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Sharp gradient estimate, rigidity and almost rigidity of Green functions on non-parabolic RCD(0, N) spaces

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Inspired by a result in T. H. Colding. (**16**). *Acta. Math.* **209**(2) (2012), 229-263 [**16**] of Colding, the present paper studies the Green function G on a non-parabolic RCD(0, N) space $(X, \mathsf{d}, \mathfrak{m})$ for some finite $N > 2$. Defining $\mathsf{b}_x = G(x, \cdot)^{\frac{1}{2-N}}$ for a point $x \in X$, which plays a role of a smoothed distance function from x, we prove that the gradient $|\nabla \mathsf{b}_x|$ has the canonical pointwise representative with the sharp upper bound in terms of the N-volume density $\nu_x = \lim_{r\to 0+} \frac{\mathfrak{m}(B_r(x))}{r^N}$ of $\mathfrak{m}$ at x;

$$|\nabla \mathsf{b}_x|(y) \leqslant (N(N-2)\nu_x)^{\frac{1}{N-2}}, \quad \text{for any } y \in X \setminus \{x\}.$$

Moreover the rigidity is obtained, namely, the upper bound is attained at a point $y \in X \setminus \{x\}$ if and only if the space is isomorphic to the N-metric measure cone over an RCD($N-2$, $N-1$) space. In the case when x is an N-regular point, the rigidity states an isomorphism to the N-dimensional Euclidean space $\mathbb{R}^N$, thus, this extends the result of Colding to RCD(0, N) spaces. It is emphasized that the almost rigidities are also proved, which are new even in the smooth framework.

Keywords: Ricci curvature; metric measure space; Green functions; rigidity; Gromov–Hausdorff convergence; Optimal transportation theory

1. Introduction

1.1. Green function and concerned problems in the smooth framework

In the classical PDE theory, the (positive) Green function G_x at the pole x of the Laplace operator on the N-dimensional Euclidean space $\mathbb{R}^N$ is the solution to the heat equation

$$\Delta u = -\delta_x \tag{1.1}$$

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as measures, where δ_x is the Dirac measure at x. In the case when $N \geqslant 2$, it is well-known that this equation is solved by

$$G_x = \begin{cases} -\dfrac{1}{2\pi} \log \mathsf{d}_x, & N = 2 \\ \dfrac{\mathsf{d}_x^{2-N}}{N(N-2)\omega_N}, & N \geqslant 3 \end{cases} \tag{1.2}$$

where $\mathsf{d}_x(\cdot)$ is the Euclidean distance function from x and $\omega_N := \pi^{\frac{N}{2}} \Gamma(\frac{N}{2} + 1)^{-1}$ is the volume of a unit ball $B_1(0_N)$ in $\mathbb{R}^N$.

We can also discuss the Green functions for more general classes of spaces along the same line. For instance, it is known that for an N-dimensional complete Riemannian manifold (M^N, g) $(N \geqslant 2)$ with non-negative Ricci curvature, the existence of the (global) Green function G is equivalent to the following non-parabolic assumption:

$$\int_1^\infty \frac{r}{\operatorname{Vol} B_r(x)} \, \mathrm{d}r < \infty, \quad \forall\, x \in M^N, \tag{1.3}$$

where $B_r(x)$ denotes the open ball centred at x of radius r with respect to the induced distance d by g, and Vol denotes the Riemannian volume measure by g. See [**56**] by Varopoulos for the details. In this case, it is well-known that the following asymptotic behaviour for the Green function G_x at the pole $x \in M^N$ holds as $\mathsf{d}_x \to 0^+$:

$$G_x = \begin{cases} -\dfrac{1}{2\pi} \log \mathsf{d}_x + o(-\log \mathsf{d}_x), & N = 2 \\ \dfrac{\mathsf{d}_x^{2-N}}{N(N-2)\omega_N} + o(\mathsf{d}_x^{2-N}), & N \geqslant 3 \end{cases} \tag{1.4}$$

This fact indicates that in the case when $N \geqslant 3$, the function

$$\mathsf{b}_x := G_x^{\frac{1}{2-N}} \tag{1.5}$$

should be a counterpart of the distance function from x up to a multiplication of a dimensional positive constant;

$$(N(N-2)\omega_N)^{\frac{1}{N-2}} \mathsf{d}_x. \tag{1.6}$$

Colding [**16**] proved the sharp gradient estimate for b_x and the rigidity as follows.

THEOREM 1.1 Theorem 3.1 of [**16**]. *Let (M^N, g) be an N-dimensional $(N \geqslant 3)$, complete and non-parabolic Riemannian manifold with non-negative Ricci curvature and let $x \in M^N$. Then we have the following.*

(1) *(Sharp gradient estimate) We have*

$$|\nabla \mathsf{b}_x|(z) \leqslant (N(N-2)\omega_N)^{\frac{1}{N-2}} \tag{1.7}$$

for any $z \in M^N \setminus \{x\}$.

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(2) *(Rigidity)* (M^N, g) *is isometric to the* N*-dimensional Euclidean space* $\mathbb{R}^N$ *with* $\mathsf{b}_x = (N(N-2)\omega_N)^{\frac{1}{N-2}}\mathsf{d}_x$ *if the equality of (1.7) holds for some* $z \in M^N \setminus \{x\}$.

Note that Colding used the normalized one, $(N(N-2)\omega_N)^{\frac{1}{2-N}}G_x^{\frac{1}{2-N}}$, as the definition of b_x. Thus the sharp upper bound in [**16**] was exactly 1 instead of the right-hand-side of (1.7). See the footnote 4 in [**16**].

In particular the rigidity indicates that b_x exactly coincides with $(N(N-2)\omega_N)^{\frac{1}{N-2}}\mathsf{d}_x$ if and only if the manifold is Euclidean. Given this rigidity result, it is natural to ask whether the quantitative almost rigidity result is satisfied or not:

(A) If $|\nabla \mathsf{b}_x|(y)$ is close to the sharp upper bound $(N(N-2)\omega_N)^{\frac{1}{N-2}}$ at some point $y \in M^N \setminus \{x\}$, then can we conclude that the manifold is pointed Gromov–Hausdorff (pGH) close to $\mathbb{R}^N$?

It is worth mentioning that

$$|\nabla \mathsf{b}_x|(y) \to (N(N-2)\omega_N)^{\frac{1}{N-2}} \tag{1.8}$$

whenever $y \to x$. Therefore in order to give a positive answer to the question (Q), we need to find an additional assumption on y.

We are now in a position to introduce the first main result of the paper.

THEOREM 1.2 Almost rigidity. *For any integer* $N \geqslant 3$, *all* $0 < \varepsilon < 1$, $0 < r < R$, $1 \leqslant p < \infty$ *and* $\varphi \in L^1([0,\infty), \mathscr{H}^1)$ *there exists* $\delta := \delta(N, \varepsilon, r, R, p, \varphi) > 0$ *such that if an* N*-dimensional* ($N \geqslant 3$) *complete Riemannian manifold with non-negative Ricci curvature* (M^N, g) *satisfies*

$$\frac{s}{\operatorname{Vol} B_s(x)} \leqslant \varphi(s), \quad \text{for } \mathscr{H}^1\text{-a.e. } s \in [1,\infty) \tag{1.9}$$

for some $x \in M^N$ *and that*

$$(N(N-2)\omega_N)^{\frac{1}{N-2}} - |\nabla \mathsf{b}_x|(y) \leqslant \delta \tag{1.10}$$

holds for some $y \in \overline{B}_R(x) \setminus B_r(x)$. *Then we have*

$$\mathsf{d}_{\mathrm{pmGH}}\left((M^N, \mathsf{d}, \mathrm{Vol}, x), (\mathbb{R}^N, \mathsf{d}_{\mathbb{R}^N}, \mathscr{H}^N, 0_N)\right) < \varepsilon \tag{1.11}$$

and

$$\begin{aligned} &\left\| \mathsf{b}_x - (N(N-2)\omega_N)^{\frac{1}{N-2}}\mathsf{d}_x \right\|_{L^\infty(B_R(x))} \\ &\quad + \left\| \mathsf{b}_x - (N(N-2)\omega_N)^{\frac{1}{N-2}}\mathsf{d}_x \right\|_{H^{1,p}(B_R(x),\mathsf{d},\mathrm{Vol}\,/\,\mathrm{Vol}\,B_R(x))} \leqslant \varepsilon \end{aligned} \tag{1.12}$$

in particular

$$\|(N(N-2)\omega_N)^{\frac{1}{N-2}} - |\nabla \mathsf{b}_x|\|_{L^p(B_R(x),\mathrm{Vol}\,/\,\mathrm{Vol}\,B_R(x))} \leqslant \varepsilon, \tag{1.13}$$

where $\mathsf{d}_{\mathrm{pmGH}}$ *denotes any fixed distance metrizing the pointed measured Gromov–Hausdorff (pmGH) convergence.*

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As explained around (1.8), the lower bound r in theorem 1.10 cannot be dropped in order to get (1.11). On the other hand, it is known that if the asymptotic N-volume V_{M^N} defined by

$$V_{M^N} := \lim_{R\to\infty} \frac{\mathrm{Vol}\, B_R(x)}{R^N} (\leqslant \omega_N) \tag{1.14}$$

is close to ω_N, then $(M^N, \mathsf{d}, \mathrm{Vol}, x)$ is pmGH close to $(\mathbb{R}^N, \mathsf{d}_{\mathbb{R}^N}, \mathscr{H}^N, 0_N)$, quantitatively. See [**15**] by Colding. Note that the converse statement is not true even in the case when the metric is Ricci flat with the maximal volume growth (namely $V_{M^N} > 0$).

In connection with this observation, it is natural to ask whether the conclusion (1.11) in the theorem above can be improved to be that V_{M^N} is close to ω_N, or not. However a simple blow-up argument on a fixed manifold which is not isometric to $\mathbb{R}^N$ allows us to conclude that the desired improvement is impossible, see also remark 4.5.

As another possible improvement in the theorem above, it is also natural to ask whether the case when $p = \infty$ in (1.13) is satisfied or not, namely

$$(N(N-2)\omega_N)^{\frac{1}{N-2}} - |\nabla \mathsf{b}_x| \leqslant \varepsilon, \quad \text{on } B_R(x)? \tag{1.15}$$

However we can also see that this improvement is impossible (thus the improvement of (1.12) to the case when $p = \infty$ is also impossible) via Gromov–Hausdorff limits. See subsection 5.1.

The observation above allows us to say that theorem 1.2 is sharp. Finally let us introduce an immediate corollary.

Corollary 1.3. *For any integer $N \geqslant 3$, all $0 < \varepsilon < 1$ and $v > 0$ there exists $\delta := \delta(N, \varepsilon, v) > 0$ such that if an N-dimensional ($N \geqslant 3$) complete Riemannian manifold with non-negative Ricci curvature (M^N, g) satisfies $V_{M^N} \geqslant v$ and (1.10) for some sequence $y_i \in M^N (i = 1, 2, \ldots)$ with $\mathsf{d}(x, y_i) \to \infty$, then*

$$|V_{M^N} - \omega_N| \leqslant \varepsilon \tag{1.16}$$

In particular, in addition, if ε is sufficiently small depending only on N and v, then M^N is diffeomorphic to $\mathbb{R}^N$.

Note that the existence of such sequence y_i in the corollary above cannot be replaced by the existence of only one point which is far from x. See remark 4.7.

The results above are justified via a non-smooth geometric analysis with Ricci curvature bounded below. Moreover the results above are generalized to such a non-smooth framework, so-called RCD spaces. In the next section let us provide a brief introduction on RCD spaces.

1.2. Non-smooth space with Ricci curvature bounded below; RCD spaces

In the first decade of this century, Lott–Villani [**46**] and Sturm [**53**, **54**] introduced the notion of $\mathrm{CD}(K, N)$ spaces independently as a concept of metric measure

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spaces with Ricci curvature bounded below by $K \in \mathbb{R}$ and dimension bounded above by $N \in [1, \infty]$ in some synthetic sense via the optimal transportation theory. For instance, in the case when N is an integer, $\mathbb{R}^N$ with any norm and the Lebesgue measure $\mathscr{L}^N$ satisfies the CD$(0, N)$ condition. Note that this is not 'Riemannian' whenever the norm does not come from an inner product and that Gigli found a 'Riemannian' notion on general metric measure spaces, so-called infinitesimally Hilbertianity, in [**32**], which allows us to meet the Dirichlet form theory from the metric measure geometry. It is worth mentioning that N is not necessarily to be an integer in general.

After a pioneer work of Gigli–Kuwada–Ohta [**27**] on Alexandrov spaces, Ambrosio–Gigli–Savaré (in the case when $N = \infty$) and Gigli (in the case when $N < \infty$) introduced RCD(K, N) spaces (or RCD spaces for short) by adding the infinitesimally Hilbertianity to the CD condition, see [**2**, **33**]. It is known that RCD spaces include weighted Riemannian manifolds with Bakry–Émery Ricci curvature bounded below, Ricci limit spaces, and Alexandrov spaces [**51**, **57**] by Petrunin and Zhang-Zhu. The study is hugely developed, see for instance [**8**, **35**] as nice surveys.

As explained in the previous subsection, we will mainly discuss an RCD$(0, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ for some finite $N > 2$ satisfying the non-parabolic assumption:

$$\int_1^\infty \frac{r}{\mathfrak{m}(B_r(x))}\,\mathrm{d}\mathfrak{m} < \infty, \quad \forall\, x \in X. \tag{1.17}$$

Then, as in the smooth case, the global Green function $G = G^X$ can be defined by the integration of the heat kernel $p(x, y, t)$:

$$G(x, y) := \int_0^\infty p(x, y, t)\,\mathrm{d}t, \tag{1.18}$$

and it is proved in [**12**] by Bruè–Semola that G is well-defined with $G(x, \cdot) \in W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ for any $x \in X$. A typical example of RCD$(0, N)$ spaces is

$$\left([0, \infty), \mathsf{d}_{\mathrm{Euc}}, r^{N-1}\,\mathrm{d}r\right) \tag{1.19}$$

whose Green function G satisfies the following expression from the pole/origin 0;

$$G(0, r) = \frac{1}{N-2} r^{2-N}. \tag{1.20}$$

See proposition 3.8. It is worth mentioning that (1.19) is the N-metric measure cone over a single point (definition 2.7).

1.3. Main results and organization of this paper

In order to introduce main results of this paper, fix an RCD$(0, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ for some finite $N > 2$ satisfying the non-parabolic assumption (1.17). Moreover we also fix a point $x \in X$ whose N-volume density ν_x is finite;

$$\nu_x := \lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{r^N} \in (0, \infty), \tag{1.21}$$

where the positivity is a direct consequence of the Bishop–Gromov inequality.

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REMARK 1.4. The origin of the RCD$(0, N)$ space (1.19) satisfies (1.21), more generally (1.21) is satisfied at the pole of any N-metric measure cone over an RCD$(N-2, N-1)$ space. It is worth mentioning that (1.21) is also satisfied at any point if the space is non-collapsed, namely $\mathfrak{m} = \mathscr{H}^N$ because of the Bishop inequality, where $\mathscr{H}^N$ denotes the N-dimensional Hausdorff measure (see definition 2.4).

Defining b_x by (1.5) in this setting, let us ask whether a similar rigidity result as in theorem 1.1 is justified even in this setting, or not. The main difficulty to realize this consist of two parts;

- a priori, $|\nabla \mathsf{b}_x|$ makes only $\mathfrak{m}$-a.e. sense;
- as observed in (1.20), in general, the constancy of $|\nabla \mathsf{b}_x|$ does not imply an isomorphism to a Euclidean space.

The first main result in this setting are stated as follows, which allow us to overcome the first issue above.

THEOREM 1.5 Pointwise properties on $|\nabla \mathsf{b}_x|$; theorems 3.15 and 3.22. *We have the following.*

(1) *(Canonical representative of $|\nabla \mathsf{b}_x|$) For any $z \in X$, the limit;*

$$\lim_{r\to 0^+} \fint_{B_r(z)} |\nabla \mathsf{b}_x| \mathrm{d}\mathfrak{m} \in [0, \infty) \tag{1.22}$$

exists. Denoting by $|\nabla \mathsf{b}_x|^*(z)$ (or $|\nabla \mathsf{b}_x|(z)$ for short if there is no confusion) the limit, we see that any point is a Lebesgue point of $|\nabla \mathsf{b}_x|$, namely

$$\lim_{r\to 0^+} \fint_{B_r(z)} ||\nabla \mathsf{b}_x| - |\nabla \mathsf{b}_x|^*(z)| \, \mathrm{d}\mathfrak{m} = 0, \quad \forall\, z \in X. \tag{1.23}$$

(2) *(Upper semicontinuity) The function $|\nabla \mathsf{b}_x|^*$ is upper semicontinuous on X.*

(3) *(Sharp pointwise gradient estimate) We have*

$$|\nabla \mathsf{b}_x|^*(z) \leqslant \mathscr{C}_N \nu_x^{\frac{1}{N-2}}, \quad \forall\, z \in X \tag{1.24}$$

and

$$|\nabla \mathsf{b}_x|^*(x) = \mathscr{C}_N \nu_x^{\frac{1}{N-2}}, \tag{1.25}$$

where

$$\mathscr{C}_N := (N(N-2))^{\frac{1}{N-2}}. \tag{1.26}$$

In particular b_x with $\mathsf{b}_x(x) := 0$ is $\mathscr{C}_N \nu_x^{\frac{1}{N-2}}$-Lipschitz on X with the (global) Lipschitz constant $\mathscr{C}_N \nu_x^{\frac{1}{N-2}}$.

Let us introduce the second main result overcoming the second issue above, see definition 2.7 for N-metric measure cones.

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THEOREM 1.6 Rigidity; theorem 4.1. *If*

$$|\nabla \mathsf{b}_x|^*(z) = \mathscr{C}_N \nu_x^{\frac{1}{N-2}} \tag{1.27}$$

for some $z \in X \setminus \{x\}$*, then* $(X, \mathsf{d}, \mathfrak{m})$ *is isomorphic to the* N*-metric measure cone over an* $\mathrm{RCD}(N-2, N-1)$ *space, in particular,* $|\nabla \mathsf{b}_x|^* \equiv \mathscr{C}_N \nu_x^{\frac{1}{N-2}}$.

Combining the results above with the compactness of non-parabolic $\mathrm{RCD}(0, N)$ spaces with respect to the pmGH convergence (theorem 3.11), we obtain the following almost rigidity result

THEOREM 1.7 Almost rigidity; theorem 4.3. *For all* $N > 2$*,* $0 < \varepsilon < 1$*,* $v > 0$*,* $0 < r < R < \infty$ *and* $\varphi \in L^1([0, \infty), \mathscr{H}^1)$ *there exists* $\delta := \delta(N, \varepsilon, r, R, \varphi) > 0$ *such that if a pointed non-parabolic* $\mathrm{RCD}(0, N)$ *space* $(X, \mathsf{d}, \mathfrak{m}, x)$ *satisfies* (1.21)*,*

$$\frac{s}{\mathfrak{m}(B_s(x))} \leqslant \varphi(s), \quad \text{for } \mathscr{H}^1 - a.e.\ s \in [1, \infty) \tag{1.28}$$

and

$$\mathscr{C}_N \nu_x^{\frac{1}{N-2}} - |\nabla \mathsf{b}_x|^*(z) \leqslant \delta \tag{1.29}$$

hold for some $z \in B_R(x) \setminus B_r(x)$*, then* $(X, \mathsf{d}, \mathfrak{m}, x)$ *ε-pmGH close to the* N*-metric measure cone over an* $\mathrm{RCD}(N-2, N-1)$ *space.*

In particular, in theorem 1.7, if we further assume that N is an integer and that the point x admits an N-dimensional Euclidean tangent cone (which is trivial in the manifold case), then the N-metric measure cone stated in theorem 1.7 can be replaced by the N-dimensional Euclidean space, which gives a positive answer to the question (Q) even in the RCD setting.

In the next subsection let us provide the outlines of the proofs of the results above.

1.4. Outline of the proofs and organization of the paper

In order to prove theorem 1.5, we will study a drifted Laplace operator $\mathscr{L}$ defined by

$$\mathscr{L} u := \Delta u + 2\langle \nabla \log G_x, \nabla u \rangle. \tag{1.30}$$

Then we follow arguments by Colding in [**16**] to get the $\mathscr{L}$-subharmonicity of $|\nabla \mathsf{b}_x|^2$ and the (Δ-)subharmonicity of $|\nabla \mathsf{b}_x|^2 G(x, \cdot)$ (proposition 3.21) via the Bochner inequality appearing in the definition of RCD spaces (see (2.8)). Combining their subharmonicities with regularity results on subharmonic functions on PI spaces [**9**] proves (1) and (2) of the theorem. To prove the remaining statements, (3), we recall that the Green function from the pole on the N-metric measure cone over an $\mathrm{RCD}(N-2, N-1)$ space can be explicitly calculated (proposition 3.8) as in (1.20) and that any tangent cone at x whose N-volume density is finite is isomorphic to such a metric measure cone (corollary 2.9) because of a result of De Philippis–Gigli [**19**]. Then, combining them with blow-up arguments at the base point x based

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on the stability of the Laplacian [**3**] by Ambrosio and the first named author, we obtain (3), where the $\mathscr{L}$-subharmonicity of $|\nabla \mathsf{b}_x|^2$ plays a role here again.

In order to prove the rigidity result, theorem 1.6, we use the strong maximum principle for $\mathscr{L}$-subharmonic, upper semicontinuous functions to get the constancy of $|\nabla \mathsf{b}_x|^2$. Then the explicit calculation of $\Delta \mathsf{b}_x$ allows us to apply a rigidity result of Gigli–Violo [**31**] to prove theorem 1.6.

Let us emphasize that under realizing the results above, we are also able to obtain a convergence result of the Green functions with respect to the pmGH convergence, in particular, as a corollary, the $W^{1,p}$-strong convergence of b_x is proved for any finite $p < \infty$ (corollary 3.23). After establishing compactness results on non-parabolic RCD$(0, N)$ spaces with respect to pmGH convergence (theorem 3.11), the $W^{1,p}$-convergence result allows us to show the almost rigidity, theorem 1.7, via a contradiction. Then the main results stated in the smooth framework, theorem 1.2 and corollary 1.3, are corollaries of the results for RCD spaces.

Finally we show the sharpness of theorem 1.2 via observing the 3-metric measure cone, $C(\mathbb{S}^2(r))$ for some $r < 1$ which is close to 1, where $\mathbb{S}^2(r)$ denotes the round sphere of radius r in $\mathbb{R}^3$ centred at the origin. See subsection 5.1.

The organization of the paper is as follows. Section 2 is devoted to fixing the notations/conventions and the introduction on RCD spaces, in particular, about N-metric measure cones. In § 3, we study the Green function on a non-parabolic RCD$(0, N)$ space, where the starting point is a work by Bruè–Semola [**12**]. One of the main purposes in this section is to prove theorem 1.5. Section 4 is devoted to proving the rigidity/almost rigidity results. In § 5, we provide simple examples which show that our results are sharp. In the final section, § 6, we provide proofs of regularity results about $\mathscr{L}$-subharmonic functions directly coming from the general theory of PI spaces [**9**]. This part makes the paper to be more self-contained.

2. Preliminary

2.1. Notation and convention

Let us fix general conventions and geometric/analytic notations:

- We denote by $C(a_1, a_2, \dots, a_k)$ a positive constant only dependent on $a_1, a_2, \dots, a_k$, which may vary from line to line unless otherwise stated.
- For a metric space (X, d), denote by
  - $B_r(x) := \{y \in X \mid \mathsf{d}(x, y) < r\}$ and $\overline{B}_r(x) := \{y \in X \mid \mathsf{d}(x, y) \leqslant r\}$;
  - $\mathrm{Lip}(X, \mathsf{d})$ the collection of all Lipschitz functions on (X, d).
- We say that a triple $(X, \mathsf{d}, \mathfrak{m})$ is a metric measure space if (X, d) is a complete and separable metric space and $\mathfrak{m}$ is a locally finite Borel measure which is fully supported on X.
- Whenever we discuss on a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, we identify two objects which coincide except for a $\mathfrak{m}$-negligible set.

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- For a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, let $\mathcal{A}$ be a collection of functions defined on an open subset U of X. Then we denote by
  - $\mathcal{A}_+ := \{f \in \mathcal{A} \,|\, f \geqslant 0 \ \text{ for } \mathfrak{m}\text{-a.e.}\}$;
  - $\mathcal{A}_{\mathrm{loc}} := \{f : U \to \mathbb{R} \,|\, f\chi_{B_r(x)} \in \mathcal{A} \text{ for any } B_r(x) \text{ with } \overline{B}_r(x) \subset U\}$;
  - $\mathcal{A}_c := \{f \in \mathcal{A} \,|\, \text{The support of } f, \ \operatorname{supp} f, \ \text{is compact and is included in } U\}$.

  For instance, $L^p_{\mathrm{loc}}(U, \mathsf{d}, \mathfrak{m})$, $\mathrm{Lip}_c(U, \mathsf{d})$, $W^{1,p}_{\mathrm{loc}}(U, \mathsf{d}, \mathfrak{m})$, etc., make sense.

2.2. Definition of RCD(K, N) space and heat kernel

Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space. We define the local Lipschitz constant at x of a function f defined on X as follows:

$$\operatorname{lip} f(x) := \limsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)}, \tag{2.1}$$

where $\operatorname{lip} f(x)$ is interpreted as 0 if x is isolated. For any $f \in L^2(X, \mathfrak{m})$, the Cheeger energy of f is defined by

$$\begin{aligned}\mathsf{Ch}(f) := \inf \Bigg\{ \frac{1}{2} \liminf_{i \to \infty} \int_X (\operatorname{lip} f)^2 \, \mathrm{d}\mathfrak{m} \ \Bigg| \ & f_i \\ & \in \mathrm{Lip}(X, \mathsf{d}) \cap (L^\infty \cap L^2)(X, \mathfrak{m}), \|f_i - f\|_{L^2} \to 0 \Bigg\}. \end{aligned} \tag{2.2}$$

The Sobolev space $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$[1] is the collection of $L^2(X, \mathfrak{m})$-functions with finite Cheeger energy, equipped with the $W^{1,2}$-norm

$$\|f\|_{W^{1,2}(X,\mathsf{d},\mathfrak{m})} := \sqrt{\|f\|^2_{L^2} + 2\mathsf{Ch}(f)}. \tag{2.3}$$

For any $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$, by taking a minimizing sequence $\{f_i\}_i$ in the right-hand-side of (2.2), we can find the optimal L^2-function denoted by $|\nabla f|$, called the minimal relaxed slope of f, realizing the Cheeger energy, namely

$$\mathsf{Ch}(f) = \frac{1}{2} \int_X |\nabla f|^2 \, \mathrm{d}\mathfrak{m}. \tag{2.4}$$

We say that $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian if $W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ is a Hilbert space. In this case, we set

$$\langle \nabla f_1, \nabla f_2 \rangle := \lim_{t \to 0} \frac{|\nabla(f_1 + t f_2)|^2 - |\nabla f_1|^2}{2t} \in L^1(X, \mathfrak{m}), \quad \forall \ f_1, f_2 \in W^{1,2}(X, \mathsf{d}, \mathfrak{m}), \tag{2.5}$$

which is symmetric and bi-linear in $\mathfrak{m}$-a.e. sense (see for instance [**29**, Theorem 4.3.3] for several equivalent definitions of infinitesimal Hilbertianity). Moreover then

[1]Similarly we can define $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$. See for instance [**5**].

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we can define the (linear) Laplacian as follows; we denote by $D(\Delta)$ the set of all $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ such that there exists $h \in L^2(X, \mathfrak{m})$ such that

$$-\int_X \langle \nabla f, \nabla \varphi \rangle \, \mathrm{d}\mathfrak{m} = \int_X \varphi h \, \mathrm{d}\mathfrak{m}, \quad \forall \varphi \in W^{1,2}(X, \mathsf{d}, \mathfrak{m}). \tag{2.6}$$

Since such h is unique whenever it exists, we shall denote by Δf. We are now in a position to give the definition of RCD spaces.

Definition 2.1 RCD-space. *We say that* $(X, \mathsf{d}, \mathfrak{m})$ *is an* $\mathrm{RCD}(K, N)$ *space for some* $K \in \mathbb{R}$ *and* $N \geqslant 1$ *if the following four conditions are satisfied.*

(1) *(Volume growth bound) There exist* $C > 0$ *and* $x \in X$ *such that*

$$\mathfrak{m}(B_r(x)) \leqslant Ce^{Cr^2}, \quad \forall\, r > 0. \tag{2.7}$$

(2) *(Infinitesimal Hilbertianity)* $(X, \mathsf{d}, \mathfrak{m})$ *is infinitesimally Hilbertian.*

(3) *(Sobolev-to-Lipschitz property) Any* $f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ *with* $|\nabla f| \leqslant L$ *for* $\mathfrak{m}$*-a.e. admits an* L*-Lipschitz representative.*

(4) *(Bochner's inequality) For any* $f \in D(\Delta)$ *with* $\Delta f \in W^{1,2}(X, \mathsf{d}, \mathfrak{m})$ *and any* $\varphi \in D(\Delta) \cap L^\infty_+(X, \mathfrak{m})$ *with* $\Delta \varphi \in L^\infty(X, \mathfrak{m})$*, it holds that*

$$\frac{1}{2}\int_X \Delta\varphi |\nabla f|^2 \, \mathrm{d}\mathfrak{m} \geqslant \int_X \varphi \left(\frac{(\Delta f)^2}{N} + \langle \nabla f, \nabla \Delta f \rangle + K|\nabla f|^2 \right) \mathrm{d}\mathfrak{m}. \tag{2.8}$$

There are also several equivalent characterizations of $\mathrm{RCD}(K, N)$-conditions, see [7, 13, 25]. We refer to [8] as a good survey for the theory of RCD spaces.

Let us also mention that there exist local notions above, including the domain $D(\Delta, U)$ of the local Laplacian defined on an open subset U of X, the local Sobolev space $W^{1,2}(U, \mathsf{d}, \mathfrak{m})$ and so on.[2] In the sequel we immediately use them, see for instance [3, 9, 38] for the details.

We here recall the precise definitions of the heat flow and the heat kernel on an $\mathrm{RCD}(K, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ for some $K \in \mathbb{R}$ and some finite $N \geqslant 1$. For any $f \in L^2(X, \mathfrak{m})$, there exists a unique locally absolutely continuous (or equivalently, smooth, in this setting, (see [29])) curve $h.f : (0, \infty) \to L^2(X, \mathfrak{m})$, called the heat flow starting at f, such that $h_t f \to f$ in $L^2(X, \mathfrak{m})$ as $t \to 0^+$ and that $h_t f \in D(\Delta)$ for any $t > 0$ with

$$\frac{\mathrm{d}}{\mathrm{d}t} h_t f = \Delta h_t f. \tag{2.9}$$

Then, thanks to [53, 54] with the Bishop–Gromov inequality and the Poincaré inequality which will be explained in the next subsection 2.3, the heat flow can be written by the integral of a unique continuous kernel $p = p_X : X \times X \times (0, \infty) \to$

[2] For instance, for any $f \in D(\Delta, U)$ we have Δf, $|\nabla f|$, $f \in L^2(U, \mathfrak{m})$.

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$(0, \infty)$, called the heat kernel of $(X, \mathsf{d}, \mathfrak{m})$. Namely, for all $f \in L^2(X, \mathfrak{m})$, we have $h_t f \in C(X)$ with

$$\mathrm{h}_t f(x) = \int_X p(x, y, t) f(y)\, \mathrm{d}\mathfrak{m}(y), \quad \forall\, x \in X. \tag{2.10}$$

Note that the heat kernel p can be characterized by using the dual heat flow $\tilde{h}_t$ acting on the space of all Borel probability measures with finite quadratic moments $\mathcal{P}_2(X)$;

$$\tilde{h}_t \delta_x = p(x, \cdot, t)\mathfrak{m}, \quad \forall\, x \in X, \quad \forall\, t > 0, \tag{2.11}$$

where δ_x is the Dirac measure at x.

Let us write a formula on the heat kernel under a rescaling, which directly follows from the definition; for all $a, b > 0$, the $\mathrm{RCD}(a^{-2}K, N)$ space

$$(\tilde{X}, \tilde{\mathsf{d}}, \tilde{\mathfrak{m}}) := (X, a\mathsf{d}, b\mathfrak{m}) \tag{2.12}$$

satisfies

$$p_{\tilde{X}}(x, y, t) = \frac{1}{b} p_X(x, y, a^{-2}t). \tag{2.13}$$

In order to keep our presentation short, we assume that the readers are familiar with basics on the RCD theory, including pointed measured Gromov–Hausdorff (pmGH) convergence, its metrization $\mathsf{d}_{\mathrm{pmGH}}$, stability/compactness of RCD spaces with respect to $\mathsf{d}_{\mathrm{pmGH}}$, and functional convergence with respect to $\mathsf{d}_{\mathrm{pmGH}}$. We refer a recent nice survey [**35**] about this topic (see also [**3**, **5**, **28**]).

Let us end this subsection by introducing the following two notions with related results.

Definition 2.2 Tangent cone. *A pointed* $\mathrm{RCD}(0, N)$ *space* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y, y)$ *is said to be a tangent cone of* $(X, \mathsf{d}, \mathfrak{m})$ *at* $x \in X$ *(or tangent cone at infinity of* $(X, \mathsf{d}, \mathfrak{m})$ *in the case when* $K = 0$*, respectively) if there exists a sequence* $r_i \to 0^+$ *(or* $r_i \to \infty$*, respectively) such that*

$$\left(X, \frac{1}{r_i}\mathsf{d}, \frac{\mathfrak{m}}{\mathfrak{m}(B_{r_i}(x_i))}, x\right) \xrightarrow{\mathrm{pmGH}} (Y, \mathsf{d}_Y, \mathfrak{m}_Y, y). \tag{2.14}$$

Moreover a point $x \in X$ *is called* k*-regular if any tangent cone at* x *is isomorphic to the* k*-dimensional Euclidean space* $(\mathbb{R}^k, \mathsf{d}_{\mathrm{Euc}}, \omega_k^{-1}\mathscr{H}^k, 0_k)$.

Remark 2.3. We often use $\frac{\mathfrak{m}}{r_i^k}$ for some $k \geqslant 1$ instead of using $\frac{\mathfrak{m}}{\mathfrak{m}(B_{r_i}(x))}$ in the definition above, and we also call such limit a tangent cone.

It is proved in [**12**] if X is not a single point, then there exists a unique integer k at most N such that for $\mathfrak{m}$-a.e. $x \in X$, x is k-regular. We call k the essential dimension of $(X, \mathsf{d}, \mathfrak{m})$ (see also [**18**, **22**]). It is known that the essential dimension is at most the Hausdorff dimension, however in general they do not coincide. See [**49**]. The following is defined in [**20**] as a synthetic counterpart of volume non-collapsed Ricci limit spaces.

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DEFINITION 2.4 Non-collapsed space. *We say that* $(X, \mathsf{d}, \mathfrak{m})$ *is non-collapsed if* $\mathfrak{m} = \mathscr{H}^N$.

It is known that any non-collapsed RCD space has nicer properties rather than that of general RCD spaces, including a fact that N must be an integer, and the Bishop inequality in the case when $K = 0$;

$$\frac{\mathscr{H}^N(B_r(x))}{\omega_N r^N} \leqslant 1, \quad \forall\, r > 0. \tag{2.15}$$

It is worth mentioning that $(X, \mathsf{d}, \mathfrak{m})$ is non-collapsed, up to multiplying a positive constant to the reference measure, if the essential dimension is equal to N, or N is an integer with the existence of an N-regular point. See [**10**] (and [**40**]).

2.3. Geometric and analytic inequalities on RCD(0, N) spaces

Let us recall several inequalities on an RCD$(0, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ for some finite $N \geqslant 1$. Fix $x \in X$. The Bishop–Gromov inequality states

$$\frac{\mathfrak{m}(B_r(x))}{r^N} \geqslant \frac{\mathfrak{m}(B_s(x))}{s^N}, \quad \forall\, r < s. \tag{2.16}$$

See [**46**] and [**53**, **54**] for the proof. Based on this inequality we introduce;

DEFINITION 2.5 N-volume density and asymptotic N-volume. *The N-volume density at x, denoted by ν_x, is defined by*

$$\nu_x := \lim_{r \to 0} \frac{\mathfrak{m}(B_r(x))}{r^N} \in (0, \infty]. \tag{2.17}$$

Moreover the asymptotic N-volume, denoted by V_X, is defined by

$$V_X := \lim_{r \to \infty} \frac{\mathfrak{m}(B_r(x))}{r^N} \in [0, \infty). \tag{2.18}$$

Note that the Bishop–Gromov inequality (2.16) *implies the existence of the both right-hand-sides of* (2.17) *and* (2.18)*, that V_X does not depend on the choice of $x \in X$ and that $\nu_x \geqslant V_X$.*

REMARK 2.6. Let us provide a formula on the N-volume density under a rescaling; for all $a, b > 0$, the N-volume density $\tilde{\nu}_{\tilde{x}}$ of the pointed non-parabolic RCD$(0, N)$ space $(\tilde{X}, \tilde{\mathsf{d}}, \tilde{\mathfrak{m}}, \tilde{x}) := (X, a\mathsf{d}, b\mathfrak{m}, x)$ satisfies

$$\tilde{\nu}_{\tilde{x}} = \lim_{r \to 0+} \frac{b\mathfrak{m}(\tilde{B}_r(x))}{r^N} = \frac{b}{a^N} \lim_{r \to 0+} \frac{\mathfrak{m}(B_{a^{-1}r}(x))}{(a^{-1}r)^N} = \frac{b}{a^N} \nu_x. \tag{2.19}$$

In particular $\tilde{\nu}_{\tilde{x}} = \nu_x$ if $b = a^N$, which will play a role later. Note that similarly we have $V_{\tilde{X}} = \frac{b}{a^N} V_X$.

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Next let us recall Gaussian estimates on the heat kernel p established in [**43**]; for any $0 < \varepsilon < 1$, there exists $C(N, \varepsilon) > 1$ such that

$$\frac{1}{C(N,\varepsilon)\mathfrak{m}(B_{t^{1/2}}(x))}\exp\left(-\frac{\mathsf{d}^2(x,y)}{4(1-\epsilon)t}\right)\leqslant p(x,y,t)\leqslant\frac{C(N,\varepsilon)}{\mathfrak{m}(B_{t^{1/2}}(x))}\exp\left(-\frac{\mathsf{d}^2(x,y)}{4(1+\epsilon)t}\right) \tag{2.20}$$

and

$$|\nabla_x p(x,y,t)| \leqslant \frac{C(N,\varepsilon)}{t^{1/2}\mathfrak{m}(B_{t^{1/2}}(x))}\exp\left(-\frac{\mathsf{d}^2(x,y)}{(4+\epsilon)t}\right). \tag{2.21}$$

Finally let us recall the following Poincaré inequality proved in [**52**];

$$\fint_{B_r(x)}\left|f-\fint_{B_r(x)}f\,\mathrm{d}\mathfrak{m}\right|\mathrm{d}\mathfrak{m}\leqslant 4r\fint_{B_{2r}(x)}|\nabla f|\,\mathrm{d}\mathfrak{m},\quad\forall\, r>0,\ \forall\, f\in W^{1,2}(X,\mathsf{d},\mathfrak{m}), \tag{2.22}$$

where

$$\fint_A\cdot\,\mathrm{d}\mathfrak{m}:=\frac{1}{\mathfrak{m}(A)}\int_A\cdot\,\mathrm{d}\mathfrak{m} \tag{2.23}$$

denotes the integral average for any measurable set A with positive and finite measure.

When $N > 2$, the Poincaré inequality combining with the Bishop–Gromov inequality (2.16) implies the self-improved Poincaré inequality:

$$\left(\fint_{B_r(x)}\left|f-\fint_{B_r(x)}f\,\mathrm{d}\mathfrak{m}\right|^{\frac{2N}{N-2}}\mathrm{d}\mathfrak{m}\right)^{\frac{N-2}{2N}}\leqslant C(N)r\left(\fint_{B_{2r}(x)}|\nabla f|^2\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{2}}. \tag{2.24}$$

Moreover if $f \in W_0^{1,2}(B_r(x), \mathsf{d}, \mathfrak{m})$ and X is non-compact, then we have a more convenient corollary usually referred as Sobolev inequality:

$$\left(\fint_{B_r(x)}|f|^{\frac{2N}{N-2}}\,\mathrm{d}\mathfrak{m}\right)^{\frac{N-2}{2N}}\leqslant C(N)r\left(\fint_{B_r(x)}|\nabla f|^2\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{2}} \tag{2.25}$$

which plays a central role to get various properties on differential operators including the Laplacian and a drifted Laplace operator $\mathscr{L}$ in § 6. See for instance [**9**, **37**] for the details.

2.4. Metric measure cone and rigidity

In this subsection, we introduce known rigidity results to an N-metric measure cone whose definition is as follows. In the sequel, we fix a finite $N > 2$.

Definition 2.7 N-metric measure cone. *The (N-)metric measure cone* $(C(Y), \mathsf{d}_{C(Y)}, \mathfrak{m}_{C(Y)})$ *over an* $\mathrm{RCD}(N-2, N-1)$ *space* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *is defined*

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by

$$\begin{aligned} C(Y) &:= [0,\infty)\times Y/(\{0\}\times Y),\\ \mathsf{d}_{C(Y)}\left((r_1,y_1),(r_2,y_2)\right) &:= \sqrt{r_1^2+r_2^2-2r_1r_2\cos(\mathsf{d}_Y(y_1,y_2))},\\ \mathrm{d}\mathfrak{m}_{C(Y)}(r,y) &:= r^{N-1}\mathrm{d}r\otimes \mathrm{d}\mathfrak{m}_Y(y), \end{aligned} \tag{2.26}$$

where $\mathrm{d}r=\mathcal{L}^1$ *is the* 1*-dimensional Lebesgue measure. Denote by* $O_Y:=[(0,\,y)]$ *the pole of* $C(Y)$.

Note that in the definition above, if $(Y,\,\mathsf{d}_Y,\,\mathfrak{m}_Y)$ is non-collapsed, then $(C(Y),\,\mathsf{d}_{C(Y)},\,\mathfrak{m}_{C(Y)})$ is also non-collapsed because we can easily check by definition

$$\lim_{s\to 0+}\frac{\mathfrak{m}_{C(Y)}(B_s(r,x))}{\omega_{N+1}s^{N+1}}=1,\quad \text{for any } r>0 \text{ and any } (N-1)\text{-regular point } x \text{ of } X. \tag{2.27}$$

This remark will play a role in subsection 5.1.

The following results are fundamental results for N-metric measure cones, where (1) is due to [**44**, Corollary 1.3] and (2) is obtained in [**19**, Theorem 1.1] and [**31**, Theorem 5.1].

THEOREM 2.8 Rigidity. *We have the following.*

(1) *The* N*-metric measure cone over an* $\mathrm{RCD}(N-2,\,N-1)$ *space is an* $\mathrm{RCD}(0,\,N)$ *space.*

(2) *Let* $(X,\,\mathsf{d},\,\mathfrak{m},\,x)$ *be a pointed* $\mathrm{RCD}(0,\,N)$ *space. Then the following three conditions are equivalent:*
   (a) $(X,\,\mathsf{d},\,\mathfrak{m},\,x)$ *is isomorphic to the* N*-metric measure cone over an* $\mathrm{RCD}(N-2,\,N-1)$ *space with the pole;*
   (b) *there exists* $f\in D_{\mathrm{loc}}(\Delta)$ *such that* $\Delta f=2N$ *holds (in particular* f *must be locally Lipschitz because of [**6**, **42**]), that* f *is positive on* $X\setminus\{x\}$ *with* $f(x)=0$, *and that* $|\nabla\sqrt{2f}|^2=1$ *(moreover then* f *is equal to* $\frac{1}{2}\mathsf{d}(x,\,\cdot)^2$*);*
   (c) *the function*

$$R\mapsto\frac{\mathfrak{m}(B_R(x))}{R^N} \tag{2.28}$$

   is constant.

The following is a well-known result which will play a central role in the paper.

COROLLARY 2.9. *We have the following.*

(1) *If an* $\mathrm{RCD}(0,\,N)$ *space* $(X,\,\mathsf{d},\,\mathfrak{m})$ *has the finite* N*-volume density* $\nu_x<\infty$ *at a point* $x\in X$, *then any tangent cone at* x *is isomorphic to the* N*-metric measure cone over an* $\mathrm{RCD}(N-2,\,N-1)$ *space.*

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(2) *If an* $\mathrm{RCD}(0, N)$ *space* $(X, \mathsf{d}, \mathfrak{m})$ *has the positive asymptotic* N*-volume* $V_X > 0$*, then any tangent cone at infinity is isomorphic to the* N*-metric measure cone over an* $\mathrm{RCD}(N-2, N-1)$ *space.*

Proof. We give only a proof of (1) because (2) is similar. Denoting by $\tilde{\mathfrak{m}}$, $\tilde{x}$ the reference measure, the base point, respectively on a tangent cone, it easily follows from the finiteness $\nu_x < \infty$ that

$$\frac{\tilde{\mathfrak{m}}(B_R(\tilde{x}))}{\tilde{\mathfrak{m}}(B_r(\tilde{x}))} = \left(\frac{R}{r}\right)^N, \quad \forall r > 0, \quad \forall R > 0. \tag{2.29}$$

Thus (2) of theorem 2.8 allows us to conclude. □

Let us recall the explicit description on the heat kernel on an N-metirc measure cone.

Proposition 2.10. *Let* $(X, \mathsf{d}, \mathfrak{m}, x)$ *be isomorphic to the* N*-metric measure cone with the pole over an* $\mathrm{RCD}(N-2, N-1)$ *space. Then we have*

$$\frac{\mathfrak{m}(B_r(x))}{r^N} = \mathfrak{m}(B_1(x)) = \nu_x = V_X, \quad \forall\, r > 0 \tag{2.30}$$

and

$$p(x, y, t) = Ct^{-\frac{N}{2}} \exp\left(-\frac{\mathsf{d}(x, y)^2}{4t}\right), \quad \forall\, y \in X, \tag{2.31}$$

where

$$C = \frac{2^{1-N}}{N\Gamma\left(\frac{N}{2}\right)\mathfrak{m}(B_1(x))} \tag{2.32}$$

Proof. It follows by definitions 2.5 and 2.7 that (2.30) holds. On the other hand, (2.31) is a direct consequence of [**45**, Proposition 4.10] with (2.11) and (2) of theorem 2.8. See [**41**, Proposition 2.13] for a more general result (see also [**23**, Theorem 6.20] and [**55**, Section 8]). □

3. Green function

In this section we discuss the Green function on a non-parabolic $\mathrm{RCD}(0, N)$ space.

3.1. Non-parabolic RCD(0, *N*) space

Throughout the section, we fix an $\mathrm{RCD}(0, N)$ space $(X, \mathsf{d}, \mathfrak{m})$ for some finite $N > 2$, which is not necessarily an integer. Let us start by introducing the following fundamental notion due to [**12**] in our framework.

Definition 3.1 Non-parabolic $\mathrm{RCD}(0, N)$ space. $(X, \mathsf{d}, \mathfrak{m})$ *is said to be non-parabolic if for some point* $x \in X$ *(and thus for any* $x \in X$*),*

$$\int_1^\infty \frac{s}{\mathfrak{m}\big(B_s(x)\big)}\, \mathrm{d}s < \infty. \tag{3.1}$$

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It is trivial that $(X, \mathsf{d}, \mathfrak{m})$ is non-parabolic if $V_X > 0$ because of the Bishop–Gromov inequality. In the sequel we assume that $(X, \mathsf{d}, \mathfrak{m})$ is non-parabolic. Note that the diameter must be infinite, thus it is non-compact. Then we can define the Green function as follows.

Definition 3.2 Green function. *The Green function $G = G^X$ of the non-parabolic* $\mathrm{RCD}(0, N)$ *space $(X, \mathsf{d}, \mathfrak{m})$ is defined by*

$$\begin{aligned} &G: \ X \times X \setminus \mathrm{diag}(X) \to (0, \infty), \\ &(x, y) \mapsto \int_0^\infty p(x, y, t)\, \mathrm{d}t, \end{aligned} \tag{3.2}$$

where $\mathrm{diag}(X) := \{(x, x) \in X \times X | x \in X\}$. *In the following we write* $G_x^X(\cdot) = G_x(\cdot) := G(x, \cdot) : X \setminus \{x\} \to (0, \infty)$.

In the sequel, we fix $x \in X$. It is proved in (the proof of) [**12**, Lemma 2.5] that G_x is harmonic on $X \setminus \{x\}$ and that $G_x \in W^{1,1}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ holds with

$$\int_X \Delta f(y) G_x(y)\, \mathrm{d}\mathfrak{m}(y) = -f(x) \tag{3.3}$$

for any $f \in D_c(\Delta)$ with $\Delta f \in L^\infty(X, \mathfrak{m})$. In order to introduce quantitative estimates on G_x, let us prepare the following auxiliary functions for all $x \in X$ and $r \in (0, \infty)$;

$$F_x^X(r) = F_x(r) := \int_r^\infty \frac{s}{\mathfrak{m}\big(B_s(x)\big)}\, \mathrm{d}s, \quad H_x^X(r) = H_x(r) := \int_r^\infty \frac{1}{\mathfrak{m}\big(B_s(x)\big)}\, \mathrm{d}s. \tag{3.4}$$

It is easy to see that both F and H are continuous with respect to the two variables $(x, r) \in X \times (0, \infty)$. Note that Bishop–Gromov inequality (2.16) shows for any $r > 0$

$$\frac{1}{(N-2)\nu_x} \leqslant \frac{F_x(r)}{r^{2-N}} \leqslant \frac{1}{(N-2)V_X}, \quad \frac{1}{(N-1)\nu_x} \leqslant \frac{H_x(r)}{r^{1-N}} \leqslant \frac{1}{(N-1)V_X}. \tag{3.5}$$

Let us provide formulae on their asymptotics.

Lemma 3.3. *The following asymptotic properties hold as $r \to 0$:*

$$\lim_{r \to 0} \frac{F_x(r)}{r^{2-N}} = \frac{1}{(N-2)\nu_x}, \tag{3.6}$$

$$\lim_{r \to 0} \frac{H_x(r)}{r^{1-N}} = \frac{1}{(N-1)\nu_x}. \tag{3.7}$$

Moreover the following asymptotic properties hold as $r \to \infty$:

$$\lim_{r \to \infty} \frac{F_x(r)}{r^{2-N}} = \frac{1}{(N-2)V_X}, \tag{3.8}$$

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$$\lim_{r\to\infty}\frac{H_x(r)}{r^{1-N}}=\frac{1}{(N-1)V_X}, \tag{3.9}$$

where the limits of (3.8) and of (3.9) can be understood as ∞ *in the case when* $V_X=0$.

Proof. We only show (3.6) under assuming $\nu_x<\infty$ because the others can follow from similar arguments. The Bishop–Gromov inequality (2.16) shows that the map

$$I_x(r):=\nu_x-\frac{\mathfrak{m}(B_r(x))}{r^N} \tag{3.10}$$

is non-decreasing with $I_x(r)\geqslant 0$. Thus fixing $r_0>0$, we have for any $0<r<r_0$,

$$\nu_x r^N-I_x(r_0)r^N\leqslant \mathfrak{m}(B_r(x))\leqslant \nu_x r^N. \tag{3.11}$$

Write $F_x(r)$ as

$$F_x(r)=\int_r^{r_0}\frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s+\int_{r_0}^{\infty}\frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s. \tag{3.12}$$

Then by (3.11) the first term of the right-hand-side can be estimated as follows.

$$\frac{r_0^{2-N}-r^{2-N}}{(2-N)\nu_x}\leqslant\int_r^{r_0}\frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s\leqslant\frac{r_0^{2-N}-r^{2-N}}{(2-N)(\nu_x-I_x(r_0))}. \tag{3.13}$$

Thus

$$\begin{aligned}r^{N-2}\int_{r_0}^{\infty}\frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s-C(N,\nu_x)\left(\frac{r}{r_0}\right)^{N-2}&\leqslant\frac{F_x(r)}{r^{2-N}}-\frac{1}{(N-2)\nu_x}\\&\leqslant r^{N-2}\int_{r_0}^{\infty}\frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s+\frac{C(N,\nu_x)}{I_x(r_0)}\left(\frac{r}{r_0}\right)^{N-2}.\end{aligned} \tag{3.14}$$

For any $\varepsilon\in(0,1)$, we can let $r=\varepsilon r_0$ and thus

$$\begin{aligned}(\varepsilon r_0)^{N-2}\int_{r_0}^{\infty}\frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s-C(N,\nu_x)\varepsilon^{N-2}&\leqslant\frac{F_x(\varepsilon r_0)}{(\varepsilon r_0)^{2-N}}-\frac{1}{(N-2)\nu_x}\\&\leqslant(\varepsilon r_0)^{N-2}\int_{r_0}^{\infty}\frac{s}{\mathfrak{m}(B_s(x))}\,\mathrm{d}s+\frac{C(N,\nu_x)}{I_x(r_0)}\varepsilon^{N-2}.\end{aligned} \tag{3.15}$$

Letting $\varepsilon\to 0$ completes the proof of (3.6). □

We are now in a position to introduce estimates on G by F, H given in [**12**, Proposition 2.3] after [**36**] in the smooth setting.

Proposition 3.4. *There exists* $C=C(N)>1$ *such that*

$$\frac{1}{C}F_x(\mathsf{d}(x,y))\leqslant G_x(y)\leqslant CF_x(\mathsf{d}(x,y)),\quad\forall\ y\in X\setminus\{x\}, \tag{3.16}$$

and

$$|\nabla G_x|(y)\leqslant CH_x(\mathsf{d}(x,y)),\quad \textit{for } \mathfrak{m}-a.e.\ y\in X. \tag{3.17}$$

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Since $F_x(r) \to 0$ as $r \to \infty$, we have immediately the following.

COROLLARY 3.5. *We have* $G_x(y) \to 0$ *as* $\mathsf{d}(x, y) \to \infty$.

Let us define the main target on the paper.

DEFINITION 3.6 Smoothed distance function b_x. *Define a function* $\mathsf{b}_x^X = \mathsf{b}_x$ *on* $X \setminus \{x\}$ *by*

$$\mathsf{b}_x := G_x^{\frac{1}{2-N}}. \tag{3.18}$$

We provide formulae related to b_x, which will play roles later.

LEMMA 3.7. *We have*

$$\Delta \mathsf{b}_x = (N-1)\frac{|\nabla \mathsf{b}_x|^2}{\mathsf{b}_x}, \tag{3.19}$$

and

$$\Delta \mathsf{b}_x^2 = 2N|\nabla \mathsf{b}_x|^2. \tag{3.20}$$

Proof. Because

$$\Delta \mathsf{b}_x = \Delta G_x^{\frac{1}{2-N}} = \frac{N-1}{(N-2)^2} G_x^{\frac{2N-3}{2-N}} |\nabla G_x|^2 = (N-1)\frac{|\nabla \mathsf{b}_x|^2}{\mathsf{b}_x}, \tag{3.21}$$

and thus

$$\Delta \mathsf{b}_x^2 = 2\mathsf{b}_x \Delta \mathsf{b}_x + 2|\nabla \mathsf{b}_x|^2 = 2N|\nabla \mathsf{b}_x|^2, \tag{3.22}$$

where we used the fact that G_x is harmonic on $X \setminus \{x\}$. □

We give the explicit formula for the smoothed distance function for an N-metric measure cone. Although this is well-known (see for instance [**11**, Lemma 2.7]), let us provide a proof for readers' convenience.

PROPOSITION 3.8 Green function on N-metric measure cone. *If* $(X, \mathsf{d}, \mathfrak{m}, x)$ *is isomorphic to the* N*-metric measure cone with the pole over an* $\mathrm{RCD}(N-2, N-1)$ *space, then we have*

$$G_x(y) = \frac{1}{N(N-2)\mathfrak{m}(B_1(x))}\mathsf{d}(x,y)^{2-N}, \quad \forall y \in X \setminus \{x\}. \tag{3.23}$$

In particular

$$\mathsf{b}_x = \mathscr{C}_N \nu_x^{\frac{1}{N-2}} \mathsf{d}_x, \tag{3.24}$$

where

$$\mathscr{C}_N := (N(N-2))^{\frac{1}{N-2}}. \tag{3.25}$$

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Proof. Thanks to proposition 2.10, we know

$$\begin{aligned} G_x(y) &= \frac{2^{1-N}}{N\Gamma\left(\frac{N}{2}\right)\mathfrak{m}(B_1(x))}\int_0^\infty t^{-\frac{N}{2}}\exp\left(-\frac{\mathsf{d}(x,y)^2}{4t}\right)\mathrm{d}t \\ &= \frac{2^{1-N}}{N\Gamma\left(\frac{N}{2}\right)\mathfrak{m}(B_1(x))}\cdot\mathsf{d}(x,y)^{2-N}4^{\frac{N}{2}-1}\Gamma\left(\frac{N}{2}-1\right) \\ &= \frac{1}{N(N-2)\mathfrak{m}(B_1(x))}\mathsf{d}(x,y)^{2-N}. \end{aligned} \tag{3.26}$$

Finally recalling (2.30), we get (3.24). □

Finally let us provide formulae on the functions above, F, G, H and b under rescalings, which will play roles later.

LEMMA 3.9. *For all a, $b>0$, consider the rescaled non-parabolic* RCD$(0,N)$ *space;*

$$\left(\tilde{X},\tilde{\mathsf{d}},\tilde{\mathfrak{m}},\tilde{x}\right) := (X, a\mathsf{d}, b\mathfrak{m}, x)\,. \tag{3.27}$$

Then the Green function $G^{\tilde{X}}_{\tilde{x}}$, the corresponding auxiliary functions $F^{\tilde{X}}_{\tilde{x}}$, $H^{\tilde{X}}_{\tilde{x}}$, and the smoothed distance function $\mathsf{b}^{\tilde{X}}_x$ of the rescaled space satisfy

$$G^{\tilde{X}}_{\tilde{x}} = \frac{a^2}{b}G^X_x, \quad F^{\tilde{X}}_{\tilde{x}}(r) = \frac{a^2}{b}F^X_x\left(\frac{r}{a}\right), \quad H^{\tilde{X}}_{\tilde{x}}(r) = \frac{a}{b}H^X_x\left(\frac{r}{a}\right) \tag{3.28}$$

and

$$\mathsf{b}^{\tilde{X}}_{\tilde{x}} = \frac{a^{\frac{2}{2-N}}}{b^{\frac{1}{2-N}}}\mathsf{b}^X_x, \quad |\tilde{\nabla}\mathsf{b}_{\tilde{x}}| = \frac{a^{\frac{N}{2-N}}}{b^{\frac{1}{2-N}}}|\nabla\mathsf{b}_x|. \tag{3.29}$$

In particular, if $b=a^N$, then

$$G^{\tilde{X}}_{\tilde{x}}(y) = a^{2-N}G^X_x(y), \quad F^{\tilde{X}}_{\tilde{x}}(r) = a^{2-N}F_x\left(\frac{r}{a}\right), \quad H^{\tilde{X}}_{\tilde{x}}(r) = a^{1-N}H_x\left(\frac{r}{a}\right). \tag{3.30}$$

and

$$\mathsf{b}_{\tilde{x}} = a\mathsf{b}_x, \quad |\tilde{\nabla}\mathsf{b}_{\tilde{x}}| = |\nabla\mathsf{b}_x|. \tag{3.31}$$

Proof. The formula for G is a direct consequence of (2.13). Moreover it implies (3.29). On the other hand, since

$$\begin{aligned} F_{\tilde{x}}(r) &= \int_r^\infty \frac{s}{b\mathfrak{m}(\tilde{B}_s(\tilde{x}))}\mathrm{d}s = \frac{1}{b}\int_r^\infty \frac{s}{\mathfrak{m}(B_{a^{-1}s}(x))}\mathrm{d}s \\ &= \frac{a^2}{b}\int_{a^{-1}r}^\infty \frac{t}{\mathfrak{m}(B_t(x))}\mathrm{d}t = \frac{a^2}{b}F_x\left(\frac{r}{a}\right), \end{aligned} \tag{3.32}$$

we have the desired formula for F. Similarly we have the remaining results. □

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3.2. Convergence

In this subsection we discuss the convergence of non-parabolic $\mathrm{RCD}(0,N)$ spaces with respect to the pmGH topology. Let us introduce an elementary lemma.

LEMMA 3.10. *Let*

$$(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)\overset{\mathrm{pmGH}}{\to}(X,\mathsf{d},\mathfrak{m},x) \tag{3.33}$$

be a pmGH convergent sequence of $\mathrm{RCD}(0,N)$ *spaces for some finite* $N\geqslant 1$. *Then we have*

$$\liminf_{i\to\infty}\nu_{x_i}\geqslant\nu_{x_\infty} \tag{3.34}$$

and

$$\limsup_{i\to\infty}V_{X_i}\leqslant V_X. \tag{3.35}$$

Proof. We give only a proof of (3.35) because the proof (3.34) is similar (moreover this is valid even in the case of negative lower bounds on Ricci curvature. See also [**47**, Subsection 2.3]). For fixed $r>0$, we have

$$V_{X_i}\leqslant\frac{\mathfrak{m}_i(B_r(x_i))}{r^N}\to\frac{\mathfrak{m}(B_r(x))}{r^N} \tag{3.36}$$

which shows

$$\limsup_{i\to\infty}V_{X_i}\leqslant\frac{\mathfrak{m}(B_r(x))}{r^N}. \tag{3.37}$$

Then letting $r\to\infty$ completes the proof of (3.35). □

Next let us provide a compactness result as follows. In the sequel we fix a finite $N>2$. Note that if $F_x(1)\leqslant\tau<\infty$, then

$$\frac{1}{\mathfrak{m}(B_2(x))}\leqslant\int_1^2\frac{s}{\mathfrak{m}(B_s(x))}\mathrm{d}s\leqslant F_x(1)\leqslant\tau<\infty, \tag{3.38}$$

thus

$$0<\frac{1}{2^N\tau}\leqslant\mathfrak{m}(B_1(x))\leqslant\frac{\mathfrak{m}(B_s(x))}{s^N},\quad\forall\, s\leqslant 1. \tag{3.39}$$

This observation plays a role at the beginning of the proof of the following.

THEOREM 3.11 Compactness of non-parabolic $\mathrm{RCD}(0,N)$ spaces. *Let* $(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)$ *be a sequence of pointed non-parabolic* $\mathrm{RCD}(0,N)$ *spaces with*

$$\sup_i\mathfrak{m}_i(B_1(x_i))<\infty \tag{3.40}$$

and

$$\sup_i F_{x_i}(1)<\infty. \tag{3.41}$$

Then after passing to a subsequence, $(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)$ *pmGH converge to a pointed* $\mathrm{RCD}(0,N)$ *space* $(X,\mathsf{d},\mathfrak{m},x)$ *with the lower semicontinuity of* F_{x_i} *in the sense*

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that

$$\liminf_{i\to\infty} F_{x_i}(r_i) \geqslant F_x(r), \quad \forall\ r_i \to r \ \text{in}\ (0,\infty). \tag{3.42}$$

In particular $(X, \mathsf{d}, \mathfrak{m})$ *is non-parabolic.*

Proof. Note that (3.41) gives a uniform positive lower (upper, respectively) bound on $\mathfrak{m}_i(B_1(x_i))$ because of (3.39). Thus, thanks to the compactness of RCD spaces with respect to the pmGH topology (see for instance [**1**, Theorem 6.11], [**25**, Theorem 5.3.22], [**28**, Theorem 7.2], [**46**, Theorem 5.19], and [**53**, Theorem 4.20]), after passing to a subsequence, $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ pmGH converge to a pointed RCD$(0, N)$ space $(X, \mathsf{d}, \mathfrak{m}, x)$. Observe that for all $r \leqslant s$, we have

$$\int_{r_i}^{s} \frac{r}{\mathfrak{m}_i(B_t(x_i))}\,\mathrm{d}t \to \int_{r}^{s} \frac{t}{\mathfrak{m}(B_t(x))}\,\mathrm{d}t, \tag{3.43}$$

which implies

$$\int_{r}^{s} \frac{t}{\mathfrak{m}(B_t(x))}\,\mathrm{d}t = \lim_{i\to\infty} \int_{r_i}^{s} \frac{t}{\mathfrak{m}_i(B_t(x_i))}\,\mathrm{d}t \leqslant \liminf_{i\to\infty} F_{x_i}(r_i). \tag{3.44}$$

Letting $s \to \infty$, we have (3.42). □

Note that (3.40) is satisfied if

$$\sup_i \nu_{x_i} < \infty. \tag{3.45}$$

Because if $\nu_x \leqslant \nu < \infty$, then

$$\frac{\mathfrak{m}(B_s(x))}{s^N} \leqslant \nu_x \leqslant \nu, \quad \forall\ s \leqslant 1 \tag{3.46}$$

by the Bishop–Gromov inequality. Compare the following theorem with [**11**, Proposition 2.3].

THEOREM 3.12 Convergence of Green functions. *Let us consider a pmGH convergent sequence of pointed non-parabolic* RCD$(0, N)$ *spaces*

$$(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \overset{\mathrm{pmGH}}{\to} (X, \mathsf{d}, \mathfrak{m}, x). \tag{3.47}$$

Then the following conditions are equivalent.

(1) *The functions* $f_i(s) := \frac{s}{\mathfrak{m}_i(B_s(x_i))}$ *converge in* $L^1([1,\infty), \mathscr{H}^1)$ *to the function* $f(s) := \frac{s}{\mathfrak{m}(B_s(x))}$ *as* $i \to \infty$.

(2) $F_{x_i}(1) \to F_x(1)$.

(3) *For any finite* $p \geqslant 1$, G_{x_i} $W^{1,p}_{\mathrm{loc}}$*-strongly, and locally uniformly converge to* G_x *on* $X \setminus \{x\}$. *Or equivalently* b_{x_i} $W^{1,p}_{\mathrm{loc}}$*-strongly, and locally uniformly converge to* b_x *on* $X \setminus \{x\}$, *where we say that a sequence of functions* $f_i : X_i \setminus \{x_i\} \to \mathbb{R}$ *locally uniformly converge to a function* $f : X \setminus \{x\} \to \mathbb{R}$ *if under fixing*

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isometric embeddings $\iota_i : X_i \hookrightarrow Y$, $\iota : X \hookrightarrow Y$ *into a common proper metric space* (Y, d_Y) *realizing (3.47), for any compact subset* $A \subset X \setminus \{x\}$ *and any* $\epsilon \in (0, 1)$, *there exist* $\delta \in (0, 1)$ *and* $i_0 \in \mathbb{N}$ *such that* $|f_i(z_i) - f(z)| < \epsilon$ *holds for all* $z_i \in X_i$ *and* $z \in A$ *whenever* $\mathsf{d}_Y(\iota_i(z_i), \iota(z)) < \delta$ *and* $i \geqslant i_0$.

Proof. The key point is:

- it is proved in [**4**, Theorem 3.3] that

$$p_{X_i}(y_i, z_i, t_i) \to p_X(y, z, t) \tag{3.48}$$

holds for all convergent sequences of $t_i \to t$ in $(0, \infty)$ and of $y_i, z_i \in X_i \to y$, $z \in X$, respectively.

Based on the above, let us start giving the proof. The implication from (1) to (2) is trivial. Assume that (2) holds. Thanks to (3.48) and (2.20) with the assumption, we know $\int_r^R p_i(x_i, y_i, t)\mathrm{d}t \to \int_r^R p(x, y, t)\mathrm{d}t$ and thus $F_{x_i}(R) \to F_x(R)$ for any $R \geqslant 1$. In particular for any $0 < \varepsilon < 1$ there exists $R \geqslant 1$ such that $F_{x_i}(R) + F_x(R) < \varepsilon$ for any i. On the other hand, for any fixed convergent sequence $y_i \in X_i$ to $y \in X$ with $x \neq y$, by (2.20), we know that there exists $0 < r < 1$ such that

$$\int_0^r p_i(x_i, y_i, t)\mathrm{d}t + \int_0^r p(x, y, t)\mathrm{d}t < \varepsilon. \tag{3.49}$$

The observation above allows us to conclude the pointwise convergence $G_{x_i}(y_i) \to G_x(y)$. Then the locally uniform convergence comes from this with a locally uniform Lipschitz bound (3.17). Moreover since G_{x_i} is harmonic on $X_i \setminus \{x_i\}$, it follows from the stability of Laplacian, [**3**, Theorem 4.4], that the $W^{1,2}_{\mathrm{loc}}$-strong convergence of the Green functions holds. Finally the improvement to the $W^{1,p}_{\mathrm{loc}}$-strong convergence is justified by combining this with (3.17) (see also [**39**]). Thus we have (3).

Finally let us prove the remaining implication from (3) to (1). Thanks to corollary 3.5, for any $0 < \varepsilon < 1$ there exists $R \geqslant 1$ such that $G_x(y) < \varepsilon$ for any $y \in X \setminus B_R(x)$. Fix $y \in X \setminus B_{2R}(x)$ and take $y_i \in X_i$ converging to y. Then our assumption allows us to conclude $G_{x_i}(y_i) < 2\varepsilon$ for any sufficiently large i. Thus by (3.16), we have

$$F_{x_i}(R) \leqslant F_{x_i}(\mathsf{d}_i(x_i, y_i)) \leqslant C(N)G_{x_i}(y_i) \leqslant C(N)\varepsilon. \tag{3.50}$$

On the other hand, as discussed above, we can prove that f_i converges in $L^1([1, r), \mathscr{H}^1)$ to f for any finite $r > 1$. This with (3.50) implies (1) because ε is arbitrary. □

Compared with theorem 3.11, it is natural to ask whether the second condition above can be replaced by a weaker one; $\sup_i F_{x_i}(1)$, or not. However this improvement is impossible by observing a simple example discussed in subsection 5.2. In this sense theorem 3.11 is sharp.

Let us give corollaries of theorem 3.12. See also [**11**, Corollary 2.4].

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COROLLARY 3.13. *We have*

$$\lim_{y\to x}\frac{G(x,y)}{\mathsf{d}(x,y)^{2-N}}=\frac{1}{N(N-2)\nu_x} \tag{3.51}$$

and

$$\lim_{\mathsf{d}(x,y)\to\infty}\frac{G(x,y)}{\mathsf{d}(x,y)^{2-N}}=\frac{1}{N(N-2)V_X}. \tag{3.52}$$

Proof. We prove only (3.51) via a blow-up argument because the proof of (3.52) is similar via a blow-down argument, where the case when $\nu_x=\infty$ or $V_X=0$ directly follows from lemma 3.3 with (3.16).

Take a convergent sequence $y_i\in X\setminus\{x\}\to x$, let $r_i:=\mathsf{d}(x,y_i)$ and consider rescaled RCD$(0,N)$ spaces;

$$(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i):=\left(X,\frac{1}{r_i}\mathsf{d},\frac{1}{r_i^N}\mathfrak{m},x\right). \tag{3.53}$$

Then since

$$F_{x_i}(1)=\frac{F_x(r_i)}{r_i^{2-N}}\to\frac{1}{(N-2)\nu_x},\quad \nu_{x_i}=\nu_x, \tag{3.54}$$

thanks to theorem 3.11, after passing to a subsequence, $(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)$ pmGH converge to a tangent cone $(W,\mathsf{d}_W,\mathfrak{m}_W,w)$ at x, which is isomorphic to the non-parabolic N-metric measure cone over an RCD$(N-2,N-1)$ space with the finite volume density $\nu_w=\nu_x$ (see (2.30)). By proposition 3.8 we know $F_w(1)=\frac{1}{(N-2)\nu_w}(=\lim_{i\to\infty}F_{x_i}(1)$ by (3.54)) and

$$G^W(w,z)=\frac{1}{N(N-2)\nu_w}\mathsf{d}_W(w,z)^{2-N},\quad \forall z\in W\setminus\{w\}. \tag{3.55}$$

After passing to a subsequence again, we find the limit point z of $y_i\in X_i$ (thus $\mathsf{d}_W(w,z)=1$). Then theorem 3.12 shows

$$\frac{G(x,y_i)}{r_i^{2-N}}=G^{X_i}(x_i,y_i)\to G^Y(w,z)=\frac{1}{N(N-2)\nu_w}=\frac{1}{N(N-2)\nu_x} \tag{3.56}$$

which completes the proof because y_i is arbitrary. □

The next corollary gives an equi-convergent result on b. Note that this corollary can be improved later under adding a uniform upper bound on the N-volume density. See corollary 3.27.

COROLLARY 3.14. *For all* $N>2$, $0<\varepsilon<1$, $0<r<R<\infty$, $v>0$, $1\leqslant p<\infty$ *and* $\varphi\in L^1([1,\infty),\mathscr{H}^1)$ *there exists* $\delta=\delta(N,\varepsilon,r,R,v,p,\varphi)>0$ *such that if two pointed non-parabolic* RCD$(0,N)$ *spaces* $(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)(i=1,2)$ *satisfy*

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$\mathfrak{m}_i(B_1(x_i)) \leqslant v$,

$$\frac{s}{\mathfrak{m}_i(B_s(x_i))} \leqslant \varphi(s), \quad \text{for } \mathscr{H}^1\text{-a.e. } s \in [1,\infty), \tag{3.57}$$

and

$$\mathsf{d}_{\mathrm{pmGH}}\left((X_1,\mathsf{d}_1,\mathfrak{m}_1,x_1),(X_2,\mathsf{d}_2,\mathfrak{m}_2,x_2)\right) < \delta, \tag{3.58}$$

then

$$|\mathsf{b}_{x_1}(y_1) - \mathsf{b}_{x_2}(y_2)| + \left| \fint_{B_s(y_1)} |\nabla \mathsf{b}_{x_1}|^p \mathrm{d}\mathfrak{m}_1 - \fint_{B_s(y_2)} |\nabla \mathsf{b}_{x_2}|^p \mathrm{d}\mathfrak{m}_2 \right| < \varepsilon \tag{3.59}$$

for all $\delta r \leqslant s \leqslant (1-\delta)r$ *and* $y_i \in B_R(x_i) \setminus B_r(x_i)$ *satisfying that* y_1 δ*-close to* y_2 *with respect to (3.58).*

Proof. In order to simplify our arguments below, we give a proof only in the case when $p = 2$ because the general case is similar after replacing $W^{1,2}$-convergence by W^{1,p_i}-convergence for a convergent sequence $p_i \to p$.

The proof is done by a strandard contradiction argument based on the compactness of RCD spaces with respect to the pmGH convergence. Namely if the assertion is not satisfied, then there exist sequences of;

- pointed non-parabolic RCD$(0, N)$ spaces $(X_{j,i}, \mathsf{d}_{j,i}, \mathfrak{m}_{j,i}, x_{j,i})$ with $\mathfrak{m}_{j,i}(B_1(x_{j,i})) \leqslant v$,

$$f_{j,i}(s) := \frac{s}{\mathfrak{m}_{j,i}(B_s(x_{j,i}))} \leqslant \varphi(s), \quad \text{for } \mathscr{H}^1\text{-a.e. } s \in [1,\infty) \tag{3.60}$$

  and

$$\mathsf{d}_{\mathrm{pmGH}}\left((X_{1,i},\mathsf{d}_{1,i},\mathfrak{m}_{1,i},x_{1,i}),(X_{2,i},\mathsf{d}_{2,i},\mathfrak{m}_{2,i},x_{2,i})\right) \to 0; \tag{3.61}$$

- points $y_{j,i} \in B_R(x_{j,i}) \setminus B_r(x_{j,i})$ satisfying that $y_{1,i}$ is ε_i-close to $y_{2,i}$ for some $\varepsilon_i \to 0^+$ and that

$$\inf_i \left(|\mathsf{b}_{x_{1,i}}(y_{1,i}) - \mathsf{b}_{x_{2,i}}(y_{2,i})| + \left| \fint_{B_s(y_{1,i})} |\nabla \mathsf{b}_{x_{1,i}}|^2 \mathrm{d}\mathfrak{m}_{1,i} - \fint_{B_s(y_{2,i})} |\nabla \mathsf{b}_{x_{2,i}}|^2 \mathrm{d}\mathfrak{m}_{2,i} \right| \right) > 0. \tag{3.62}$$

Theorem 3.11 shows that after passing to a subsequence, $(X_{j,i}, \mathsf{d}_{j,i}, \mathfrak{m}_{j,i}, x_{j,i})$ pmGH-converge to a pointed non-parabolic RCD$(0, N)$ space $(X, \mathsf{d}, \mathfrak{m}, x)$. With no loss of generality we can assume that $y_{j,i}$ converge to a point $y \in \overline{B}_R(x) \setminus B_r(x)$. Moreover the dominated convergence theorem with (3.60) yields that $f_{j,i}(s)$ L^1-strongly converge to $f(s) := \frac{s}{\mathfrak{m}(B_s(x))}$ in $L^1([1,\infty), \mathscr{H}^1)$. Thus theorem 3.12 allows

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us to conclude

$$\begin{aligned}&\left|\mathsf{b}_{x_{1,i}}(y_{1,i})-\mathsf{b}_{x_{2,i}}(y_{2,i})\right|+\left|\fint_{B_s(y_{1,i})}|\nabla \mathsf{b}_{x_{1,i}}|^2\mathrm{d}\mathfrak{m}_{1,i}-\fint_{B_s(y_{2,i})}|\nabla \mathsf{b}_{x_{2,i}}|^2\mathrm{d}\mathfrak{m}_{2,i}\right|\\&\to|\mathsf{b}_x(y)-\mathsf{b}_x(y)|+\left|\fint_{B_s(y)}|\nabla \mathsf{b}_x|^2\mathrm{d}\mathfrak{m}-\fint_{B_s(y)}|\nabla \mathsf{b}_x|^2\mathrm{d}\mathfrak{m}\right|=0\end{aligned}\tag{3.63}$$

which contradicts (3.62). □

3.3. Canonical representative of $|\nabla \mathsf{b}_x|$ and drifted Laplace operator $\mathscr{L}$

Throughout this subsection we continue to argue under the same assumptions as in the previous subsection, namely we fix a pointed non-parabolic RCD$(0, N)$ space $(X, \mathsf{d}, \mathfrak{m}, x)$. A main result of this subsection is the following.

THEOREM 3.15 Canonical pointwise representative of $|\nabla \mathsf{b}_x|$. *The limit*

$$\lim_{r\to 0^+}\fint_{B_r(z)}|\nabla \mathsf{b}_x|^2\mathrm{d}\mathfrak{m}\in[0,\infty)\tag{3.64}$$

exists for any $z\in X\setminus\{x\}$. Denoting by $|\nabla \mathsf{b}_x|^*(z)$ the square root of the limit, we have the following.

(1) *$|\nabla \mathsf{b}_x|^*$ is upper semicontinuous.*

(2) *Any point $z\in X\setminus\{x\}$ is a Lebesgue point of $|\nabla \mathsf{b}_x|^*$;*

$$\fint_{B_r(z)}\left||\nabla \mathsf{b}_x|^*-|\nabla \mathsf{b}_x|^*(z)\right|\mathrm{d}\mathfrak{m}\to 0.\tag{3.65}$$

(3) *We see that*

$$|\nabla \mathsf{b}_x|^*(z)=\limsup_{y\to z}|\nabla \mathsf{b}_x|^*(y),\quad \forall z\in X\setminus\{x\}.\tag{3.66}$$

It is worth mentioning that in the proof of the theorem above, we immediately show

$$|\nabla \mathsf{b}_x|^*(z)=\lim_{r\to 0^+}\operatorname*{ess\,sup}_{y\in B_r(z)}|\nabla \mathsf{b}_x|(y),\quad \forall\, z\in X\setminus\{x\}.\tag{3.67}$$

Actually theorem 3.15 with (3.67) is a direct consequence of the subharmonicity of $|\nabla \mathsf{b}_x|^2G_x$ stated in proposition 3.21 and general results on PI spaces in [**9**, Section 8.5]. Thus in the rest of this subsection, we focus on introducing the subharmonicity and its drifted ones which play important roles later.

REMARK 3.16. Let us recall the following well-known fact; if x is a Lebesgue point of a locally bounded function f defined on an open subset U of a PI space $(X, \mathsf{d}, \mathfrak{m})$,

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then

$$⨍_{B_r(x)} \psi\left(\varphi \circ f - \varphi(f(x))\right) \mathrm{d}\mathfrak{m} \to 0 \tag{3.68}$$

for all φ, $\psi \in C(\mathbb{R})$ with $\psi(0) = 0$. In particular x is also a Lebesgue point of $\varphi \circ f$. In order to prove the theorem above, we will apply this fact as $f = |\nabla \mathsf{b}_x|^2$, $\varphi(t) = \sqrt{|t|}$ and $\psi(t) = |t|$ with the following arguments.

Consider the following drifted Laplace operator $\mathscr{L}$ by

$$\mathscr{L} := \Delta + 2\langle \nabla \log G_x, \nabla \cdot \rangle. \tag{3.69}$$

See definition 3.18 for the precise definition. It should be emphasized that Colding studied $\mathscr{L}$ in [**16**] deeply in the smooth framework in order to prove the pointwise rigidity result, (2) of theorem 1.1 (see [**17**] for applications).

In the sequel, we follow his arguments, but extra delicate treatments on $\mathscr{L}$ are necessary in our setting because of lack of the smoothness. Firstly let us estimate the drifted term of (3.69) as follows.

Proposition 3.17. *We have*

$$|\nabla \log G_x|(y) \leqslant \frac{C(N)}{\mathsf{d}(x,y)}, \quad \textit{for } \mathfrak{m}\textit{-a.e. } y \in X \setminus \{x\}. \tag{3.70}$$

In particular

$$\|\nabla \log G_x\|_{L^\infty(X \setminus B_r(x))} \leqslant \frac{C(N)}{r}, \quad \forall\ r > 0. \tag{3.71}$$

Proof. By theorem 3.4,

$$|\nabla \log G_x| = \frac{|\nabla G_x|}{G_x} \leqslant C(N) \frac{H_x(\mathsf{d}(x,\cdot))}{F_x(\mathsf{d}(x,\cdot))}. \tag{3.72}$$

On the other hand by definition we have

$$F_x(s) = \int_s^\infty \frac{t}{\mathfrak{m}(B_t(x))} \mathrm{d}t \geqslant \int_s^\infty \frac{s}{\mathfrak{m}(B_t(x))} \mathrm{d}t = s H_x(s). \tag{3.73}$$

Combining this with (3.72) completes the proof. □

Let us recall the *sub/super harmonicity* of a function f on an open subset Ω of X. We say that f is *sub (or super, respectively) harmonic* on Ω if $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathsf{d}, \mathfrak{m})$ with

$$\int_\Omega -\langle \nabla u, \nabla \varphi \rangle \mathrm{d}\mathfrak{m} \geqslant 0, \quad (\text{or } \leqslant 0, \text{ respectively}) \tag{3.74}$$

for any $\varphi \in (\mathrm{Lip}_c)_+(\Omega, \mathsf{d})$. It directly follows that f is sub (or super, respectively) harmoninc on Ω if $f \in D_{\mathrm{loc}}(\Delta, \Omega)$ with $\Delta f \geqslant 0$ (or $\Delta f \leqslant 0$, respectively). See also [**50**].

Based on this observation, we are now in a position to define the $\mathscr{L}$-operator precisely as follows.

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DEFINITION 3.18 $\mathscr{L}$-operator and $\mathscr{L}$-sub/super harmonicity. *Let Ω be an open subset in $X \setminus \{x\}$.*

(1) *($\mathscr{L}$-operator) For $u \in D_{\mathrm{loc}}(\Delta, \Omega)$, let*

$$\mathscr{L}u := \Delta u + 2\langle \nabla \log G_x, \nabla u \rangle \in L^2_{\mathrm{loc}}(\Omega, \mathfrak{m}). \tag{3.75}$$

(2) *($\mathscr{L}$-sub/super harmonicity) A function $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathsf{d}, \mathfrak{m})$ is said to be $\mathscr{L}$-sub (or super, respectively) harmonic on Ω if*

$$\int_\Omega -\langle \nabla u, \nabla \varphi \rangle \mathrm{d}\mathfrak{m}_{G_x} \geqslant 0, \quad (\text{or } \leqslant 0, \text{ respectively}) \tag{3.76}$$

for any $\varphi \in (\mathrm{Lip}_c)_+(\Omega, \mathsf{d})$, where $\mathfrak{m}_{G_x}$ is the weighted Borel measure on X defined by

$$\mathfrak{m}_{G_x}(A) := \int_A G_x^2 \mathrm{d}\mathfrak{m}. \tag{3.77}$$

REMARK 3.19. The $\mathscr{L}$-operator can be defined as a measure valued one; for any $u \in D(\mathbf{\Delta}, \Omega)$, define

$$\mathscr{L}u := \mathbf{\Delta} u + 2\langle \nabla \log G_x, \nabla u \rangle \mathrm{d}\mathfrak{m}, \tag{3.78}$$

where $D(\mathbf{\Delta}, \Omega)$ is the domain of the measure valued Laplacian, see [**29**, **34**] for the detail. Then, even in the measure valued case, $\mathscr{L}$-sub/super harmonicity are also well-defined, and weak/strong maximum principles are justified. See also [**30**, **31**]. Although we avoid to use the measure valued Laplacian/$\mathscr{L}$-operator for simplicity in our presentation, however, for our main target in the sequel, $|\nabla \mathsf{b}_x|^2$, the measure valued $\mathscr{L}$-operator, $\mathscr{L}|\nabla \mathsf{b}_x|^2$ is well-defined.

In connection with this, it is easy to see that u is $\mathscr{L}$-sub (or $\mathscr{L}$-super, respectively) harmonic on Ω if and only if for any $\varphi \in (\mathrm{Lip}_c)_+(\Omega, \mathsf{d})$,

$$\int_\Omega -\langle \nabla \varphi, \nabla u \rangle + 2\varphi \langle \nabla \log G, \nabla u \rangle \mathrm{d}\mathfrak{m} \geqslant 0, \quad (\text{or } \leqslant 0, \text{ respectively}). \tag{3.79}$$

This observation will be a starting point in § 6.

Let us introduce a standard integration-by-parts formula for $\mathfrak{m}_{G_x}$.

PROPOSITION 3.20. *Let Ω be an open subset in $X \setminus \{x\}$ and let $u \in D_{\mathrm{loc}}(\Delta, \Omega)$. Then*

$$\int_\Omega \langle \nabla \varphi, \nabla u \rangle \mathrm{d}\mathfrak{m}_{G_x} = -\int_\Omega \varphi \cdot \mathscr{L}u \mathrm{d}\mathfrak{m}_{G_x} \tag{3.80}$$

for any $\varphi \in \mathrm{Lip}_c(\Omega, \mathsf{d})$. In particular u is $\mathscr{L}$-sub (or super, respectively) harmonic on Ω if $\mathscr{L}u \geqslant 0$ (or $\mathscr{L}u \leqslant 0$, respectively).

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Proof. The proof is quite standard;

$$\begin{aligned}\int_\Omega \langle \nabla\varphi, \nabla u\rangle \mathrm{d}\mathfrak{m}_{G_x} &= \int_\Omega \langle \nabla(G_x^2\varphi), \nabla u\rangle \mathrm{d}\mathfrak{m} - \int_\Omega \varphi\langle \nabla G_x^2, \nabla u\rangle \mathrm{d}\mathfrak{m}\\ &= -\int_\Omega G_x^2\varphi\Delta u \mathrm{d}\mathfrak{m} - \int_\Omega 2\varphi G_x\langle \nabla G_x, \nabla u\rangle \mathrm{d}\mathfrak{m}\\ &= -\int_\Omega \varphi\cdot \mathscr{L}u \mathrm{d}\mathfrak{m}_{G_x}. \end{aligned} \tag{3.81}$$

□

We are now in a position to prove a main result in this subsection, recall that the subharmonicity of $|\nabla \mathsf{b}_x|^2 G_x$ with results in [**9**, Section 8.5] implies theorem 3.15 (we can find the corresponding regularity results for the $\mathscr{L}$-operator in § 6).

Proposition 3.21 Subharmonicity of gradient of b_x. *We see that* $|\nabla \mathsf{b}_x|^2 G_x$ *is subharmonic on* $X\setminus\{x\}$ *and that* $|\nabla \mathsf{b}_x|^2$ *is* $\mathscr{L}$*-subharmonic on* $X\setminus\{x\}$.

Proof. First of all, we claim that $|\nabla \mathsf{b}_x|^2 \in W^{1,2}_{\mathrm{loc}}(X\setminus\{x\}, \mathsf{d}, \mathfrak{m})$. For any compact set $K\subset X\setminus\{x\}$, we can take a good cut-off function $\eta \in (\mathrm{Lip}_c)_+(X, \mathsf{d})\cap D(\Delta)$ such that $\eta\equiv 1$ in K, $\operatorname{supp}\eta \subset X\setminus\{x\}$ and $|\nabla\eta| + |\Delta\eta| < C$ (see [**48**, Lemma 3.2] for such an existence). Letting $h := \eta \mathsf{b}_x$, then [**34**, Propositions 3.3.18 and 3.3.22] shows that $|\nabla h|^2 \in W^{1,2}(X\setminus\{x\}, \mathsf{d}, \mathfrak{m})$, which implies $|\nabla \mathsf{b}_x|^2 \in W^{1,2}_{\mathrm{loc}}(X\setminus\{x\}, \mathsf{d}, \mathfrak{m})$ because K is arbitrary.

Next recalling

$$\begin{aligned}-\int_{X\setminus\{x\}} \langle \nabla\psi, \nabla(fh)\rangle \mathrm{d}\mathfrak{m} &= \int_{X\setminus\{x\}} \psi\left(h\Delta f + 2\langle \nabla f, \nabla h\rangle\right)\mathrm{d}\mathfrak{m}\\ &\quad - \int_{X\setminus\{x\}} \langle \nabla(\psi f), \nabla h\rangle \mathrm{d}\mathfrak{m}\end{aligned} \tag{3.82}$$

for all $\psi \in \mathrm{Lip}_c(X\setminus\{x\}, \mathsf{d})$, $f\in D_{\mathrm{loc}}(\Delta, X\setminus\{x\})$ and $h\in W^{1,2}_{\mathrm{loc}}(X\setminus\{x\}, \mathsf{d}, \mathfrak{m})$, we apply this as $f = \mathsf{b}_x^2$, $h = |\nabla \mathsf{b}_x|^2$ to get

$$\begin{aligned}&-\int_{X\setminus\{x\}} \langle \nabla\psi, \nabla(\mathsf{b}_x^2|\nabla \mathsf{b}_x|^2)\rangle \mathrm{d}\mathfrak{m}\\ &= \int_{X\setminus\{x\}} \psi\left(|\nabla \mathsf{b}_x|^2\Delta \mathsf{b}_x^2 + 2\langle \nabla \mathsf{b}_x^2, \nabla|\nabla \mathsf{b}_x|^2\rangle\right)\mathrm{d}\mathfrak{m} - \int_{X\setminus\{x\}} \langle \nabla(\psi \mathsf{b}_x^2), \nabla|\nabla \mathsf{b}_x|^2\rangle \mathrm{d}\mathfrak{m}\\ &= \int_{X\setminus\{x\}} \psi\left(2N|\nabla \mathsf{b}_x|^4 + 2\langle \nabla \mathsf{b}_x^2, \nabla|\nabla \mathsf{b}_x|^2\rangle\right)\mathrm{d}\mathfrak{m} - \int_{X\setminus\{x\}} \langle \nabla(\psi \mathsf{b}_x^2), \nabla|\nabla \mathsf{b}_x|^2\rangle \mathrm{d}\mathfrak{m},\end{aligned} \tag{3.83}$$

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where we used (3.20). On the other hand, since $\mathsf{b}_x^2|\nabla \mathsf{b}_x|^2 = \frac{|\nabla \mathsf{b}_x^2|^2}{4}$, the Bochner inequality allows us to estimate the left-hand-side above as follows;

$$\begin{aligned} -\int_{X\setminus\{x\}} \langle \nabla\psi, \nabla(\mathsf{b}_x^2|\nabla \mathsf{b}_x|^2)\rangle \mathrm{d}\mathfrak{m} &= -\frac{1}{4}\int_{X\setminus\{x\}} \langle \nabla\psi, \nabla|\nabla \mathsf{b}_x^2|^2\rangle \mathrm{d}\mathfrak{m} \\ &\geqslant \frac{1}{2}\int_{X\setminus\{x\}} \psi\left(\frac{(\Delta \mathsf{b}_x^2)^2}{N} + \langle \nabla\Delta \mathsf{b}_x^2, \nabla \mathsf{b}_x^2\rangle\right) \mathrm{d}\mathfrak{m} \\ &= \int_{X\setminus\{x\}} \psi\left(2N|\nabla \mathsf{b}_x|^4 + N\langle \nabla|\nabla \mathsf{b}_x|^2, \nabla \mathsf{b}_x^2\rangle\right) \mathrm{d}\mathfrak{m}, \end{aligned} \tag{3.84}$$

where we used (3.20) again. Therefore it follows from (3.83) and (3.84) that

$$\int_{X\setminus\{x\}} (2-N)\psi\langle \nabla \mathsf{b}_x^2, \nabla|\nabla \mathsf{b}_x|^2\rangle \mathrm{d}\mathfrak{m} - \int_{X\setminus\{x\}} \langle \nabla(\psi \mathsf{b}_x^2), \nabla|\nabla \mathsf{b}_x|^2\rangle \mathrm{d}\mathfrak{m} \geqslant 0. \tag{3.85}$$

Let us prove that this inequality (3.85) implies the conclusions. Actually as done in (3.83) and (3.84), it follows from Leibniz' rule that[3]

$$\begin{aligned} &-\int_{X\setminus\{x\}} \langle \nabla\psi, \nabla|\nabla \mathsf{b}_x|^2\rangle \mathrm{d}\mathfrak{m}_{G_x} \\ &= \int_{X\setminus\{x\}} \left(-\langle \nabla(\varphi \mathsf{b}_x^2), \nabla|\nabla \mathsf{b}_x|^2\rangle + (2-N)\varphi\langle \nabla \mathsf{b}_x^2, \nabla|\nabla \mathsf{b}_x|^2\rangle\right) \mathrm{d}\mathfrak{m} \geqslant 0 \end{aligned} \tag{3.87}$$

holds, where $\varphi = \mathsf{b}_x^{-2} G_x^2 \psi$, which proves the $\mathscr{L}$-subharmonicity of $|\nabla \mathsf{b}_x|^2$ on $X \setminus \{x\}$.

Similarly we have[4]

$$\begin{aligned} &-\int_{X\setminus\{x\}} \left\langle \nabla\left(|\nabla \mathsf{b}_x|^2 G_x\right), \nabla\psi\right\rangle \mathrm{d}\mathfrak{m} \\ &= \int_{X\setminus\{x\}} \left(-\langle \nabla(\varphi \mathsf{b}_x^2), \nabla|\nabla \mathsf{b}_x|^2\rangle + (2-N)\varphi\langle \nabla \mathsf{b}_x^2, \nabla|\nabla \mathsf{b}_x|^2\rangle\right) \mathrm{d}\mathfrak{m} \geqslant 0, \end{aligned} \tag{3.89}$$

where $\varphi = \psi \mathsf{b}_x^{-N}$, which proves the subharmonicity of $|\nabla \mathsf{b}_x|^2 G_x$ on $X \setminus \{x\}$. □

[3]This is also justified by using the measure valued $\mathscr{L}$-operator because

$$\begin{aligned} \mathscr{L}|\nabla \mathsf{b}_x|^2 &= \mathbf{\Delta}|\nabla \mathsf{b}_x|^2 + 2\langle \nabla \log G, \nabla|\nabla \mathsf{b}_x|^2\rangle \mathrm{d}\mathfrak{m} \\ &= \mathbf{\Delta}|\nabla \mathsf{b}_x|^2 + (2-N)\mathsf{b}_x^{-2}\langle \nabla \mathsf{b}_x^2, \nabla|\nabla \mathsf{b}_x|^2\rangle \mathrm{d}\mathfrak{m}. \end{aligned} \tag{3.86}$$

[4]This is also justified by using the measure valued Laplacian because

$$\begin{aligned} \mathbf{\Delta}(|\nabla \mathsf{b}_x|^2 G_x) &= G\mathbf{\Delta}|\nabla \mathsf{b}_x|^2 + 2\langle \nabla G, \nabla|\nabla \mathsf{b}_x|^2\rangle \\ &= \mathsf{b}_x^{2-N}\mathbf{\Delta}|\nabla \mathsf{b}_x|^2 + (2-N)\mathsf{b}_x^{-N}\langle \nabla \mathsf{b}_x^2, \nabla|\nabla \mathsf{b}_x|^2\rangle \\ &= \mathsf{b}_x^{-N}\left(\mathsf{b}_x^2\mathbf{\Delta}|\nabla \mathsf{b}_x|^2 + (2-N)\langle \nabla \mathsf{b}_x^2, \nabla|\nabla \mathsf{b}_x|^2\rangle\right). \end{aligned} \tag{3.88}$$

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In the sequel, we will use the simplified notation $|\nabla \mathsf{b}_x| = |\nabla \mathsf{b}_x|^*$ for the simplicity on our presentations. In § 6, we will also provide the fundamental properties on the $\mathscr{L}$-operator with the proofs, including the strong maximum principle, which comes from the general theory on PI spaces. They will play important roles in the sequel.

3.4. Sharp gradient estimate on $\mathbf{b}_x$

Fix a pointed non-parabolic RCD(0, N) space $(X, \mathsf{d}, \mathfrak{m}, x)$ and assume that the N-volume density ν_x at x is finite;

$$\nu_x = \lim_{r\to 0^+} \frac{\mathfrak{m}(B_r(x))}{r^N} < \infty. \tag{3.90}$$

Note that this condition does not imply that the essential dimension is equal to N because of an example; $([0, \infty), \mathsf{d}_{\mathrm{Euc}}, r^{N-1}\mathrm{d}r)$ is an RCD(0, N) space with the finite N-volume density at the origin.

The main result of this section is the following. Recall $\mathscr{C}_N = (N(N-2))^{\frac{1}{N-2}}$.

THEOREM 3.22 Sharp gradient estimate of b_x. *We have*

$$|\nabla \mathsf{b}_x|(y) \leqslant \mathscr{C}_N \nu_x^{\frac{1}{N-2}}, \quad \forall\, y \in X \setminus \{x\}. \tag{3.91}$$

Proof. The proof is divided into several steps as follows.

Step 1. Let us prove that $|\nabla \mathsf{b}_x| \in L^\infty(X, \mathfrak{m})$. By calculus rules

$$|\nabla \mathsf{b}_x| = \frac{1}{N-2} G_x^{\frac{N-1}{2-N}} |\nabla G_x|. \tag{3.92}$$

Let

$$C_{\mathrm{opt},x} := \lim_{r\to 0} \operatorname*{ess\,sup}_{y\in B_r(x)} |\nabla \mathsf{b}_x|(y) \left(= \lim_{r\to 0} \sup_{y\in B_r(x)} |\nabla \mathsf{b}_x|(y) \right). \tag{3.93}$$

Plugging in (3.51), (3.17) and (3.7) we see that

$$C_{\mathrm{opt},x} \leqslant C(N) \nu_x^{\frac{1}{N-2}}. \tag{3.94}$$

Thus we can take r sufficiently small such that

$$|\nabla \mathsf{b}_x|(y) \leqslant C(N) \nu_x^{\frac{1}{N-2}}, \quad \mathfrak{m}\text{-a.e. } y \in B_r(x) \setminus \{x\}. \tag{3.95}$$

On the other hand, combining (3.92) with theorem 3.4 yields that for $\mathfrak{m}$-a.e. $y \in X \setminus \{x\}$, we have

$$|\nabla \mathsf{b}_x|(y) \leqslant C(N) F_x(\mathsf{d}(x,y))^{\frac{N-1}{2-N}} H_x(\mathsf{d}(x,y)). \tag{3.96}$$

Note that it holds that $sH_x(s) \leqslant F_x(s)$ for any $s \in [1, \infty)$. Thus, choosing R sufficiently large, then for any $y \in X \setminus B_R(x)$, we have

$$|\nabla \mathsf{b}_x|(y) \leqslant C(N) \frac{H_x(\mathsf{d}(x,y))}{F_x(\mathsf{d}(x,y))^{\frac{N-1}{N-2}}} \leqslant C(N) V_X + 1 \tag{3.97}$$

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because of (3.8). Moreover it follows from the continuity of right-hand-side of (3.96) that $|\nabla \mathsf{b}_x|$ is bounded $\mathfrak{m}$-a.e. in $\overline{B_R(x)} \setminus B_r(x)$, which is a compact subset. Therefore $|\nabla \mathsf{b}_x| \in L^\infty(X, \mathfrak{m})$.

Step 2. Let us prove that $|\nabla \mathsf{b}_x|(y) \leqslant C_{\mathrm{opt},\,x}$ for any $y \in X \setminus \{x\}$.

Fix $0 < r < R$ and choose arbitrarily $\varepsilon > 0$. By corollary 3.5, we can find $R_0 > R$ sufficiently large such that

$$G_x(y) < \varepsilon, \quad \forall\, y \in X \setminus B_{R_0}(x) \tag{3.98}$$

Take $0 < r_0 < r$ with

$$|\nabla \mathsf{b}_x|^2(y) < C^2_{\mathrm{opt},x} + \varepsilon, \quad \forall\, y \in \overline{B}_{r_0}(x) \setminus \{x\}. \tag{3.99}$$

Let $L := \left\| |\nabla \mathsf{b}_x| \right\|_{L^\infty(X,\mathfrak{m})}$ and set

$$u_x := |\nabla \mathsf{b}_x|^2 G_x - (C^2_{\mathrm{opt},x} + \varepsilon) G_x - L^2 \varepsilon, \tag{3.100}$$

which is upper semicontinuous and subharmonic on $X \setminus \{x\}$ because of proposition 3.21.

Let $\Omega := B_{R_0}(x) \setminus \overline{B_{r_0}(x)}$. Applying the weak maximum principle for upper semicontinuous subharmonic functions [**31**, Proposition 1.15], we see that

$$\sup_{\Omega} u_x = \sup_{\partial\Omega} u_x \leqslant 0 \tag{3.101}$$

which proves

$$|\nabla \mathsf{b}_x|^2(y) \leqslant C^2_{\mathrm{opt},x} + \frac{L^2 \varepsilon}{G_x(y)} + \varepsilon, \quad \forall\, y \in \Omega. \tag{3.102}$$

This observation allows us to conclude that $|\nabla \mathsf{b}_x| \leqslant C_{\mathrm{opt},\,x}$ for $\mathfrak{m}$-a.e. after letting $\varepsilon \to 0^+$ under fixing r, R. Thus by theorem 3.15 we know that $|\nabla \mathsf{b}_x|(y) \leqslant C_{\mathrm{opt},\,x}$ for any $y \in X \setminus \{x\}$.

Step 3. We claim that $C_{\mathrm{opt},\,x} = \mathscr{C}_N \nu_x^{\frac{1}{N-2}}$, where this completes the proof. Thanks to (2.30), proposition 3.8 and theorem 3.12, we know that for any $0 < \delta < 1$

$$\fint_{B_r(x) \setminus B_{\delta r}(x)} \left| |\nabla \mathsf{b}_x|^2 - \mathscr{C}_N^2 \nu_x^{\frac{2}{N-2}} \right| \mathrm{d}\mathfrak{m} \to 0, \quad \text{as } r \to 0^+. \tag{3.103}$$

Take $y_i \to x$ satisfying $|\nabla \mathsf{b}_x|(y_i) \to C_{\mathrm{opt},\,x}$ and let $r_i := \mathsf{d}(x, y_i)$ and consider rescaled spaces;

$$(X_i, \mathsf{d}_i, \mathfrak{m}_i) := \left(X, \frac{1}{r_i}\mathsf{d}, \frac{1}{r_i^N}\mathfrak{m} \right). \tag{3.104}$$

Applying the weak Harnack inequality for $\mathscr{L}$-superharmonic functions, proposition 6.5, to a lower semicontinuous $\mathscr{L}$-superharmonic function $C^2_{\mathrm{opt},\,x} - |\nabla \mathsf{b}_x^{X_i}|^2 \geqslant 0$, we

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have

$$\left(\fint_{B^{\mathsf{d}_i}_{\frac{1}{4}}(y_i)} \left|C^2_{\mathrm{opt},x} - |\nabla \mathsf{b}^{X_i}_x|^2\right|^p \mathrm{d}\mathfrak{m}_i\right)^{1/p} \leqslant C(N)\left(C^2_{\mathrm{opt},x} - |\nabla \mathsf{b}^{X_i}_x|^2(y_i)\right) = C(N)\left(C^2_{\mathrm{opt},x} - |\nabla \mathsf{b}_x|^2(y_i)\right) \to 0, \tag{3.105}$$

where $p = p(N) > 0$. Thus recalling $|\nabla \mathsf{b}_x| \in L^\infty(X \setminus \{x\}, \mathfrak{m})$, it holds that

$$\fint_{B^{\mathsf{d}_i}_{\frac{1}{4}}(y_i)} \left|C^2_{\mathrm{opt},x} - |\nabla \mathsf{b}^{X_i}_x|^2\right| \mathrm{d}\mathfrak{m}_i \to 0, \tag{3.106}$$

namely

$$\fint_{B_{\frac{r_i}{4}}(y_i)} \left|C^2_{\mathrm{opt},x} - |\nabla \mathsf{b}_x|^2\right| \mathrm{d}\mathfrak{m} \to 0. \tag{3.107}$$

On the other hand, (3.103) implies

$$\fint_{B_{\frac{r_i}{4}}(y_i)} \left|\mathscr{C}^2_N \nu_x^{\frac{2}{N-2}} - |\nabla \mathsf{b}_x|^2\right| \mathrm{d}\mathfrak{m} \to 0. \tag{3.108}$$

Thus by (3.107) and (3.108) we have $C_{\mathrm{opt},\,x} = \mathscr{C}_N \nu_x^{\frac{1}{N-2}}$. □

We provide direct consequences of theorem 3.22. Firstly we improve theorems 3.12 and 3.15 removing the singular base point.

Corollary 3.23 Improvement of the convergence of b. *Let us consider a pmGH convergent sequence of pointed non-parabolic* $\mathrm{RCD}(0, N)$ *spaces*

$$(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i) \overset{\mathrm{pmGH}}{\to} (X, \mathsf{d}, \mathfrak{m}, x) \tag{3.109}$$

with (3.45) *and* $F_{x_i}(1) \to F_x(1)$. *Then* b_{x_i} $W^{1,p}_{\mathrm{loc}}$*-strongly, and locally uniformly converge to* b_x *on* X *for any* $p < \infty$.

Proof. Note

$$\frac{1}{\mathfrak{m}_i(B_1(x_i))}\int_{B_\varepsilon(x_i)} |\nabla \mathsf{b}_{x_i}|^p \mathrm{d}\mathfrak{m}_i \leqslant \frac{\mathfrak{m}_i(B_\varepsilon(x_i))}{\mathfrak{m}_i(B_1(x_i))}\mathscr{C}^p_N \cdot \sup_j \nu_{x_j}^{\frac{p}{N-2}} \to \frac{\mathfrak{m}(B_\varepsilon(x))}{\mathfrak{m}(B_1(x))}\mathscr{C}^p_N \cdot \sup_j \nu_{x_j}^{\frac{p}{N-2}}. \tag{3.110}$$

Since the right-hand-side of (3.110) is small if ε is small, combining this with theorem 3.12 completes the proof. □

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COROLLARY 3.24. *Let* $\mathsf{b}_x(x) := 0$. *Then* b_x *is* $\mathscr{C}_N \nu_x^{\frac{1}{N-2}}$*-Lipschitz with the Lipschitz constant* $\mathscr{C}_N \nu_x^{\frac{1}{N-2}}$. *Moreover letting*

$$|\nabla \mathsf{b}_x|(x) := \mathscr{C}_N \nu_x^{\frac{1}{N-2}}, \tag{3.111}$$

we have the same conclusions as in theorem 3.15 with (3.67) even for the base point $z = x$. *More strongly, we have for any* $p < \infty$.

$$\fint_{B_r(x)} \left| \nabla \left(\mathsf{b}_x - \mathscr{C}_N \nu_x^{\frac{1}{N-2}} \mathsf{d}_x \right) \right|^p \mathrm{d}\mathfrak{m} \to 0, \quad r \to 0^+. \tag{3.112}$$

In particular, $|\nabla \mathsf{b}_x|(y_i) \to \mathscr{C}_N \nu_x^{\frac{1}{N-2}}$ *for some convergent sequence* $y_i \to x$.

Proof. Thanks to (3.16) and (3.6), putting $\mathsf{b}_x(x) := 0$ gives a unique continuous extention of b_x on X. Since $\{x\}$ is null with respect to the 2-Sobolev capacity because of the finiteness of ν_x (see [**9**]), we know $\mathsf{b}_x \in W^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$. Then the first statement comes from arguments in the last step in the proof of theorem 3.22 with the (local) Sobolev-to-Lipschitz property and corollary 3.23. Moreover corollary 3.23 with proposition 3.8 allows us to obtain (3.112). □

As the final application of theorem 3.22, we determine the small scale asymptotics of the gradient of the Green function.

COROLLARY 3.25. *We have*

$$\lim_{r \to 0^+} \sup_{y \in B_r(x)} \frac{|\nabla G_x|(y)}{\mathsf{d}(x,y)^{1-N}} = \frac{1}{N\nu_x}. \tag{3.113}$$

Proof. Since (3.51), (3.92) and corollary 3.24 yield (under a suitable limit $y \to x$)

$$\begin{aligned} \frac{|\nabla G_x|(y)}{\mathsf{d}(x,y)^{1-N}} &= (N-2) \left(\mathsf{d}(x,y) G_x(y)^{\frac{1}{N-2}} \right)^{N-1} |\nabla \mathsf{b}_x|(y) \\ &\to (N-2) \left(\frac{1}{\mathscr{C}_N \nu_x^{\frac{1}{N-2}}} \right)^{N-1} \mathscr{C}_N \nu_x^{\frac{1}{N-2}} \\ &= \frac{1}{N\nu_x} \end{aligned} \tag{3.114}$$

we conclude. □

Next we provide an asymptotic formula as $y \to \infty$.

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COROLLARY 3.26 Sharp gradient asymptotics. *For any $p < \infty$ we have*

$$\fint_{B_r(x)} \left| \nabla \left(\mathsf{b}_x - \mathscr{C}_N V_X^{\frac{1}{N-2}} \mathsf{d}_x \right) \right|^p \mathrm{d}\mathfrak{m} \to 0, \quad r \to \infty, \tag{3.115}$$

therefore

$$\fint_{B_r(x)} \left| |\nabla \mathsf{b}_x|^2 - \mathscr{C}_N^2 V_X^{\frac{2}{N-2}} \right| \mathrm{d}\mathfrak{m} \to 0, \quad r \to \infty. \tag{3.116}$$

In particular

$$|\nabla \mathsf{b}_x|(y_i) \to \mathscr{C}_N V_X^{\frac{1}{N-2}}, \tag{3.117}$$

equivalently

$$\frac{|\nabla G_x|(y_i)}{\mathsf{d}(x, y_i)^{1-N}} \to \frac{1}{N V_X} \tag{3.118}$$

holds for some sequence $y_i \in X$ with $\mathsf{d}(x, y_i) \to \infty$.

Proof. Firstly we discuss the case when $V_X = 0$. Then we can follow the same arguments as in the proof of [**16**, Theorem 2.12]. Namely we can estimate as $\mathsf{d}(x, y) \to \infty$,

$$\begin{aligned} |\nabla \mathsf{b}_x|(y) &= \frac{G_x^{\frac{N-1}{2-N}}}{N-2} |\nabla G_x|(y) \\ &\leqslant \frac{G_x(y)^{\frac{N-1}{2-N}}}{N-2} \cdot C(N) \cdot G_x(y) \cdot \frac{1}{\mathsf{d}(x, y)} \\ &\leqslant C(N) \left(\frac{F_x(\mathsf{d}(x, y))}{\mathsf{d}(x, y)^{2-N}} \right)^{\frac{1}{2-N}} \to 0, \end{aligned} \tag{3.119}$$

where we used the gradient estimates on positive harmonic functions obtained in [**42**, Theorem 1.2] in the first inequality above and we also used (3.8) and (3.16) in the last inequality and in the limit. Thus we obtain the conclusion in this case.

Next we consider the case when $V_X > 0$. The first statement, (3.115), is a direct consequence of corollary 3.23 and (2) of corollary 2.9. The remaining one (3.117) (or (3.118)) follows from an argument similar to the proof of corollary 3.25. □

Based on corollary 3.23, we can prove the following whose proof is the same to that of corollary 3.14. Thus we omit the proof.

COROLLARY 3.27. *For all $N > 2$, $0 < \varepsilon < 1$, $0 < \tau < 1$, $v > 0$, $1 \leqslant p < \infty$ and $\varphi \in L^1([1, \infty), \mathscr{H}^1)$ there exists $\delta = \delta(N, \varepsilon, \tau, v, p, \varphi) > 0$ such that if two pointed*

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non-parabolic $\mathrm{RCD}(0,N)$ *spaces* $(X_i,\mathsf{d}_i,\mathfrak{m}_i,x_i)(i=1,2)$ *satisfy* $\nu_{x_i}\leqslant v<\infty$,

$$\frac{s}{\mathfrak{m}_i(B_s(x_i))}\leqslant\varphi(s),\quad \text{for } \mathscr{H}^1\text{-a.e. } s\in[1,\infty) \tag{3.120}$$

and

$$\mathsf{d}_{\mathrm{pmGH}}\left((X_1,\mathsf{d}_1,\mathfrak{m}_1,x_1),(X_2,\mathsf{d}_2,\mathfrak{m}_2,x_2)\right)<\delta, \tag{3.121}$$

then

$$|\mathsf{b}_{x_1}(y_1)-\mathsf{b}_{x_2}(y_2)|+\left|\fint_{B_s(y_1)}|\nabla\mathsf{b}_{x_1}|^p\mathrm{d}\mathfrak{m}_1-\fint_{B_s(y_2)}|\nabla\mathsf{b}_{x_2}|^p\mathrm{d}\mathfrak{m}_2\right|<\varepsilon \tag{3.122}$$

for all $\tau\leqslant s\leqslant\tau^{-1}$ *and* $y_i\in B_{\tau^{-1}}(x_i)$ *satisfying that* y_1 δ*-close to* y_2 *with respect to (3.121).*

Finally let us end this section by giving the following corollary which generalizes [**26**, Theorem 3.5] to the RCD setting. This corollary is pointed out by the reviewer. We thank the reviewer.

COROLLARY 3.28. *The function* $f_x:(0,\infty)\to[0,\infty)$ *defined by*

$$f_x(t):=\sup_{\{\mathsf{b}_x=t\}}|\nabla\mathsf{b}_x| \tag{3.123}$$

is monotone non-increasing, where recall that we assume $\nu_x<\infty$. *Moreover we have*

$$\lim_{t\to0^+}f_x(t)=\mathscr{C}_N\nu_x^{\frac{1}{N-2}} \tag{3.124}$$

and

$$\lim_{t\to\infty}f_x(t)=\mathscr{C}_N V_X^{\frac{1}{N-2}}. \tag{3.125}$$

Proof. For the first statement, we just follow the proof of [**26**, Theorem 3.5]. Namely our goal is to prove for any $t>0$

$$|\nabla\mathsf{b}_x|\leqslant\sup_{\{\mathsf{b}_x=t\}}|\nabla\mathsf{b}_x|,\quad \text{on } \{\mathsf{b}_x\geqslant t\} \tag{3.126}$$

because then for any $s\leqslant t$ we have

$$f_x(s)=\sup_{\{\mathsf{b}_x\geqslant s\}}|\nabla\mathsf{b}_x|\geqslant\sup_{\{\mathsf{b}_x\geqslant t\}}|\nabla\mathsf{b}_x|=f_x(t) \tag{3.127}$$

which completes the proof of the first statement. Thus let us focus on the proof of (3.126). Note that G_x is $\mathcal{L}$-harmonic on $X\setminus\{x\}$. Fixing a sufficiently large $T>t$, consider an $\mathcal{L}$-subharmonic function φ on $X\setminus\{x\}$ defined by

$$\varphi(y):=|\nabla\mathsf{b}_x|^2(y)-f_x(t)-\frac{\mathscr{C}_N^2\nu_x^{\frac{2}{N-2}}}{T^{N-2}G_x(y)}. \tag{3.128}$$

It is trivial that $\varphi\leqslant0$ holds on $\{\mathsf{b}_x=t\}\cup\{\mathsf{b}_x=T\}$. Thus the strong maximum principle, proposition 6.7, shows $\varphi\leqslant0$ on $\{t\leqslant\mathsf{b}_x\leqslant T\}$. Then letting $T\to\infty$ proves (3.126).

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On the other hand (3.124) is a direct consequence of theorem 3.22 and corollary 3.24. Thus let us focus on the proof of (3.125). The proof is divided into the following 2 cases.

Case 1: $V_X = 0$.

By the same argument as in (3.97), we see that for any $\epsilon > 0$, there exists $R > 1$ such that

$$|\nabla \mathsf{b}_x|(y) \leqslant C(N)V_X + \epsilon = \epsilon \tag{3.129}$$

holds for any $y \in X \setminus B_R(x)$, namely we have $\limsup_{\mathsf{d}_x(y)\to\infty} |\nabla \mathsf{b}_x|(y) \leqslant C(N)V_X = 0$ which proves (3.124) in this case.

Case 2: $V_X > 0$.

In this case the proof is similar to *Step 3* of the proof of theorem 3.22 with (3.126). Let $C_{\mathrm{opt},\infty} := \lim_{t\to\infty} f_x(t)$. Take $x_t \in \{\mathsf{b}_x = t\}$ with $|\nabla \mathsf{b}_x|(x_t) = f_x(t)$, and let $0\epsilon_t := f_x(t)^2 - C^2_{\mathrm{opt},\infty} \geqslant 0$. Note that $C^2_{\mathrm{opt},\infty} - |\nabla \mathsf{b}_x|^2 + \varepsilon_t \geqslant 0$ holds on $\{\mathsf{b}_x \geqslant t\}$.

On the other hand, thanks to (3.52), for any sufficiently large $t > 1$, we have $B_{\delta t}(x_t) \subset \{\mathsf{b}_x \geqslant \frac{t}{2}\}$, where $\delta = \delta(N, V_X)$ is a positive constant depending only on N, V_X. In particular

$$C^2_{\mathrm{opt},\infty} - |\nabla \mathsf{b}_x|^2 + \varepsilon_{\frac{t}{2}} \geqslant 0, \quad \text{on } B_{\delta t}(x_t). \tag{3.130}$$

Then, after a rescaling $t^{-1}\mathsf{d}$, $\mathfrak{m}(B_t(x))^{-1}\mathfrak{m}$, applying the weak Harnack inequality for $\mathcal{L}$-superharmonic function, proposition 6.5, to $C^2_{\mathrm{opt},\infty} - |\nabla \mathsf{b}_x|^2 + \varepsilon_{\frac{t}{2}} \geqslant 0$, we have as $t \to \infty$

$$\left(\fint_{B_{\frac{t}{4}}(x_t)} \left| C^2_{\mathrm{opt},\infty} - |\nabla \mathsf{b}_x|^2 + \varepsilon_{\frac{t}{2}} \right|^p \mathrm{d}\mathfrak{m} \right)^{1/p} \leqslant C(N) \left(C^2_{\mathrm{opt},\infty} - |\nabla \mathsf{b}_x|^2(x_t) + \varepsilon_{\frac{t}{2}} \right) \to 0, \tag{3.131}$$

for some $p = p(N) > 0$, namely

$$\fint_{B_{\frac{t}{4}}(x_t)} \left| C^2_{\mathrm{opt},\infty} - |\nabla \mathsf{b}_x|^2 \right| \mathrm{d}\mathfrak{m} \to 0. \tag{3.132}$$

Combining this with (3.116) yields $C_{\mathrm{opt},\infty} = \mathscr{C}_N V_X^{\frac{1}{N-2}}$ which completes the proof of (3.125). □

4. Rigidity to N-metric measure cone

We are now in a position to prove the main results. Fix a finite $N > 2$.

4.1. Rigidity

Let us prove the desired rigidity result based on theorems 3.15 and 3.22 (see (3.25) for the definition of $\mathscr{C}_N$).

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THEOREM 4.1 Rigidity to N-metric measure cone. *Let $(X, \mathsf{d}, \mathfrak{m}, x)$ be a pointed non-parabolic* $\mathrm{RCD}(0, N)$ *space with the finite N-volume density $\nu_x < \infty$. If there exists a point $y \in X \setminus \{x\}$ such that*

$$|\nabla \mathsf{b}_x|(y) = \mathscr{C}_N \nu_x^{\frac{1}{N-2}}, \tag{4.1}$$

then $(X, \mathsf{d}, \mathfrak{m}, x)$ is isomorphic to the N-metric measure cone with the pole over an $\mathrm{RCD}(N-2, N-1)$ *space.*

Proof. First of all, let us prove that $X \setminus \{x\}$ is connected. Let U be a connected component of $X \setminus \{x\}$. As mentioned in the beginning of the proof of corollary 3.24, since $\{x\}$ has a null 2-capacity, the indicator function χ_U of U is in $H^{1,2}_{\mathrm{loc}}(X, \mathsf{d}, \mathfrak{m})$ with $|\nabla \chi_U| = 0$. Thus the (local) Sobolev-to-Lipschitz property shows $\chi_U = 1$ for $\mathfrak{m}$-a.e., which implies $X \setminus \{x\} = U$. Thus $X \setminus \{x\}$ is connected.

Then applying the strong maximum principle, proposition 6.7, for $|\nabla \mathsf{b}_x|^2$ yields $|\nabla \mathsf{b}_x| \equiv \mathscr{C}_N \nu_x^{\frac{1}{N-2}}$ in $X \setminus \{x\}$. Letting $u := \mathsf{b}_x^2 / (2\mathscr{C}_N^2 \nu_x^{\frac{2}{N-2}})$, we have $|\nabla \sqrt{2u}|^2 = 1$ and thus, by (3.20), $\Delta u = N$ for $\mathfrak{m}$-a.e. These observations allow us to apply [**31**, Theorem 5.1] (see also Theorem 2.8) to get the conclusion. □

In particular, when restricted to points with Euclidean tangent spaces, theorem 4.1 implies an interesting corollary; compare with [**16**, Theorem 3.1].

COROLLARY 4.2. *We have the following.*

(1) *Let $(X, \mathsf{d}, \mathfrak{m})$ be a non-parabolic* $\mathrm{RCD}(0, N)$ *space for some integer $N \geqslant 3$ with the finite N-volume density $\nu_x < \infty$ at an N-regular point x. If there exists $z \in X \setminus \{x\}$ such that $|\nabla \mathsf{b}_x|(z) = \mathscr{C}_N \nu_x^{\frac{1}{N-2}}$ holds, then $(X, \mathsf{d}, \mathfrak{m})$ is isometric to $(\mathbb{R}^N, \mathsf{d}_{\mathbb{R}^N}, c\mathscr{H}^N)$ for some positive constant $c > 0$.*

(2) *If a non-parabolic non-collapsed* $\mathrm{RCD}(0, N)$ *space $(X, \mathsf{d}, \mathscr{H}^N)$ for some integer $N \geqslant 3$ with the finite N-volume density $\nu_x < \infty$ at a point $x \in X$ satisfies $|\nabla \mathsf{b}_x|(z) \geqslant \mathscr{C}_N \omega_N^{\frac{1}{N-2}}$ for some $z \in X \setminus \{x\}$, then $(X, \mathsf{d}, \mathscr{H}^N)$ is isometric to $(\mathbb{R}^N, \mathsf{d}_{\mathbb{R}^N}, \mathscr{H}^N)$.*

Proof. Let us prove (1). Theorem 4.1 yields that $(X, \mathsf{d}, \mathfrak{m})$ is isomorphic to an N-metric measure cone over an $\mathrm{RCD}(N-2, N-1)$ space. In particular it must be isomorphic to a tangent cone at x. Thus we conclude.

Next let us prove (2). The sharp gradient estimate, theorem 3.22, yields

$$\lim_{r \to 0^+} \frac{\mathscr{H}^N(B_r(x))}{\omega_N r^N} \geqslant 1, \tag{4.2}$$

thus x is an N-regular point because of [**20**, Corollary 1.7] (recall (2.15)). Then the conclusion follows from the first statement (1). □

4.2. Almost rigidity

Finally, let us prove the following almost rigidity theorem of b_x:

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THEOREM 4.3 Almost rigidity. *For all* $N > 2$, $0 < \varepsilon < 1$, $0 < r < R < \infty$, $\nu > 0$, $1 \leqslant p < \infty$ *and* $\varphi \in L^1([1, \infty), \mathscr{H}^1)$ *there exists* $\delta := \delta(N, \varepsilon, r, R, \nu, p, \varphi) > 0$ *such that if a pointed non-parabolic* $\mathrm{RCD}(0, N)$ *space* $(X, \mathsf{d}, \mathfrak{m}, x)$ *satisfies* $\nu_x \leqslant \nu$,

$$\frac{s}{\mathfrak{m}(B_s(x))} \leqslant \varphi(s), \quad \text{for } \mathscr{H}^1\text{-a.e. } s \in [1, \infty) \tag{4.3}$$

and

$$|\nabla \mathsf{b}_x|(z) \geqslant \mathscr{C}_N \nu_x^{\frac{1}{N-2}} - \delta \tag{4.4}$$

for some $z \in \overline{B}_R(x) \setminus B_r(x)$, *then we have*

$$\left\| \mathsf{b}_x - \mathscr{C}_N \nu_x^{\frac{1}{N-2}} \mathsf{d}_x \right\|_{L^\infty(B_R(x))} + \left(\fint_{B_R(x)} \left| \nabla \left(\mathsf{b}_x - \mathscr{C}_N \nu_x^{\frac{1}{N-2}} \mathsf{d}_x \right) \right|^p \mathrm{d}\mathfrak{m} \right)^{1/p} \leqslant \varepsilon \tag{4.5}$$

and there exists an $\mathrm{RCD}(N-2, N-1)$ *space* $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$ *such that*

$$\mathsf{d}_{\mathrm{pmGH}} \left((X, \mathsf{d}, \mathfrak{m}, x), (C(Y), \mathsf{d}_{C(Y)}, \mathfrak{m}_{C(Y)}, O_Y) \right) + |\nu_x - \nu_{O_Y}| < \varepsilon \tag{4.6}$$

Proof. As in the proof of corollary 3.14, it is enough to consider the case when $p = 2$. Then the proof is done by a contradiction, thus assume the conclusion fails. Then there exist sequences of positive numbers $\delta_i \to 0^+$ and of pointed $\mathrm{RCD}(0, N)$ spaces $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ such that $\nu_{x_i} \leqslant \nu$, $\frac{s}{\mathfrak{m}_i(B_s(x_i))} \leqslant \varphi(s)$ for $\mathscr{H}^1$-a.e. $s \in [1, \infty)$, that

$$|\nabla_i \mathsf{b}_{x_i}|(z_i) \geqslant \mathscr{C}_N \nu_{x_i}^{\frac{1}{N-2}} - \delta_i, \quad \text{for some } z_i \in \overline{B}_R(x_i) \setminus B_r(x_i) \tag{4.7}$$

and that for any $\mathrm{RCD}(N-2, N-1)$ space $(Y, \mathsf{d}_Y, \mathfrak{m}_Y)$, we have

$$\inf_i \left(\mathsf{d}_{\mathrm{pmGH}} \left((X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i), (C(Y), \mathsf{d}_{C(Y)}, \mathfrak{m}_{C(Y)}, O_Y) \right) + |\nu_{x_i} - \nu_{O_Y}| \right) > 0 \tag{4.8}$$

or

$$\inf_i \left(\left\| \mathsf{b}_{x_i} - \mathscr{C}_N \nu_{x_i}^{\frac{1}{N-2}} \mathsf{d}_{x_i} \right\|_{L^\infty(B_R(x_i))} + \left(\fint_{B_R(x_i)} \left| \nabla \left(\mathsf{b}_{x_i} - \mathscr{C}_N \nu_{x_i}^{\frac{1}{N-2}} \mathsf{d}_{x_i} \right) \right|^p \mathrm{d}\mathfrak{m} \right)^{1/p} \right) > 0. \tag{4.9}$$

Theorem 3.11 allows us to conclude that after passing to a subsequence, $(X_i, \mathsf{d}_i, \mathfrak{m}_i, x_i)$ pmGH converge to a pointed non-parabolic $\mathrm{RCD}(0, N)$ space $(X, \mathsf{d}, \mathfrak{m}, x)$, and $\nu_{x_i} \to \mu$ for some $\mu \in [0, \infty)$. On the other hand, the lower semicontinuity of N-volume densities (3.34) implies

$$\nu_x \leqslant \lim_{i \to \infty} \nu_{x_i} = \mu \leqslant \nu < \infty. \tag{4.10}$$

Consider $\mathscr{L}$-superharmonic lower semicontinuous functions on $X_i \setminus \{x_i\}$;

$$u_i := \mathscr{C}_N^2 \nu_{x_i}^{\frac{2}{N-2}} - |\nabla_i \mathsf{b}_{x_i}|^2 \geqslant 0. \tag{4.11}$$

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Fix $0 < s < \frac{r}{4}$. Applying the weak Harnack inequality, proposition 6.5, to the nonegatively valued $\mathscr{L}$-superharmonic function u_i, we have

$$\left(\fint_{B_{2s}(z_i)} u_i^p \, \mathrm{d}\mathfrak{m}_i \right)^{\frac{1}{p}} \leqslant C \inf_{B_s(z_i)} u_i \leqslant C u_i(z_i) \leqslant C\delta_i. \tag{4.12}$$

where $p = p(N, s) > 0$. Thus recalling that u_i is uniformly bounded, we have

$$\fint_{B_{2s}(z_i)} \left| \mathscr{C}_N^2 \nu_{x_i}^{\frac{2}{N-2}} - |\nabla_i \mathsf{b}_{x_i}|^2 \right| \mathrm{d}\mathfrak{m}_i \to 0. \tag{4.13}$$

Note that with no loss of generality we can assume that z_i converge to a point $z \in \overline{B}_R(x) \setminus B_r(x)$. Moreover the dominated convergence theorem yields the convergence of $\frac{s}{\mathfrak{m}_i(B_s(x_i))}$ to $\frac{s}{\mathfrak{m}(B_s(x))}$ in $L^1([1, \infty), \mathscr{H}^1)$. Thus theorem 3.12 shows that b_{x_i} $W^{1,2}$-strongly converge to b_x on $B_{2s}(z)$. Therefore (4.13) with (4.10) implies

$$\fint_{B_{2s}(z)} |\nabla \mathsf{b}_x|^2 \mathrm{d}\mathfrak{m} = \mathscr{C}_N^2 \mu^{\frac{2}{N-2}} \geqslant \mathscr{C}_N^2 \nu_x^{\frac{2}{N-2}}. \tag{4.14}$$

In particular theorem 3.22 shows $\mu = \lim_{i\to\infty} \nu_{x_i} = \nu_x$ and

$$|\nabla b_x| = \mathscr{C}_N \nu_x^{\frac{1}{N-2}}, \quad \text{for } \mathfrak{m}\text{-a.e. in } B_{2s}(z). \tag{4.15}$$

Thus, theorem 4.1 allows us to conclude that the limit space is isomorphic to the N-metric measure cone over an $\mathrm{RCD}(N-2, N-1)$ space, which contradicts (4.8) and (4.9). □

Next we provide an almost rigidity to a Euclidean space on a non-collapsed space.

Corollary 4.4. *For any integer $N \geqslant 3$, all $0 < \varepsilon < 1$, $0 < r < R$ and $\varphi \in L^1([1, \infty), \mathscr{H}^1)$ there exists $\delta := \delta(N, \varepsilon, r, \varphi) > 0$ such that if a pointed non-parabolic non-collapsed* $\mathrm{RCD}(0, N)$ *space $(X, \mathsf{d}, \mathscr{H}^N, x)$ satisfies*

$$\frac{s}{\mathscr{H}^N(B_s(x))} \leqslant \varphi(s), \quad \text{for } \mathscr{H}^1\text{-a.e. } s \in [1, \infty) \tag{4.16}$$

and

$$|\nabla \mathsf{b}_x|^2(y) \geqslant \mathscr{C}_N \omega_N^{\frac{1}{N-2}} - \delta \tag{4.17}$$

for some $y \in \overline{B}_R(x) \setminus B_r(x)$, then $(X, \mathsf{d}, \mathscr{H}^N, x)$ is ε- pmGH close to $(\mathbb{R}^N, \mathsf{d}_{\mathbb{R}^N}, \mathscr{H}^N, 0_N)$.

Proof. The first statement is a direct consequence of (the proof of) theorem 4.3 with the Bishop inequality $\nu_x \leqslant \omega_N$. □

Remark 4.5. Let us remark that the conclusion of corollary 4.4 cannot be replaced by a stronger one;

$$V_X \geqslant \omega_N - \varepsilon. \tag{4.18}$$

Find an open N-manifold (M^N, g) with the maximal volume growth which is not isometric to $\mathbb{R}^N$ (for instance the Eguchi-Hanson metric on the cotangent bundle

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$T^*\mathbb{S}^2$ of $\mathbb{S}^2$ gives such an example with vanishing Ricci curvature, see [**24**]). Then since the tangent cone at infinity is not isometric to $\mathbb{R}^N$, the asymptotic N-volume V_{M^N} is away from ω_N. Fix $x \in M^N$. Take any convergent sequence $x_i \to x$ with $x_i \neq x$ and then recall

$$|\nabla \mathsf{b}_x|(x_i) \to \mathscr{C}_N \omega_N^{\frac{1}{N-2}}. \tag{4.19}$$

Thus considering the rescaled distance $\mathsf{d}_i := \mathsf{d}(x, y_i)^{-1}\mathsf{d}$, the pointed Riemannian manifolds (M^N, d_i, x) satisfies (4.19) for some $y_i \in \partial B_1^{\mathsf{d}_i}(x)$, but the asymptotic N-volume is away from ω_N because of the scale invariance of V_{M^N}. In particular (4.18) is not satisfied in this case.

In connection with this remark, we prove the following.

THEOREM 4.6. *For any integer $N \geqslant 3$, all $0 < \varepsilon < 1$ and $\tau > 0$ there exists $\delta = \delta(N, \varepsilon, \tau) > 0$ such that if a pointed non-parabolic non-collapsed* RCD$(0, N)$ *space $(X, \mathsf{d}, \mathscr{H}^N, x)$ satisfies $V_X \geqslant \tau$ and*

$$\mathscr{C}_N \omega_N^{\frac{1}{N-2}} - |\nabla \mathsf{b}_x|(y_i) \leqslant \delta \tag{4.20}$$

for some sequence $y_i \in X$ with $\mathsf{d}(x, y_i) \to \infty$, then

$$V_X \geqslant \omega_N - \varepsilon \tag{4.21}$$

In particular if ε is sufficiently small depending only on N and τ, then X is homeomorphic to $\mathbb{R}^N$, moreover in addition, if X is smooth, then the homeomorphism can be improved to be a diffeomorphism.

Proof. Let $r_i := \mathsf{d}(x, y_i)$ and consider a rescaled pointed non-parabolic non-collapsed RCD$(0, N)$ space;

$$(X_i, \mathsf{d}_i, \mathscr{H}^N_{\mathsf{d}_i}, x_i) := \left(X, \frac{1}{r_i}\mathsf{d}, \frac{1}{r_i^N}\mathscr{H}^N_{\mathsf{d}}, x\right). \tag{4.22}$$

Recalling (3.30) we have for any $r \geqslant 1$

$$F^{X_i}_{x_i}(r) = \frac{F^X_x(rr_i)}{r_i^{2-N}} \leqslant \frac{r^{2-N}}{(N-1)\tau}, \tag{4.23}$$

where we used (3.5) in the final inequality. Thus corollary 4.4 allows us to conclude that $(X_i, \mathsf{d}_i, \mathscr{H}^N_{\mathsf{d}_i}, x_i)$ is pmGH close to the N-dimensional Euclidean space. In particular a tangent cone at infinity is also pmGH close to the N-dimensional Euclidean space. Therefore the volume convergence result, [**20**, Theorem 1.3], implies (4.21). Thus we get the first statement. The remaining statements come from this with the same arguments as in [**14**, Theorems A.1.11]. □

REMARK 4.7. In the theorem above, in order to get the same conclusion, we cannot replace the existence of divergent points by the existence of a point which is far from x. The reason is the same to remark 4.5.

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5. Examples

In this section we see that theorems 3.12 and 4.3 are sharp via simple examples. Moreover we also discuss a related sharpness and open problems.

5.1. Sharpness I

In this subsection we prove that theorem 4.3 is sharp, namely this cannot be improved to the case when $p = \infty$. The following arguments also allow us to conclude that (1.12) and (1.13) in theorem 1.2 are also sharp.

Step 1. Consider $\mathbb{S}^2(r) := \{x \in \mathbb{R}^3 | |x|_{\mathbb{R}^3} = r\}$ for $r < 1$ with the standard Riemannian metric and denote by $X = C(\mathbb{S}^2(r))$ the 3-metric measure cone with the 3-dimensional Hausdorff measure $\mathscr{H}^3_X (= \mathfrak{m}_{C(\mathbb{S}^2(r))})$. Assume that r is close to 1, and take a point $y \in X$ which is close to the pole $x \in X$ with $x \neq y$. Since $(X, \mathsf{d}, \mathscr{H}^3, y)$ is pmGH close to the 3-dimensional Euclidean space, theorem 4.3 yields that

$$\int_{B_1(y)} ||\nabla \mathsf{b}_y| - 4\pi| \, \mathrm{d}\mathscr{H}^3_X \tag{5.1}$$

is small, where the Lipschitz constant can be calculated by $(3(3-2)\omega_3)^{\frac{1}{3-2}} = 4\pi$.

On the other hand, we have

$$|\nabla \mathsf{b}_y|(x) = \lim_{r \to 0^+} \left(\fint_{B_r(x)} |\nabla \mathsf{b}_y|^2 \mathrm{d}\mathscr{H}^3_X \right)^{1/2} = 0 \tag{5.2}$$

because of the same trick observed in [**21**]. Namely, thanks to lemmas 3.7, 3.9 and the stability of the Laplacian [**3**, Theorem 4.4], under any blow-up at x, b_x $W^{1,2}_{\mathrm{loc}}$-strongly converge to a linear growth harmonic function on $C(\mathbb{S}^2(r))$. Recalling that any such function must be a constant because $r < 1$, we have (5.2).

Then recalling the upper semicontinuity of $|\nabla \mathsf{b}_y|$, we know that $|\nabla \mathsf{b}_y|$ is small around x. In particular

$$4\pi - |\nabla \mathsf{b}_y| \geqslant 3\pi \tag{5.3}$$

near x, thus

$$\| |\nabla \mathsf{b}_y| - 4\pi \|_{L^\infty(B_1(y))} \geqslant 3\pi. \tag{5.4}$$

Step 2. Let

$$(Z_i, \mathsf{d}_i, \mathfrak{m}_i, z_i) \overset{\mathrm{pmGH}}{\to} (Z, \mathsf{d}, \mathfrak{m}, z) \tag{5.5}$$

be a pmGH convergent sequence of pointed RCD(K, N) spaces and let $f_i \in L^\infty(B_R(x_i), \mathfrak{m}_i)$ L^p-strongly converge to $f \in L^\infty(B_R(x), \mathfrak{m})$ for any $p < \infty$ with

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$\sup_i \|f_i\|_{L^\infty} < \infty$. Then

$$\liminf_{i\to\infty} \|f_i\|_{L^\infty} \geqslant \|f\|_{L^\infty}. \tag{5.6}$$

Because for any $p < \infty$, since

$$\liminf_{i\to\infty} \|f_i\|_{L^\infty} \geqslant \lim_{i\to\infty} \left(\fint_{B_R(z_i)} |f_i|^p \mathrm{d}\mathfrak{m}_i \right)^{1/p} = \left(\fint_{B_R(z)} |f|^p \mathrm{d}\mathfrak{m} \right)^{1/p}, \tag{5.7}$$

letting $p \to \infty$ completes the proof of (5.6).

Step 3. Let us consider $X = C(\mathbb{S}^2(r))$ again as in *Step 1*. Note that it is easy to find a sequence of manifolds of dimension 3, $(M_i^3, g_i, \mathscr{H}^3, y_i)$, pmGH-converge to $(X, \mathsf{d}, \mathscr{H}^3, y)$ (actually $(X, \mathsf{d}, \mathscr{H}^3, x)$ is the tangent cone at infinity of a 3-dimensional complete Riemannian manifold (M^3, g) with $V_{M^3} = V_X > 0$). Then applying (5.6) for $f_i = 4\pi - |\nabla \mathsf{b}_{y_i}|$, $f = 4\pi - |\nabla \mathsf{b}_y|$ with corollary 3.23 implies

$$\liminf_{i\to\infty} \|4\pi - |\nabla \mathsf{b}_{y_i}|\|_{L^\infty(B_1(y_i))} \geqslant \|4\pi - |\nabla \mathsf{b}_y|\|_{L^\infty(B_1(y))} \geqslant 3\pi. \tag{5.8}$$

This observation shows that we cannot improve theorem 4.3 to the case when $p = \infty$.

5.2. Sharpness II

In the rest two subsections 5.2 and 5.3, we prove that several results we obtained previously under assuming;

$$\frac{s}{\mathfrak{m}(B_s(x))} \leqslant \varphi(s) \tag{5.9}$$

do not hold if we replace (5.9) by a weaker one;

$$F_x(1) \leqslant C < \infty \tag{5.10}$$

based on examples discussed in the previous subsection. Namely the assumption (5.9) is sharp in these results.

The first one is about theorem 3.12, namely we provide an example of pmGH convergent sequences of non-parabolic RCD(0, 3) spaces whose Green functions do not converge to the limit one, though the corresponding $F_{x_i}(1)$ are bounded.

For fixed $0 < r < 1$,[5] let $X = C(\mathbb{S}^2(r))$ with the pole x, the cone distance $\mathsf{d} = \mathsf{d}_{C(\mathbb{S}^2(r))}$ and the 3-dimensional Hausdorff measure $\mathscr{H}^3_X$. For any $R > 0$, let us denote by X_R the glued space of a closed ball $\overline{B}_R(x)$ and a cylinder $\partial B_R(x) \times [0, \infty)$ along the boundary $\partial B_R(x) := \{y \in X | \mathsf{d}(x, y) = R\}$. Then it is trivial that both X and X_R with the canonical intrinsic distance d_R can be canonically realized as boundaries ∂D, ∂D_R of closed convex subsets D, D_R in $\mathbb{R}^4$ with $x = 0_4$ and $D_R \subset D$, respectively. In particular $(X_R, \mathsf{d}_R, \mathscr{H}^3_{X_R})$ is an RCD(0, 3) space. Note that $(X_R, \mathsf{d}_R, \mathscr{H}^3_{X_R})$ is not non-parabolic.

[5]It is not difficult to see that the following arguments are also justified even in the case when $r = 1$ after choosing a suitable projection π.

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Under the conventions above, denote by $\pi_R : X \to X_R$ the canonical projection in $\mathbb{R}^4$ (thus $\pi_R|_{\overline{B}_R(x)} = \mathrm{id}_{\overline{B}_R(x)}$). For any $0 \leqslant t \leqslant 1$, define $\pi_{R,t} : X \to D$ by $\pi_{R,t}(y) := (1-t)y + t\pi_R(y)$, and put $X_{R,t} := \pi_{R,t}(X)$ which is also the boundary of a covex closed subset in $\mathbb{R}^4$.

Choose $R_0 > 1$ with

$$\int_{R_0}^{\infty} \frac{s}{\mathscr{H}^3_X(B_s(x))}\mathrm{d}s < 1 \tag{5.11}$$

because of the non-parabolicity of $(X, \mathsf{d}, \mathscr{H}^3_X)$. Fix $R \geqslant R_0^2$. Since

$$\int_R^{\infty} \frac{s}{\mathscr{H}^3_{X_{R,t}}(B_s(x))}\mathrm{d}s \to \int_R^{\infty} \frac{s}{\mathscr{H}^3_{X_R}(B_s(x))}\mathrm{d}s = \infty, \quad t \to 1^- \tag{5.12}$$

because $(X_R, \mathsf{d}_R, \mathscr{H}^3_{X_R})$ is not non-parabolic, we can find $t_R \in (0, 1)$ with

$$\int_R^{\infty} \frac{s}{\mathscr{H}^3_{X_{R,t_R}}(B_s(x))}\mathrm{d}s = 1. \tag{5.13}$$

Then let us consider a pmGH-convergent sequence of non-parabolic RCD(0, 3) spaces;

$$\left(X_{R,t_R}, \mathsf{d}_{R,t_R}, \mathscr{H}^3_{X_{R,t_R}}, x\right) \overset{\mathrm{pmGH}}{\to} \left(X, \mathsf{d}, \mathscr{H}^3_X, x\right), \quad R \to \infty, \tag{5.14}$$

where d_{R,t_R} denotes the canonical intrinsic distance on X_{R,t_R}. Since

$$\begin{aligned} F_x^{X_{R,t_R}}(1) &= \int_1^R \frac{s}{\mathscr{H}^3_{X_{R,t_R}}(B_s(x))}\mathrm{d}s + 1 \\ &= \int_1^R \frac{s}{s^3\mathscr{H}^3_X(B_1(x))}\mathrm{d}s + 1 \to F_x^X(1) + 1, \quad R \to \infty, \end{aligned} \tag{5.15}$$

Theorem 3.12 tells us that the Green functions do not converge to the limit one. Actually we can see this directly as follows.

Fix $y \in X \setminus \{x\}$. Then for any sufficiently large R, Gaussian estimates (2.20) show

$$\begin{aligned} G^{X_{R,t_R}}(x,y) &= \int_0^R p_{X_{R,t_R}}(x,y,t)\mathrm{d}t + \int_R^{\infty} p_{X_{R,t_R}}(x,y,t)\mathrm{d}t \\ &\geqslant \int_0^R p_{X_{R,t_R}}(x,y,t)\mathrm{d}t + \frac{1}{C}\int_{\sqrt{R}}^{\infty} \frac{s}{\mathscr{H}^3_{X_{R,t_R}}(B_s(x))}\mathrm{d}s \\ &= \int_0^R p_{X_{R,t_R}}(x,y,t)\mathrm{d}t + \frac{1}{C} \end{aligned} \tag{5.16}$$

for some $C > 1$. Thus letting $R \to \infty$ with Fatou's lemma and (3.48) yields

$$\liminf_{R\to\infty} G^{X_{R,t_R}}(x,y) \geqslant \int_0^{\infty} p_X(x,y,t)\mathrm{d}t + \frac{1}{C} = G^X(x,y) + \frac{1}{C}. \tag{5.17}$$

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5.3. Sharpness III

The final sharpness result is related to corollary 3.27. An immediate consequence of corollary 3.27 states that if a pointed non-parabolic RCD$(0, N)$ space $(X, \mathsf{d}, \mathfrak{m}, x)$ with the finite N-volume density $\nu_x < \infty$ is pmGH-close to the N-metric measure cone over an RCD$(N-2, N-1)$ space, then $|\nabla \mathsf{b}_x|$ attains the maximum $\mathscr{C}_N \nu_x^{\frac{1}{N-2}}$ almostly at some point which is bounded and away from x, whenever (5.9) holds. In the sequel we prove that (5.9) cannot be replaced by (5.10) to get the same conclusion.

To do so, under the same notations as in subsection 5.2, let us discuss the behaviour of $\mathsf{b}_x^{X_{R,t_R}}$ as $R \to \infty$. Our claim is that if r is small, then there exists no sequence $y_{R_i} \in X_{R_i,t_{R_i}}$ as $R_i \to \infty$ such that y_{R_i} is bounded and is away from x and that

$$|\nabla \mathsf{b}_x^{X_{R_i,t_{R_i}}}|(y_{R_i}) \to \mathscr{C}_3 \nu_x (=3\nu_x) \tag{5.18}$$

The proof is done by a contradiction. If such sequence y_{R_i} exists, then after passing to a subsequence, with no loss of generality we can assume that y_{R_i} converge to some $y \in X \setminus \{x\}$ and that $\mathsf{b}_x^{X_{R_i,t_{R_i}}}$ locally uniformly converge to some $\tilde{\mathsf{b}} \in \mathrm{Lip}(X, \mathsf{d})$. Note that thanks to lemma 3.7 and the stability of the Laplacian [**3**, Theorem 4.4], we know that $\tilde{\mathsf{b}} \in D(\Delta, X \setminus \{x\})$ and $\tilde{\mathsf{b}}^2 \in D(\Delta)$ hold with

$$\Delta \tilde{\mathsf{b}} = 2\frac{|\nabla \tilde{\mathsf{b}}|^2}{\tilde{\mathsf{b}}}, \quad \Delta \tilde{\mathsf{b}}^2 = 6|\nabla \tilde{\mathsf{b}}|^2. \tag{5.19}$$

On the other hand, applying a weak Harnack inequality, theorem 6.5, for $(3\nu_x)^2 - |\nabla \mathsf{b}_x^{X_{R_i,t_{R_i}}}|^2$ as in (3.105) proves

$$\int_{B_R(x)\setminus B_r(x)} \left| (3\nu_x)^2 - |\nabla \mathsf{b}_x^{X_{R_i,t_{R_i}}}|^2 \right| \mathrm{d}\mathscr{H}^3_{X_{R_i,t_{R_i}}} \to 0 \tag{5.20}$$

for all $0 < r < R < \infty$. Thus we know

$$\Delta \tilde{\mathsf{b}}^2 = 6(3\nu_x)^2. \tag{5.21}$$

Then since $\tilde{\mathsf{b}}^2 - (3\nu_x)^2 \mathsf{d}_x^2$ is a harmonic function on X with polynomial growth of degree at most 2, any such function must be a constant if r is small. Thus we have $\tilde{\mathsf{b}}^2 = (3\nu_x)^2 \mathsf{d}_x^2 + d$ for some $d \in \mathbb{R}$. Since $\tilde{\mathsf{b}}(x) = 0$ by definition, we know $d = 0$, namely

$$\tilde{\mathsf{b}} = 3\nu_x \mathsf{d}_x. \tag{5.22}$$

In particular we have $\tilde{\mathsf{b}}^{-1} = G_x^X$ which contradicts (5.17). Thus the observation above allows us to conclude for all $0 < r_1 < r_2 < \infty$

$$\liminf_{R\to\infty} \left(\inf_{B_{r_2}(x)\setminus B_{r_2}(x)} \left(\mathscr{C}_3 \nu_x - |\nabla \mathsf{b}_x^{X_{R,t_R}}| \right) \right) > 0. \tag{5.23}$$

Remark 5.1. We do not know whether we can replace (5.9) by (5.10) to get the same conclusions in theorems 1.2 and 1.7.

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6. Appendix; analysis on a drifted Laplace operator $\mathscr{L}$

In this appendix, we provide detailed proofs of the regularity results for $\mathscr{L}$-sub/super harmonic functions (recall definition 3.18), coming directly from the general theory on PI spaces. It is emphasized that these techniques can be applied to more general operators, including, of course, our Laplacian Δ. We refer [**9**, Section 8.5] as a main reference on this topic.

Let $(X, \mathsf{d}, \mathfrak{m}, x)$ be a pointed non-parabolic RCD$(0, N)$ space. The first result is about the weak Harnack inequality for $\mathscr{L}$-subharmonic functions. This is justified by applying [**9**, Theorem 8.4] to an (incomplete) metric measure space $(B_{100r}(y), \mathsf{d}, \mathfrak{m}_{G_x})$ because of (3.16) and (3.70).

PROPOSITION 6.1 Weak Harnack inequality for $\mathscr{L}$-subharmonic functions. *Let u be an $\mathscr{L}$-subharmonic function on a ball $B_{100r}(y)$ for some $r \leqslant 1$ with $B_{100r}(y) \subset X \setminus B_s(x)$ for some $s \leqslant 1$. Then for all $k \in \mathbb{R}$ and $p > 1$, there exists $C = C(N, s, p) > 0$ such that*[6]

$$\operatorname*{ess\,sup}_{B_{\frac{r}{2}}(y)} u + k \leqslant C \left(\fint_{B_r(y)} (u+k)_+^p \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{p}}. \tag{6.2}$$

Proof. First let us assume that $p > 2$. For arbitrary $l > 0$, let $\hat{u} = \hat{u}_k := (u+k)_+$, and denote by $\bar{u} = \bar{u}_{k,l} := \min\{\hat{u}_k, l\}$. For $\frac{1}{2}r \leqslant r_1 < r_2 \leqslant r$, take the cutoff function

$$\eta(z) := \min\left\{1, \frac{r_2 - \mathsf{d}(z,y)}{r_2 - r_1}\right\}_+. \tag{6.3}$$

Then we have $0 \leqslant \eta \leqslant 1$, $|\nabla \eta| \leqslant \frac{1}{r_2 - r_1}$ in $B_{r_2}(y)$, $\eta \equiv 1$ in $B_{r_1}(y)$, and $\eta \equiv 0$ outside $B_{r_2}(y)$. For any $\beta > p - 2 > 0$, define

$$v = v_{k,l,\beta} := \eta^2 \bar{u}^\beta \hat{u} \in W^{1,2}_{0,+}(B_{100r}(y)). \tag{6.4}$$

By direct calculation

$$\nabla v = 2\eta \bar{u}^\beta \hat{u} \nabla \eta + \eta^2 \bar{u}^\beta (\beta \nabla \bar{u} + \nabla \hat{u}). \tag{6.5}$$

Substitute $\varphi = v$ in (3.79),

$$\int_{B_{r_2}(y)} \left(2\eta \bar{u}^\beta \hat{u} \langle \nabla \eta, \nabla u \rangle + \eta^2 \bar{u}^\beta (\beta |\nabla \bar{u}|^2 + |\nabla \hat{u}|^2) - 2\eta^2 \bar{u}^\beta \hat{u} \langle \nabla \log G_x, \nabla u \rangle \right) \mathrm{d}\mathfrak{m} \leqslant 0. \tag{6.6}$$

[6] When we apply directly [**9**, Theorem 8.4], then the conclusion should be written in terms of $\mathfrak{m}_{G_x}$ instead of $\mathfrak{m}$ in (6.2). The difference can be understood as follows. Recalling $\mathrm{d}\mathfrak{m}_{G_x} = G_x^2 \mathrm{d}\mathfrak{m} = \exp(2 \log G_x) \mathrm{d}\mathfrak{m}$, for fixed y as in the statement, under assuming $\|\nabla \log G_x\|_{L^\infty} \leqslant L$ on the domain, we have

$$\frac{1}{L} \leqslant \frac{G_x(z)}{G_x(w)} \leqslant L \tag{6.1}$$

because of applying the upper gradient inequality for $\log G_x$. This inequality, (6.1), allows us to compare $\mathfrak{m}$ with $\mathfrak{m}_G$ quantitatively. Thus we can state the conclusion in terms of $\mathfrak{m}$. The same observation can be applied in the sequel.

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Let $E_k := \{u \geqslant -k\}$, noting that $\hat{u} \equiv 0$ outside E_k, the integrand above vanishes in $B_{r_2}(y) \setminus E_k$. Using Young's inequality, noticing that fact that $|\nabla \hat{u}| = |\nabla u|$ in E_k,

$$\chi_{E_k} |2\eta \bar{u}^{\beta} \hat{u} \langle \nabla \eta, \nabla u \rangle| \leqslant \chi_{E_k} \left(\frac{1}{3} \eta^2 \bar{u}^{\beta} |\nabla \hat{u}|^2 + 3 \bar{u}^{\beta} \hat{u}^2 |\nabla \eta|^2 \right), \tag{6.7}$$

$$\chi_{E_k} \left| 2\eta^2 \bar{u}^{\beta} \hat{u} \langle \nabla \log G_x, \nabla u \rangle \right| \leqslant \chi_{E_k} \left(\frac{1}{3} \eta^2 \bar{u}^{\beta} |\nabla \hat{u}|^2 + 3\eta^2 \bar{u}^{\beta} \hat{u}^2 |\nabla \log G_x|^2 \right). \tag{6.8}$$

Combining (6.6), (6.7) and (6.8), it holds that

$$\int_{B_{r_2}(y)} \eta^2 \bar{u}^{\beta} (\beta |\nabla \bar{u}|^2 + |\nabla \hat{u}|^2) \, \mathrm{d}\mathfrak{m} \leqslant 9 \int_{B_{r_2}(y)} \bar{u}^{\beta} \hat{u}^2 (|\nabla \eta|^2 + \eta^2 |\nabla \log G_x|^2) \, \mathrm{d}\mathfrak{m}. \tag{6.9}$$

Set $w := \bar{u}^{\frac{\beta}{2}} \hat{u}$, noting that $\hat{u} \geqslant \bar{u}$ and $|\nabla \bar{u}|$ vanishes in $\{\hat{u} > l\}$, it is easy to check

$$|\nabla w|^2 \leqslant \left(\frac{\beta}{2} \bar{u}^{\frac{\beta}{2} - 1} \hat{u} |\nabla \bar{u}| + \bar{u}^{\frac{\beta}{2}} |\nabla \hat{u}| \right)^2 \leqslant (1 + \beta) \bar{u}^{\beta} \left(\beta |\nabla \bar{u}|^2 + |\nabla \hat{u}|^2 \right). \tag{6.10}$$

Therefore,

$$\int_{B_{r_2}(y)} \eta^2 |\nabla w|^2 \, \mathrm{d}\mathfrak{m} \leqslant 9(1 + \beta) \int_{B_{r_2}(y)} w^2 (|\nabla \eta|^2 + \eta^2 |\nabla \log G_x|^2) \, \mathrm{d}\mathfrak{m}. \tag{6.11}$$

By the Sobolev inequality

$$\begin{aligned} \left(\int_{B_{r_2}(y)} (\eta w)^{2\xi} \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{\xi}} &\leqslant C(N) \frac{r_2^2}{\mathfrak{m}(B_{r_2}(y))^{\frac{2}{N}}} \int_{B_{r_2}(y)} |\nabla(\eta w)|^2 \, \mathrm{d}\mathfrak{m} \\ &\leqslant C(N) \frac{(1 + \beta) r_2^2}{\mathfrak{m}(B_{r_2}(y))^{\frac{2}{N}}} \int_{B_{r_2}(y)} w^2 (|\nabla \eta|^2 + \eta^2 |\nabla \log G_x|^2) \, \mathrm{d}\mathfrak{m}, \end{aligned} \tag{6.12}$$

where $\xi = \xi(N) := \frac{N}{N-2}$. Recall that $|\nabla \log G_x|^2$ is bounded above by $C(N, s)$, by the choice of η,

$$\|w\|_{L^{2\xi}(B_{r_1}(y), \mathfrak{m})} \leqslant C(N, s) \frac{(1 + \beta)^{\frac{1}{2}}}{\mathfrak{m}(B_{r_2}(y))^{\frac{1}{N}}} \frac{r_2}{r_2 - r_1} \|w\|_{L^2(B_{r_2}(y), \mathfrak{m})}. \tag{6.13}$$

For any $\gamma > 2$ and $t \in [\frac{1}{2} r, r]$, set the quantity

$$A(\gamma, t) := \left(\int_{B_r(y)} \bar{u}^{\gamma - 2} \hat{u}^2 \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{\gamma}}. \tag{6.14}$$

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Then (6.13) yields that

$$\begin{aligned} A(\xi(\beta+2), r_1) &= \left(\int_{B_{r_1}(y)} \bar{u}^{\xi(\beta+2)-2} \hat{u}^2 \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{\xi(\beta+2)}} \\ &\leqslant \left(\int_{B_{r_1}(y)} w^{2\xi} \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{\xi(\beta+2)}} \\ &\leqslant \left(C(N,s) \frac{(1+\beta)^{\frac{1}{2}}}{\mathfrak{m}(B_r(y))^{\frac{1}{N}}} \frac{r}{r_2 - r_1} \right)^{\frac{2}{\beta+2}} A(\beta+2, r_2), \end{aligned} \tag{6.15}$$

where we used the following fact from Bishop–Gromov inequality:

$$\frac{r_2}{\mathfrak{m}(B_{r_2}(y))^{\frac{1}{N}}} \leqslant \frac{r}{\mathfrak{m}(B_r(y))^{\frac{1}{N}}}. \tag{6.16}$$

Let $\beta_n = p\xi^n - 2 > 0$, $\gamma_n := \beta_n + 2 > 2$, $t_n := (2^{-1} + 2^{-n-1})r$, iterating (6.15),

$$A(\gamma_n, t_n) \leqslant \prod_{i=1}^{n} \left(2^{i+1} C(N,s) (\gamma_i - 1)^{\frac{1}{2}} \right)^{\frac{2}{\gamma_i}} \mathfrak{m}(B_r(y))^{-\frac{2(\xi - \xi^{-n})}{Np(1-\xi)}} A(p, r), \tag{6.17}$$

Let $n \to \infty$, recalling by definition

$$\|\bar{u}\|_{L^\gamma(B_t(y),\mathfrak{m})} \leqslant A(\gamma, t) \leqslant \|\hat{u}\|_{L^\gamma(B_t(y),\mathfrak{m})}, \tag{6.18}$$

it holds that

$$\|\bar{u}_{k,l}\|_{L^\infty(B_{\frac{r}{2}}(t),\mathfrak{m})} \leqslant C(N,s,p) \mathfrak{m}(B_r(y))^{-\frac{1}{p}} \|\hat{u}_k\|_{L^p(B_r(y),\mathfrak{m})}. \tag{6.19}$$

Since the right-handed side of the inequality is independent on l, we may let $l \to \infty$, namely

$$\|\hat{u}_k\|_{L^\infty(B_{\frac{r}{2}}(t),\mathfrak{m})} \leqslant C(N,s,p) \mathfrak{m}(B_r(y))^{-\frac{1}{p}} \|\hat{u}_k\|_{L^p(B_r(y),\mathfrak{m})}, \tag{6.20}$$

which finishes the proof under the assumption $p > 2$.

The case where $1 < p \leqslant 2$ can be treated via a similar iteration process provided $\hat{u} \in L^\infty(B_r(y), \mathfrak{m})$. Indeed, for any $\beta > p - 1 > 0$, we can choose a simpler test function without truncation:

$$v_{k,\beta} := \eta^2 (u + k)_+^\beta \in W^{1,2}_{0,+}(B_{100r}(y)), \tag{6.21}$$

which can further simplify the proof. Here we omit the details. □

Next let us discuss about $\mathscr{L}$-superharmonic functions. The corresponding results for PI spaces can be found in [**9**, Theorem 8.10]. In order to establish a weak Harnack inequality for $\mathscr{L}$-superharmonic functions, we first recall the definition of bounded mean oscillating (BMO) functions on metric measure spaces and John–Nirenberg's lemma.

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DEFINITION 6.2 BMO. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a measure-doubling metric measure space, namely there exists a constant $C_d > 0$ such that*

$$\mathfrak{m}(B_{2r}(x)) \leqslant C_d \mathfrak{m}(B_r(x)), \quad \forall\, x \in X, \quad \forall\, r > 0. \tag{6.22}$$

For any open subset $\Omega \subset X$ and any function $f \in L^1_{\mathrm{loc}}(\Omega, \mathfrak{m})$, we set

$$\|f\|_{\mathrm{BMO}(\Omega)} := \sup_B \fint_B |f - f_B| \, \mathrm{d}\mathfrak{m}, \tag{6.23}$$

where the supremum is taken over all open balls B with $B \subset \Omega$. The class of BMO functions on Ω is the collection

$$\mathrm{BMO}(\Omega) := \left\{ f \,\middle|\, \|f\|_{\mathrm{BMO}(\Omega)} < \infty \right\}. \tag{6.24}$$

REMARK 6.3. Note that the doubling assumption is naturally fulfilled for $\mathrm{RCD}(0, N)$ spaces because of the Bishop–Gromov inequality. In particular, in this case, the doubling constant C_d is only dependent on N.

John–Nirenberg's lemma we refer is stated as follows, see [**9**, Theorem 3.20].

THEOREM 6.4 John–Nirenberg's lemma. *Let $(X, \mathsf{d}, \mathfrak{m})$ be as in definition 6.2 and let $f \in \mathrm{BMO}(B_{5r}(x))$ for some $x \in X$ and $r > 0$. Then for any $0 < \varepsilon \leqslant A := \log 2/(4C_d^{15})$,*

$$\fint_{B_r(x)} \exp\left(\frac{\varepsilon |f - f_{B_r(x)}|}{\|f\|_{\mathrm{BMO}(B_{5r}(x))}} \right) \mathrm{d}\mathfrak{m} \leqslant \frac{A + \varepsilon}{A - \varepsilon}. \tag{6.25}$$

We are now in a position to prove a weak Harnack inequality (see [**9**, Theorem 8.10]).

PROPOSITION 6.5 Weak Harnack inequality for $\mathscr{L}$-superharmonic functions. *Let u be a non-negatively valued $\mathscr{L}$-superharmonic function on a ball $B_{100r}(y)$ for some $r \leqslant 1$ with $B_{100r}(y) \subset X \setminus B_s(x)$ for some $s \leqslant 1$. Then there exist $p = p(N, s) > 0$ and $C = C(N, s) > 1$ such that*

$$C \operatorname*{ess\,inf}_{B_r(y)} u \geqslant \left(\fint_{B_{2r}(y)} u^p \mathrm{d}\mathfrak{m} \right)^{\frac{1}{p}}. \tag{6.26}$$

Proof. First let us assume u is bounded away from 0. For any fixed $\varepsilon > 0$, there exists a piecewise linear function ψ in the form

$$\psi(t) := \max_{1 \leqslant i \leqslant k} (a_i t + b_i), \tag{6.27}$$

where $a_i < 0$, such that

$$\frac{1}{t} \leqslant \psi(t) \leqslant \frac{1}{t} + \varepsilon, \quad \forall t > \varepsilon. \tag{6.28}$$

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Applying proposition 6.1 to $\psi \circ (u+\varepsilon)$, we obtain that for any $p > 0$,

$$\frac{1}{\operatorname*{ess\,inf}_{B_r(y)} u+\varepsilon} \leqslant \operatorname*{ess\,sup}_{B_r(y)} \psi\circ(u+\varepsilon) \leqslant C(N,s,p)\left(\fint_{B_{2r}(y)} (\psi\circ(u+\varepsilon))^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}} \leqslant$$
$$C(N,s,p)\left(\fint_{B_{2r}(y)} \left(\frac{1}{u+\varepsilon}+\varepsilon\right)^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}} \leqslant C(N,s,p)\left(\fint_{B_{2r}(y)} \left(\frac{1}{u}+\varepsilon\right)^p \,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}. \tag{6.29}$$

Letting $\varepsilon \to 0$, we obtain that

$$\begin{aligned}\operatorname*{ess\,inf}_{B_r(y)} u &\geqslant C(N,s,p)\left(\fint_{B_{2r}(y)} u^{-p}\,\mathrm{d}\mathfrak{m}\right)^{-\frac{1}{p}} \\ &= C(N,s,p)\left(\fint_{B_{2r}(y)} u^{-p}\,\mathrm{d}\mathfrak{m}\fint_{B_{2r}(y)} u^{p}\,\mathrm{d}\mathfrak{m}\right)^{-\frac{1}{p}}\left(\fint_{B_{2r}(y)} u^{p}\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{p}}.\end{aligned} \tag{6.30}$$

Now it suffices to show that there exists some $p = p(N,\, s) > 0$ such that

$$\fint_{B_{2r}(y)} u^{-p}\,\mathrm{d}\mathfrak{m}\fint_{B_{2r}(y)} u^{p}\,\mathrm{d}\mathfrak{m} \leqslant C(N,s). \tag{6.31}$$

On the other hand, recalling (3.79), we can establish an analogue of (6.11) similarly as in the proof of proposition 6.1 for $w := \log u$ treating the test function $v := \eta^2$

$$\int_{B_{100r}(y)} \eta^2|\nabla w|^2\,\mathrm{d}\mathfrak{m} \leqslant C(N)\int_{B_{100r}(y)} \left(|\nabla\eta|^2+\eta^2|\nabla\log G_x|^2\right)\,\mathrm{d}\mathfrak{m}, \tag{6.32}$$

where η is any cut-off function with $\operatorname{supp}\eta \subset B_{100r}(y)$. Thus choosing η with $\eta \equiv 1$ on $B_{50r}(y)$, we obtain

$$\fint_{B_{50r}(y)} |\nabla w|^2\,\mathrm{d}\mathfrak{m} \leqslant \frac{C(N,s)}{r^2}. \tag{6.33}$$

Thus using the Poincaré inequality (2.22), for any $z \in B_{10r}(y)$ and $r' < 20r$, we have

$$\fint_{B_{r'}(z)} \left|w - \fint_{B_{r'}(z)} w\,\mathrm{d}\mathfrak{m}\right|\,\mathrm{d}\mathfrak{m} \leqslant C(N)r\left(\fint_{B_{r'}(z)} |\nabla w|^2\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{2}} \leqslant C(N,s), \tag{6.34}$$

which implies

$$\|w\|_{\mathrm{BMO}(B_{10r}(x))} \leqslant C(N,s). \tag{6.35}$$

On the other hand, for this $C(N,\, s)$ in (6.35), applying John–Nirenberg's lemma, theorem 6.4, to w with $\varepsilon := A\cdot C(N,\, s)/2$ (recall A is taken as a dimensional

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constant in this setting), we have

$$\fint_{B_{2r}(y)} e^{\varepsilon|w-w_0|}\,\mathrm{d}\mathfrak{m} \leqslant 3, \tag{6.36}$$

where

$$w_0 := \fint_{B_{2r}(y)} w\,\mathrm{d}\mathfrak{m}. \tag{6.37}$$

Thus

$$\begin{aligned}
&\fint_{B_{2r}(y)} u^{-\varepsilon}\,\mathrm{d}\mathfrak{m} \fint_{B_{2r}(y)} u^{\varepsilon}\,\mathrm{d}\mathfrak{m} \\
&\quad = \fint_{B_{2r}(y)} e^{\varepsilon w}\,\mathrm{d}\mathfrak{m} \fint_{B_{2r}(y)} e^{-\varepsilon w}\,\mathrm{d}\mathfrak{m} \;=\; \fint_{B_{2r}(y)} e^{\varepsilon(w-w_0)}\,\mathrm{d}\mathfrak{m} \fint_{B_{2r}(y)} e^{\varepsilon(w_0-w)}\,\mathrm{d}\mathfrak{m} \\
&\quad \leqslant \left(\fint_{B_{2r}(y)} e^{\varepsilon|w-w_0|}\,\mathrm{d}\mathfrak{m}\right)^2 \leqslant 9.
\end{aligned} \tag{6.38}$$

By (6.30),

$$\operatorname*{ess\,inf}_{B_r(y)} u \geqslant C(N,s)\left(\fint_{B_{2r}(y)} u^{\varepsilon}\,\mathrm{d}\mathfrak{m}\right)^{\frac{1}{\varepsilon}}. \tag{6.39}$$

Therefore, recalling our choice of ε and (6.35), we have the desired inequality.

Finally let us assume u is not bounded away from 0. We can consider $u_\delta = u + \delta$ ($\delta > 0$) instead, and then let $\delta \downarrow 0$. □

We are now in a position to introduce a regularity result on ($\mathscr{L}$-)superharmonic functions. See [**9**, Subsection 8.5], in particular, proposition 8.24 therein.

Proposition 6.6. *Let u be as in proposition 6.1. Assume that u is locally bounded. Then there exists a unique representative $\bar{u}$ of u such that every $z \in B_{100r}(y)$ is a Lebesgue point of $\bar{u}$. Moreover $\bar{u}$ is upper semi-continuous satisfying that for any $z \in B_{100r}(y)$*

$$\bar{u}(z) = \limsup_{w\to z} \bar{u}(w). \tag{6.40}$$

Indeed, such a representative can be realized by

$$\bar{u}(z) := \lim_{\rho\to 0} \operatorname*{ess\,sup}_{w\in B_\rho(z)} u(w). \tag{6.41}$$

Proof. Let $\bar{u}$ be as in (6.41). Firstly, since it is easily checked that the set $\{z \mid \bar{u}(z) < a\}$ is open for any $a \in \mathbb{R}$, $\bar{u}$ is upper semicontinuous.

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Secondly, let us check $\bar{u} = u$ $\mathfrak{m}$-a.e. Take a Lebesgue point $z \in B_{100r}(y)$ of u. Since u is locally bounded, we have

$$\lim_{\rho \to 0} \fint_{B_\rho(z)} |u - u(z)|^q \, \mathrm{d}\mathfrak{m} = 0, \quad \forall\, q > 0. \tag{6.42}$$

For any $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $0 < \rho < r_0$, $B_\rho(z) \subset B_{100r}(y)$ and

$$\left(\fint_{B_\rho(z)} |u - u(z)|^p \, \mathrm{d}\mathfrak{m} \right)^{1/p} < \varepsilon, \tag{6.43}$$

where p is as in proposition 6.1. Applying proposition 6.1 proves

$$\bar{u}(z) - u(z) \leqslant \operatorname*{ess\,sup}_{w \in B_{\frac{\rho}{2}}(z)} (u(w) - u(z)) \leqslant C \left(\fint_{B_\rho(z)} |u - u(z)|^p \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{p}} \leqslant C\varepsilon. \tag{6.44}$$

Thus $\bar{u}(z) \leqslant u(z)$. On the other hand

$$\bar{u}(z) = \lim_{\rho \to 0} \operatorname*{ess\,sup}_{w \in B_\rho(z)} u(w) \geqslant \lim_{\rho \to 0} \fint_{B_\rho(z)} u \, \mathrm{d}\mathfrak{m} = u(z). \tag{6.45}$$

We obtain that $\bar{u}(z) = u(z)$. Thus the Lebesgue differentiation theorem allows us to conclude $\bar{u} = u$ for $\mathfrak{m}$-a.e. Moreover, observe that

$$\bar{u}(z) \geqslant \limsup_{w \to z} \bar{u}(w) \geqslant \lim_{\rho \to 0} \operatorname*{ess\,sup}_{w \in B_\rho(z)} \bar{u}(w) = \bar{u}(z), \tag{6.46}$$

therefore we obtain (6.40).

Thirdly, let us show that every point in the domain is a Lebesgue point of $\bar{u}$. Without loss of generality we may assume $0 < \bar{u} < 1$ in $B_{100r}(y)$ since u is locally bounded. Fix any $z \in B_{100r}(y)$. For any $\varepsilon > 0$ with $u(z) < 1 - \varepsilon$, by the upper semicontinuity, there exists $\rho > 0$ sufficiently small such that $0 < u(w) < u(z) + \varepsilon < 1$ for any $w \in B_\rho(z)$. Let $v := -u + u(z) + \varepsilon$, then $0 < v < 1$ in $B_\rho(z)$. Applying the weak Harnack inequality (6.26) to v, there exist $p = p(N, s) > 0$ and $C = C(N, s) > 0$ such that

$$\begin{aligned} \fint_{B_\rho(z)} |\bar{u} - \bar{u}(z)| \, \mathrm{d}\mathfrak{m} &= \fint_{B_\rho(z)} |v - v(z)| \, \mathrm{d}\mathfrak{m} \leqslant \varepsilon + \fint_{B_\rho(z)} v^{1-p} v^p \, \mathrm{d}\mathfrak{m} \\ &\leqslant \varepsilon + \fint_{B_\rho(z)} v^p \, \mathrm{d}\mathfrak{m} \leqslant \varepsilon + C \left(\inf_{B_\rho(z)} v \right)^p \leqslant \varepsilon + C\varepsilon^p. \end{aligned} \tag{6.47}$$

Letting $\rho \to 0$, we conclude because ε is arbitrary.

Finally, the uniqueness is obvious because every point in the domain is a Lebesgue point. □

In view of this, we always assume that any $\mathscr{L}$-subharmonic function u is actually the canonical representative as obtained in proposition 6.6. Therefore, for example,

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by (6.40), the 'ess sup' in proposition 6.1 ('ess inf' in proposition 6.5, respectively) can be replaced by 'sup' ('inf', respectively). Under this convention, finally let us provide the strong maximum principle for $\mathscr{L}$-subharmonic functions.

PROPOSITION 6.7 Strong maximum principle for $\mathscr{L}$-subharmonic functions. *Let u be a $\mathscr{L}$-subharmonic function on a connected open subset Ω in $X \setminus \{x\}$. If its supremum in Ω attains at a point in Ω, then u is constant.*

Proof. Denote by $A := \sup_\Omega u$ and put $D := \{x \in \Omega | u(x) = A\}$. It is trivial that D is closed in Ω. On the other hand, for any $y \in D$, applying proposition 6.5 to $\bar{u} = A - u$ proves

$$0 = \inf_{B_r(y)} \bar{u} \geqslant C \left(\fint_{B_{2r}(y)} \bar{u}^p \, \mathrm{d}\mathfrak{m} \right)^{\frac{1}{p}}, \tag{6.48}$$

namely $\bar{u} = 0$ $\mathfrak{m}$-a.e. in $B_{2r}(y)$ for any sufficiently small $r > 0$, thus $\bar{u} \equiv A$ on $B_{2r}(y)$ because of the lower semicontinuity of $\bar{u}$. This shows that D is open in Ω, Thus $D = \Omega$ because Ω is connected. □

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