial. *J. London Math. Soc.* (2) **14**, 1 (1976), 13–20.

[**2**] R. Callejas–Bedregal, V. H. Jorge Pérez and M. D. Silva. On coefficient modules and Ratliff–Rush closure. Preprint.

[**3**] J. Herzog, T. Puthenpurakal and J. K. Verma. Hilbert polynomials and powers of ideals. *Math. Proc. Camb. Phil. Soc.* **145** (2008), 623–642.

[**4**] C. Huneke and I. Swanson. Integral closure of ideals, rings and modules. London Math. Soc. Lecture Note Series (Cambridge University Press, 2006).

[**5**] S. McAdam. Asymptotic prime divisors. Lecture Notes in Math. 1023 (Springer-Verlag, Berlin, 1983).

[**6**] L. J. Ratliff and D. Rush. Two notes on reductions of ideals. *Indiana University Math J.* **27** (1978), 929–934.

https://doi.org/10.1017/S0305004118000324 Published online by Cambridge University Press

[**7**] D. REES. $\mathcal{A}$-transforms of local rings and a theorem on multiplicities of ideals. *Cambridge Philos. Soc*. **57** (1961), 8–17.
[**8**] D. REES. Amao's theorem and reduction criteria. *J. London Math. Soc*. (2) **32** (1985), no. 3, 404–410.
[**9**] D. REES. Lectures on the asymptotic theory of ideals. London Math. Soc. Lecture Note Series, **113** (Cambridge University Press, Cambridge, 1988).
[**10**] K. SHAH. K. Coefficient ideals *Trans. Math. Soc.* **327** (1991), 373–384.
[**11**] P. SAMUEL and O. ZARISKI. *Commutative Algebra*. Vol. II (Springer-Verlag, Berlin, 1975).

https://doi.org/10.1017/S0305004118000324 Published online by Cambridge University Press