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Geometric algebra in plasma electrodynamics

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Abstract. Geometric algebra (GA) is a recent broad mathematical framework incorporating synthetic and coordinate geometry, complex variables, quarternions, vector analysis, matrix algebra, spinors, tensors, and differential forms. It has been claimed to be a unified language for physics. GA is presented in the context of the Maxwell-Plasma system. In this formalism the divergence and curl differential operators are united in a single vector derivative, which is invertible, in the form of a first-order Green function. The four Maxwell equations can be combined into a single equation (for homogeneous and constant media) or into two equations involving the invertible vector derivative for more complex media. GA is applied to simple examples to illustrate the compactness of the notation and coordinate-free computations.

1. Introduction

Geometric algebra (GA) and geometric calculus (GC), based on the Clifford product, have been developed as a universal geometric language for mathematics and physics (Hestenes and Sobczyk 1987). It has the advantage of unifying synthetic and coordinate geometry, matrix, Grassmann and Clifford algebra, complex, vector, and tensor analysis as well as differential forms into a single unified mathematical system. Our purpose is to formulate a phase-space representation of Maxwell's non-relativistic electrodynamics using GA/GC. An introductory reference to this topic may be found in Jancewicz (1988), while a relativistic approach to this subject may be found in Baylis (2002).

In this work, we present a soft introduction to the ideas of GA/GC and apply them to phase-space electrodynamics. The motivation for this research is that a more compact (e.g. reduction of the four Maxwell equations to a single/double equation), coordinate-free approach, embedded in a more powerful algebraic system will ultimately lead to the solution of problems which are difficult to tackle through vector calculus alone as well as lead to new insights.

We begin with the vector Maxwell equations, which are reinterpreted in this new formalism. Due to its simplicity, the case of free fields in vacuum is explored in detail to show the advantages of the formalism. Then the formalism is applied to the derivation of the phase-space field equations in a scalar inhomogeneous variable medium. The Liouville approximation to this equation is presented.

2. Maxwell equations

According to the Helmholtz theorem a vector field is determined, to within a constant, by giving its divergence and curl everywhere in space. Thus, to determine the electric and magnetic fields, four vectorial (Maxwell) equations are necessary:

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \tag{2.1}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{2.2}$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}, \tag{2.3}$$

$$\vec{\nabla} \cdot \vec{D} = \rho. \tag{2.4}$$

It should be noted that the divergence and the curl differential operators are not invertible individually; both are necessary to determine a vector field. It is, however, highly desirable to have a vector differential operator which is invertible. Extending the usual vectorial calculus to GC allows one to define such an invertible operator; it is, in fact, given by the linear combination of the divergence and the curl for any Clifford valued function,

$$\vec{\nabla}\vec{F} = \vec{\nabla} \cdot \vec{F} + I(\vec{\nabla} \times \vec{F}), \tag{2.5}$$

where I is the unit trivector. It will be seen shortly that, in terms of this differential operator, the set of four Maxwell equations may be reduced to a single differential equation thereby affording a great compactness of notation.

3. The geometric product of two vectors

We start with a consideration of the usual products of two vectors $\vec{a}, \vec{b}$, namely the dot and cross products, and note that either product by itself does not completely describe the relative directions of the vectors. Since the cross product, $\times$, does not generalize to spaces of

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dimension greater than three, we consider instead the closely related outer product, $\wedge$, which does. The vector resulting from the cross product is perpendicular to the plane defined by $\vec{a} \wedge \vec{b}$ and has the same magnitude as the outer product. Geometrically, the outer product represents the oriented plane area ($\vec{a}$ rotates into $\vec{b}$) with magnitude $|\vec{a} \wedge \vec{b}| = |\vec{a}||\vec{b}|sin(\theta)$. The resulting oriented plane area is called a bivector. The outer product may be generalized to arbitrary multivectors: trivectors, quadrivectors, etc. An outer product between a vector and a simple multivector (blade) of any grade results in a new geometric object with grade one higher. The inner product of a vector with a blade, on the other hand, lowers the grade of the resulting blade by one. Both the outer product of a vector with a trivector and the inner product of a vector with a scalar result in objects outside the algebra in three-dimensional Euclidean space and are defined to be equal to zero. The geometric product of two vectors, defined as the sum of their inner and outer products, completely determines the relative direction between the vectors,

$$\vec{a}\,\vec{b} = \vec{a} \cdot \vec{b} + \vec{a} \wedge \vec{b}. \tag{3.1}$$

In this equation, the scalar $\vec{a} \cdot \vec{b}$ has been added to the bivector $\vec{a} \wedge \vec{b}$. It may seem strange at first to add two different geometric objects, although it not unlike a complex number consisting of a real and an imaginary part. The justification for this definition of the geometric product may be given in Grassmannian fashion in which all that is required is that the indicated relations and operations be well defined and consistently employed. Geometrically, the product $\vec{a}\,\vec{b}$ completely specifies the relative directions between the two vectors $\vec{a}$, $\vec{b}$, i.e. the geometric product of these two vectors consists of the projection of $\vec{b}$ on $\vec{a}$ and the rejection of $\vec{b}$ from $\vec{a}$. In particular, $\vec{a}\,\vec{b} = \vec{a} \cdot \vec{b}$, if $\vec{a} \wedge \vec{b} = 0$ or equivalently the two vectors are parallel, and $\vec{a}\,\vec{b} = \vec{a} \wedge \vec{b}$, if $\vec{a} \cdot \vec{b} = 0$ or equivalently the two vectors are perpendicular. In general two vectors fall between these extremes but are fully characterized by their geometric product. More fundamentally, one may consider the geometric product as basic and construct the inner and outer products according to $\vec{a} \cdot \vec{b} = 1/2(\vec{a}\,\vec{b} + \vec{b}\,\vec{a})$ and $\vec{a} \wedge \vec{b} = 1/2(\vec{a}\,\vec{b} - \vec{b}\,\vec{a})$, respectively. It may then be verified that the inner and outer products so defined have the expected properties. GA of multivectors is an associative algebra with the additional condition that the geometric product of a vector with itself results in a scalar. The resulting algebra allows for multiplication (and division) of multivectors which permits the calculation of multiplicative inverses.

4. The equations of Maxwell in geometric algebra

From the Lorentz force law, $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$, it is clear that the electric field is a vector whereas the magnetic field vector is more properly interpreted as a bivector related to $\vec{B}$ by a duality operation $\hat{B} = -I\vec{B}$, where the hat above capital B is our notation for a bivector and I represents the unit trivector in three-dimensional Euclidean space. In geometrical terms, the bivector $\hat{B}$ is the oriented plane area orthogonal to the vector $\vec{B}$. Using the above relation between the cross and the outer products, the Maxwell equations may be written as (Jancewicz 1988)

$$\vec{\nabla} \wedge \vec{E} + \frac{\partial \hat{B}}{\partial t} = 0, \tag{4.1}$$

$$\vec{\nabla} \wedge \hat{B} = 0, \tag{4.2}$$

$$\vec{\nabla} \cdot \hat{H} + \frac{\partial \vec{D}}{\partial t} = -\vec{J}, \tag{4.3}$$

$$\vec{\nabla} \cdot \vec{D} = \rho. \tag{4.4}$$

The above Maxwell equations have been grouped into two homogeneous equations, containing outer derivatives with no sources and two inner derivative equations containing the sources of the fields. Each equation has a different multivector grade.

For the sake of simplicity, let us assume scalar constitutive relations which may depend on space and time, $\epsilon = \epsilon(\vec{x}, t)$, $\mu = \mu(\vec{x}, t)$. Let us further introduce the following normalization for the fields

$$\vec{e} = \sqrt{\epsilon}\vec{E}, \tag{4.5}$$

$$\hat{b} = \frac{1}{\sqrt{\mu}}\hat{B}. \tag{4.6}$$

With this normalization, both quantities $\vec{e}$, $\hat{b}$ have physical dimensions of $\sqrt{J/m^3}$ in SI units. In terms of $\vec{D}$ and $\hat{H}$, we have $\vec{e} = (1/\sqrt{\epsilon})\vec{D}$ and $\hat{b} = \sqrt{\mu}\hat{H}$. We now form the electromagnetic cliffor

$$f = \vec{e} + \hat{b}. \tag{4.7}$$

In terms of the cliffor f, the Maxwell equations may be written as a single equation by making use of the first-order vector derivative,

$$D \equiv \vec{\nabla} + c^{-1}\frac{\partial}{\partial t}. \tag{4.8}$$

The resulting equation is

$$Df + \vec{e}D\,ln(\sqrt{\epsilon}) + \hat{b}D\,ln(\sqrt{\mu}) = \tilde{j}, \tag{4.9}$$

where the source term $\tilde{j}$ is given by

$$\tilde{j} = \frac{1}{\sqrt{\epsilon}}\rho - \sqrt{\mu}\vec{j}. \tag{4.10}$$

Equation (4.9) is a combination of terms of all grades in a three-dimensional Euclidean space. Like the Maxwell equations, it is expressed in coordinate-free form. However, the multivector structure is much more developed than the usual vectorial structure which should allow much further development or even complete solutions without resorting to coordinates. This has been achieved at the cost of going from a vector algebra to a Cifford algebra with the consequent increase of dimensionality

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from 3 to $2^3 = 8$. The geometric product of the cliffor f with its reverse $f^\dagger$ gives the energy–momentum cliffor for the electromagnetic field,

$$\frac{1}{2}ff^\dagger = \frac{1}{2}|f|^2 + \hat{b}\cdot\vec{e} = \frac{1}{2}(|\vec{e}|^2 + |\hat{b}|^2) + \sqrt{\epsilon\mu}\,\vec{S}$$
$$= w + c(\vec{x},t)\vec{g}, \tag{4.11}$$

which is a linear combination of the electromagnetic energy and momentum densities. Here $\vec{S}$ is the Poynting vector, w is the energy density, $c = 1/\sqrt{\epsilon\mu}$ is the local speed, and $\vec{g}$ is the momentum density. An evolution equation for the energy–momentum cliffor, particularly in phase-space, is highly desirable as the cliffor is more directly related to measurements. Such an equation is the principal result of this research.

As a simple example of this formalism, we consider free space. In this case

$$Df = 0. \tag{4.12}$$

Let us attempt a travelling plane wave solution of the form

$$f = A_0 \exp(I(\vec{k}\cdot\vec{r} - \omega t)). \tag{4.13}$$

The argument of the exponential involves the unit trivector I which commutes with all cliffor numbers and whose square equals –1. It thus plays the role of a complex imaginary and is useful in generalizing the definition of the usual Fourier transform to GA. Substituting the trial solution in the equation and simplifying, we get $(\vec{k} - \omega)A_0 = 0$. We have set $c = 1$ in this free field example. Since f is a combination of a vector and a bivector, let us take $A_0 = \vec{e}_0 + \hat{b}_0$ and substitute this form in the above expression to get

$$0 = (\vec{k} - \omega)(\vec{e}_0 + \hat{b}_0) = \vec{k}\cdot\vec{e}_0 + (-\omega\vec{e}_0 + \vec{k}\cdot\hat{b}_0)$$
$$+ (\vec{k}\wedge\vec{e}_0 - \omega\hat{b}_0) + \vec{k}\wedge\hat{b}_0. \tag{4.14}$$

Each term in this expression has a different grade and the sum must add to zero, hence

$$0 = \vec{k}\cdot\vec{e}_0, \tag{4.15}$$
$$0 = -\omega\vec{e}_0 + \vec{k}\cdot\hat{b}_0, \tag{4.16}$$
$$0 = \vec{k}\wedge\vec{e}_0 - \omega\hat{b}_0, \tag{4.17}$$
$$0 = \vec{k}\wedge\hat{b}_0. \tag{4.18}$$

The first equation is a scalar and states that the vector k is perpendicular to $\vec{e}_0$. The second, vector equation, $\vec{e}_0 = \vec{k}\cdot\hat{b}_0/\omega$, and the fourth trivector equation imply that the vectors $\vec{e}_0$ and $\vec{k}$ lie in the $\hat{b}_0$ plane. The oriented $\hat{b}_0$ plane is defined by the third bivector equation, $\hat{b}_0 = \vec{k}\wedge\vec{e}_0/\omega$. In addition, the second and third equations imply $|\vec{k}| = \omega$. It should be noted that no coordinates have been introduced.

5. Phase-space representations of wave equations

In this section we present a phase-space representation of the geometric Maxwell equation in the so-called ordinary symbol formalism. In (4.9) we assume the medium properties are slowly varying so as to be able to neglect the logarithmic derivatives of ϵ and μ. Neglect of the logarithmic derivatives of the medium amounts to a new Wentzel–Kramers–Brillouin (WKB) approximation. This is because, while the Maxwell equation consists of terms of all grades, each term has the dimensions of the square root of an energy density, $\sqrt{J/m^3}$. This is to say, the Maxwell equation implements a different normalization to the usual set of four Maxwell equations. For example, the equation obtained from the Maxwell equation corresponding to $\nabla\cdot\vec{D} = \nabla\cdot(\epsilon\vec{E}) = \rho$ becomes, in this approximation, $\nabla\cdot(\sqrt{\epsilon}\vec{E}) = \rho/\sqrt{\epsilon}$. It is worth noting that the product of the Maxwell equation with its adjoint leads to an equation where the various terms have the dimension of an energy density which should prove particularly useful in studying energy propagation. Under this approximation

$$Df = (\vec{\nabla} + c(\vec{x},t)^{-1}\partial_t)f = \tilde{j}(\vec{x},t). \tag{5.1}$$

Its kernel representation is given by

$$\int d\vec{x}\,'dt' D(\vec{x},t;\vec{x}\,',t')f(\vec{x}\,',t') = \tilde{j}(\vec{x},t). \tag{5.2}$$

One can construct an operator equation for the cliffor f by multiplying this equation by its reverse

$$(Df)(f^\dagger D^\dagger) = \tilde{j}\tilde{j}^\dagger. \tag{5.3}$$

The multiplication operations in this equation are of two types: Hilbert space application of a linear differential operator D on the Hilbert space vector field f, and geometric multiplication in the three-dimensional vector space of all multivector quantities.

The ordinary symbol, or the phase-space representative, for the dispersion operator D is defined as the Fourier transform of the dispersion kernel,

$$D(\vec{x},t;\vec{x}\,',t') = -\delta(t-t')\delta'(\vec{x}-\vec{x}')$$
$$- c(\vec{x},t)^{-1}\delta'(t-t')\delta(\vec{x}-\vec{x}'), \tag{5.4}$$

which, for this kernel is

$$\tilde{D}(\vec{x},t;\vec{k},\omega) \equiv \int d\vec{s}d\tau D(\vec{x},t;\vec{x}-\vec{s},t-\tau)\exp(-I(\vec{k}\cdot\vec{s}-\omega\tau))$$
$$= -I\vec{k} + I\omega c(\vec{x},t)^{-1}. \tag{5.5}$$

Similarly, the symbol for the electromagnetic field is given by

$$\tilde{F}(\vec{x},t;\vec{k},\omega) = \int d\vec{s}d\tau F(\vec{x},t;\vec{x}-\vec{s},t-\tau)$$
$$\times \exp(-I(\vec{k}\cdot\vec{s}-\omega\tau)), \tag{5.6}$$

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which is again just the Fourier transform of

$$F(\vec{x},t;\vec{x}\,',t') = \frac{1}{(2\pi)^4}\int d\vec{k}d\omega\tilde{F}(\vec{x},t;\vec{k},\omega) \times \exp\left(I(\vec{k}\cdot(\vec{x}-\vec{x}') - \omega(t-t')\right). \quad (5.7)$$

Both the symbol and its Fourier transform have a scalar (energy density) and a vector part (linear momentum density). The inverse speed is a function of both position and time.

In analogy to McDonald (1988), the phase-space equation for the symbol $\tilde{F}(\vec{x},t;\vec{k},\omega)$ of the operator $f(\vec{x},t)f^{\dagger}(\vec{x}',t') \equiv F(\vec{x},t;\vec{x}',t')$ then becomes

$$\tilde{D}(\vec{x},t;\vec{k},\omega)\exp(-I\mathscr{L})\tilde{F}(\vec{x},t;\vec{k},\omega) = \tilde{j}(\vec{x},t;\vec{k},\omega)\exp(-I\mathscr{L})(\tilde{D}^{\dagger})^{-1}(\vec{x},t;\vec{k},\omega), \quad (5.8)$$

where

$$\mathscr{L} \equiv \overleftarrow{\partial_{\vec{k}}}\cdot\overrightarrow{\partial_{\vec{x}}} - \overleftarrow{\partial_{\omega}}\,\overrightarrow{\partial_t}, \quad (5.9)$$

and $\tilde{j}$ is the symbol for the current source. This equation is the principal result of this research. In the Liouville approximation with no sources present, one finds

$$(-I\vec{k} + I\omega c(\vec{x},t)^{-1})(1 - I\mathscr{L} + \ldots)\tilde{F}(\vec{x},t;\vec{k},\omega) = 0, \quad (5.10)$$

which implies

$$(\overrightarrow{\partial_{\vec{x}}} + c(\vec{x},t)^{-1}\overrightarrow{\partial_t})\tilde{F} = -I(\vec{k} - \omega c(\vec{x},t)^{-1})\tilde{F}. \quad (5.11)$$

This equation is a linear first-order equation which admits a solution in general, by the method of characteristics, for example.

6. Conclusion

We have derived, using the GA approach, a coordinate free, phase-space equation describing the propagation of both electromagnetic energy and momentum densities (Poynting vector) and which contains, at each space-time point, full spectral information. It is noteworthy that this equation is a first-order differential equation. To the lowest order, the Liouville approximation leads to a kinetic equation in a weakly inhomogeneous and non-stationary medium for the energy–momentum cliffor. While the ordinary symbol is probably the simplest phase-space representation one can derive for wave equations, it lacks the symmetry properties of the Weyl symbol which has received much attention from the physics community. Extension and application of the GA-/GC-based formalism to phase-space representations using the Weyl symbol is reserved for future research.

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