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SURVEY

Arithmeticity of discrete subgroups

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Abstract. The topic of this course is the discrete subgroups of semisimple Lie groups. We discuss a criterion that ensures that such a subgroup is arithmetic. This criterion is a joint work with Sébastien Miquel, which extends previous work of Selberg and Hee Oh and solves an old conjecture of Margulis. We focus on concrete examples like the group $\mathrm{SL}(d,\mathbb{R})$ and we explain how classical tools and new techniques enter the proof: the Auslander projection theorem, the Bruhat decomposition, the Mahler compactness criterion, the Borel density theorem, the Borel–Harish-Chandra finiteness theorem, the Howe–Moore mixing theorem, the Dani–Margulis recurrence theorem, the Raghunathan–Venkataramana finite-index subgroup theorem and so on.

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0. *Introduction*

This text is the written version of a series of four lectures I gave at the Fields Institute in August 2018 and at the IHES in July 2019. Videos of these lectures are available on the web [www.fields.utoronto.ca/video-archive/static/2018/08/2384-19365/mergedvideo.ogv, www.youtube.com/watch?v=QsR0-5R9uJE&feature=youtu.be]. Most of the students in the audience were either graduate students or postdocs. I tried to keep the informal style of the lectures, giving only complete proof of representative examples, focusing on the main ideas, pointing out those ideas that are often useful in this subject, recalling shortly the proof of preliminary classical results and leaving the technical issues to my joint paper [3] with Sébastien Miquel.

Lecture 1 is a short survey of the arithmeticity question of discrete subgroups Γ of a semisimple Lie group G, focusing on a few historical landmarks that will be useful in the next lectures. It also presents our criterion, when $\mathrm{rank}_{\mathbb{R}}(G) \geq 2$, ensuring that Γ is an (irreducible and non-cocompact) arithmetic subgroup of G. This criterion is as follows.

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The discrete subgroup Γ is Zariski dense and intersects cocompactly and irreducibly a non-trivial horospherical subgroup U of G.

We will discuss this criterion in the remaining three lectures.

Lectures 2 and 3 deal only with the group $G = \mathrm{SL}(4, \mathbb{R})$ and the horospherical subgroup U of G that stabilizes the 2-plane $\mathbb{R}^2$ of $\mathbb{R}^4$. This horospherical subgroup is both commutative and reflexive, i.e. it is conjugate to an 'opposite horospherical' subgroup U^-.

Lecture 2 is elementary and accessible to an undergraduate student. We introduce the intersection L of the normalizers of U and U^- and the intersection L_0 of the unimodular normalizers of U and U^-. The connected component of this group L_0 is isomorphic to $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. We explain why the L_0-orbit of the horospherical lattice $\Gamma \cap U$ is closed in the space of lattices of U.

Lecture 3 relies on various classical results of this subject like the Howe–Moore mixing theorem, the Dani–Margulis recurrence theorem and the Raghunathan–Venkataramana finite-index subgroup theorem. We do not enter the proof of these nice results but explain why those results are useful. We show that, since L_0 is semisimple and non-compact, the closedness of this L_0-orbit allows us to assume that the intersection $L_0 \cap \Gamma$ is a lattice in L_0. We then explain why, when the intersection $L_0 \cap \Gamma$ is a lattice, the group Γ is arithmetic.

Lecture 4 presents five other examples of horospherical subgroups U: two in the group $G = \mathrm{SL}(3, \mathbb{R})$, two in the group $G = \mathrm{SL}(4, \mathbb{R})$ and one in the product $G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$. The proofs given in Lectures 2 and 3 work only for a horospherical group U which is reflexive and commutative and for which the derived group $[L, L]$ is non-compact. These five examples are intended to explain the strategy when these assumptions are not satisfied.

- Case 4.1 with U not reflexive: it is an application of the Auslander theorem.
- Case 4.2 with U Heisenberg and $[L, L]$ not compact: it is dealt with exactly as in Lectures 2 and 3.
- Case 4.3 with U not commutative and not Heisenberg: it reduces to the commutative case thanks to the structure theorem for nilpotent lattices.
- Case 4.4 with U commutative and $[L, L]$ compact: by Lecture 2, the L_0-orbit is still closed. We check that it is compact and conclude as in Lecture 3.
- Case 4.5 with U Heisenberg and $[L, L]$ compact: it is similar to Case 4.4.

These four lectures can easily be read independently, even though they logically depend on one another.

1. *First lecture. Arithmetic groups*

I begin this first lecture by quoting the main objective of this series, which is the following theorem. You are not supposed to understand right now the statement of this theorem. Indeed, the aim of this first lecture is to explain the meaning of the words that appear in this statement. Moreover, in the next lectures, we will be dealing with explicit examples and we will re-explain concretely this statement.

THEOREM 1.1. *Let G be a semisimple algebraic Lie group of real rank at least two and U be a non-trivial horospherical subgroup of G. Let Γ be a discrete Zariski dense subgroup*

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of G that contains an irreducible lattice Δ of U. Then Γ is commensurable to an arithmetic lattice $G_{\mathbb{Z}}$ of G.

This theorem is the main result of my article [3] with Sébastien Miquel. It solves a conjecture of Margulis. Many cases of this conjecture were already handled in Hee Oh's PhD thesis; for instance, all cases but one where G is simple and $\mathbb{R}$-split (see [18–20] and the missing $\mathbb{R}$-split case, which is in [4]). The main feature in Theorem 1.1 is that 'Γ is a lattice' is a conclusion, not an assumption.

This first lecture can be seen as a survey of this topic or a motivation for Theorem 1.1. No proof will be given in this first lecture.

1.1. *Examples.* We first recall the definition of *lattices*. Let G be a Lie group. The group G is endowed with a measure λ_G called the *Haar measure* which is invariant by right-multiplication. This measure is unique up to a scalar. G is said to be *unimodular* if this measure is also invariant by left-multiplication.

For instance, the Haar measure on $\mathbb{R}^d$ is the Lebesgue measure on the vector space $\mathbb{R}^d$. The Haar measure on the group $G = \mathrm{SL}(d, \mathbb{R})$ is also easy to construct. For an open subset $A \subset G$ we introduce the truncated cone $C_A := \{tg \mid 0 < t < 1,\ g \in A\}$ and set $\lambda_G(A) := \mathrm{Leb}(C_A)$, where Leb is the Lebesgue measure on the vector space $\mathcal{M}(d, \mathbb{R})$.

A subgroup Γ of G is *discrete* if it is discrete for the induced topology. It is an exercise to check that a discrete subgroup is always closed. The quotient space $X = G/\Gamma$ is endowed with a measure λ_X which is locally equal to λ_G. This measure is also called the Haar measure on X. When G is unimodular, this is the unique G-invariant measure on X, up to scalar.

Definition 1.2. A discrete subgroup Γ of G is a *lattice* if the volume $\lambda_X(X)$ of the quotient space $X = G/\Gamma$ is finite. This means that there exists a measurable subset D of G such that $G = D\Gamma$ and $\lambda_G(D) < \infty$.

A discrete subgroup Γ of G is *cocompact* if the quotient $X = G/\Gamma$ is compact. This means that there exists a compact subset D of G such that $G = D\Gamma$.

A discrete cocompact subgroup is always a lattice. A Lie group G that contains a lattice Γ is always unimodular and therefore the Haar measure on $X = G/\Gamma$ is G-invariant. We normalize this measure to have total mass one.

Two subgroups Γ_1 and Γ_2 of G are said to be *commensurable* when the intersection $\Gamma_1 \cap \Gamma_2$ has finite index in both of them. In this case:

- Γ_1 is discrete if and only if Γ_2 is discrete;
- Γ_1 is a lattice if and only if Γ_2 is a lattice; and
- Γ_1 is cocompact if and only if Γ_2 is cocompact.

The first examples of lattices are very familiar.

Example 1.3. The group $\mathbb{Z}^d$ is a lattice in $\mathbb{R}^d$.

This lattice is cocompact and one can choose D to be the cube $[0, 1]^d$.

The first family of interesting lattices is due to Minkowski (see [7]).

Example 1.4. (Minkowski) The group $\mathrm{SL}(d, \mathbb{Z})$ is a lattice in $\mathrm{SL}(d, \mathbb{R})$.

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In this example the group G is the group of $d \times d$ matrices with real coefficients and determinant one and Γ is the subgroup of matrices with integer coefficients. In this case the quotient space $X = G/\Gamma$ can be seen as the space

$$X := \text{lattices } \Lambda \subset \mathbb{R}^d \text{ of covolume one.}$$

This space X is very useful in number theory. The finiteness of its volume is a key fact that allows us to apply methods of dynamical systems and ergodic theory to problems in number theory. This space X is not compact. Indeed, the compact subsets Y of X are described by the following Mahler criterion. This criterion tells us that one can detect the non-compactness of Y by the existence of arbitrarily small non-zero vectors in a lattice belonging to Y.

Fact 1.5. (Mahler) A closed subset $Y \subset X$ is compact if and only if

$$\inf_{\Lambda \in Y} \inf_{v \in \Lambda \setminus 0} \|v\| > 0.$$

There are many more examples of lattices. Indeed, Siegel discovered that similar examples can be constructed with orthogonal groups. In these examples the group G is a group of orthogonal matrices with real coefficients and Γ is the subgroup of matrices with integer coefficients.

Example 1.6. (Siegel) Let $Q := \sum_{i,j \le d} Q_{ij} x_i x_j$ be a non-degenerate integral quadratic form in $d \ge 3$ variables.

Then the group $\Gamma := \mathrm{SO}(Q, \mathbb{Z})$ is a lattice in $G := \mathrm{SO}(Q, \mathbb{R})$.

This group Γ is cocompact in G if and only if Q does not represent 0 over $\mathbb{Z}$, i.e. $Q^{-1}(0) \cap \mathbb{Z}^d = \{0\}$.

This theorem is not true for $d = 2$ when the quadratic form represents 0 over $\mathbb{Z}$. Indeed, when $Q(x_1, x_2) = x_1 x_2$, the group $\Gamma = \mathrm{SO}(Q, \mathbb{Z})$ is a cyclic group of order two while the group $G = \mathrm{SO}(Q, \mathbb{R})$ is isomorphic to the multiplicative group $\mathbb{R}^*$, which is non-compact.

Note also that, by Meyer's theorem, for $d \ge 5$, a non-definite quadratic form always represents 0 over $\mathbb{Z}$ and therefore the lattice Γ is not cocompact.

These examples were greatly extended by replacing the ring $\mathbb{Z}$ by any ring of integers in a number field. To make it simple, we just quote three instances of this extension. Here is an example in a complex quadratic field.

Example 1.7. The group $\Gamma = \mathrm{SL}(d, \mathbb{Z}[\sqrt{-1}])$ is a lattice in $G = \mathrm{SL}(d, \mathbb{C})$.

Here is an example in a real quadratic field.

Example 1.8. $\Gamma := \mathrm{SL}(d, \mathbb{Z}[\sqrt{2}])$ is a lattice in $G := \mathrm{SL}(d, \mathbb{R}) \times \mathrm{SL}(d, \mathbb{R})$.

The embedding $\Gamma \hookrightarrow G$ is given by the map $g \mapsto (g, g^\sigma)$, where σ is the Galois automorphism: when $g = a + b\sqrt{2}$ with a, b integral matrices, one has $g^\sigma := a - b\sqrt{2}$. It is an exercise to check that Γ is discrete in G. The main point of this example is that Γ has finite covolume.

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Combining Examples 1.6 and 1.8, one gets more examples. Here is one important instance.

Example 1.9. Let $Q_0 := -\sqrt{2}\, x_0^2 + x_1^2 + \cdots + x_d^2$ with $d \geq 2$. Then the group $\Gamma = \mathrm{SO}(Q_0, \mathbb{Z}[\sqrt{2}])$ is a cocompact lattice in $G = \mathrm{SO}(Q_0, \mathbb{R})$.

Indeed, the general theory (Fact 1.12(a)) will tell us that the map $g \mapsto (g, g^\sigma)$ embeds Γ as a lattice in the product group $G' := G \times K$, where $K := \mathrm{SO}(Q_0^\sigma, \mathbb{R})$. Since this group K is compact, the image of Γ in G is also discrete. Moreover, since $\Gamma \setminus 1$ does not contain unipotent matrices, the general theory (Fact 1.12(b)) will tell us that Γ is cocompact in G' and hence also cocompact in G.

These examples were historically very important. They are the first instances of cocompact lattices in the group $\mathrm{SO}(d, 1)$ for all $d \geq 2$. Since these groups act properly cocompactly by isometries on the hyperbolic space $\mathbb{H}^d$, this construction gave the first examples of periodic tilings in the hyperbolic space $\mathbb{H}^d$ for all $d \geq 2$.

1.2. *Arithmetic groups.* We now give a definition of *arithmetic groups*. We will use two topologies on the group $\mathrm{SL}(d_0, \mathbb{R})$: the topology induced by its injection in the real vector space of matrices $\mathcal{M}(d_0, \mathbb{R})$ and the *Zariski topology*. By definition, the *Zariski closed* sets are the algebraic subvarieties, i.e. the set of zeros of a family of polynomials. The *Zariski dense* sets are those subsets which are not included in a proper Zariski closed subset. The *Zariski connected* sets are those subsets which are not a disjoint union of two proper Zariski closed subsets and so on. By definition, an *algebraic subgroup* $G \subset \mathrm{SL}(d_0, \mathbb{R})$ is a Zariski closed subgroup. This group G is also a Lie subgroup of $\mathrm{SL}(d_0, \mathbb{R})$. We will always denote by the corresponding Gothic letter $\mathfrak{g}$ the Lie algebra of G.

One can check that the Zariski closure H of a subgroup Γ of $\mathrm{SL}(d_0, \mathbb{R})$ is always an algebraic subgroup. It is clear that H is an algebraic variety. What we claim, and leave as an exercise, is that H is a subgroup of $\mathrm{SL}(d_0, \mathbb{R})$. This simple claim explains why the study of discrete subgroups of $\mathrm{SL}(d_0, \mathbb{R})$ can be reduced to the study of discrete Zariski dense subgroups of algebraic groups.

When one chooses the embedding $G \subset \mathrm{SL}(d_0, \mathbb{R})$ such that the defining polynomial equations have rational coefficients, we say that G is *defined over* $\mathbb{Q}$ or that G is endowed with a $\mathbb{Q}$-*structure* or a $\mathbb{Q}$-*form*.

Definition 1.10. The group G is *quasisimple* if the Lie algebra $\mathfrak{g}$ is simple. The group G is *semisimple* if the Lie algebra $\mathfrak{g}$ is semisimple, i.e. $\mathfrak{g}$ is a direct sum $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ of simple ideals $\mathfrak{g}_i$.

One often uses abusively the word 'simple' instead of 'quasisimple'. For G semisimple, the choice of a particular embedding of G in $\mathrm{SL}(d_0, \mathbb{R})$ is not very important, and we will not discuss the subtlety coming from a change of linear embedding. They will play no role here. Indeed, we will work with the adjoint map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, where

$$\mathrm{Aut}(\mathfrak{g}) := \{\varphi \in \mathrm{GL}(\mathfrak{g}) \mid [\varphi(x), \varphi(y)] = [x, y] \text{ for all } x, y \in \mathfrak{g}\}$$

is the group of automorphisms of the Lie algebra $\mathfrak{g}$. This group is an algebraic group. The adjoint map is not always an embedding but, for G semisimple, it has finite kernel and

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finite cokernel. The Zariski connected component of the group $\mathrm{Aut}(\mathfrak{g})$ is called the *adjoint group* of G. We will denote it by AdG. Hence, for Theorem 1.1, we can always replace the group G by its adjoint group AdG. This is why we will use this representation to define the arithmetic group $G_{\mathbb{Z}}$.

Definition 1.11. A $\mathbb{Q}$-*form* of $\mathfrak{g}$ is a $\mathbb{Q}$-vector subspace $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$ such that:

- $\mathfrak{g}_{\mathbb{Q}}$ is a Lie subalgebra of $\mathfrak{g}$;
- the natural map $\mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{g}_{\mathbb{Q}} \to \mathfrak{g}$ is an isomorphism.

Similarly, one defines a $\mathbb{Z}$-*form* of $\mathfrak{g}$ to be a lattice $\mathfrak{g}_{\mathbb{Z}}$ in $\mathfrak{g}$ which is stable by the Lie bracket. It is also called a *Lie lattice*. A $\mathbb{Q}$-form $\mathfrak{g}_{\mathbb{Q}}$ always contains a $\mathbb{Z}$-form $\mathfrak{g}_{\mathbb{Z}}$. Note that a $\mathbb{Q}$-form $\mathfrak{g}_{\mathbb{Q}}$ of $\mathfrak{g}$ induces a $\mathbb{Q}$-structure on the algebraic group AdG whose $\mathbb{Q}$-points are $(\mathrm{AdG})_{\mathbb{Q}} := \mathrm{AdG} \cap \mathrm{Aut}(\mathfrak{g}_{\mathbb{Q}})$. We set $G_{\mathbb{Q}} := \mathrm{Ad}^{-1}(\mathrm{Aut}(\mathfrak{g}_{\mathbb{Q}}))$ and

$$G_{\mathbb{Z}} := \mathrm{Ad}^{-1}(\mathrm{Aut}(\mathfrak{g}_{\mathbb{Z}})).$$

Given the $\mathbb{Q}$-form $\mathfrak{g}_{\mathbb{Q}}$, the Lie lattice $\mathfrak{g}_{\mathbb{Z}}$ is well defined up to finite index. The arithmetic group $G_{\mathbb{Z}}$ is also well defined up to finite index. We leave this fact as exercise (see [7]). The following fact, due to Borel and Harish-Chandra in [8], encompasses all the examples we have discussed so far.

Fact 1.12. (Borel and Harish-Chandra) Let G be a semisimple algebraic Lie group and $\mathfrak{g}_{\mathbb{Q}}$ be a $\mathbb{Q}$-form of $\mathfrak{g}$.

(a) (Finiteness theorem) The subgroup $G_{\mathbb{Z}}$ is a lattice in G.

(b) (Godement compactness criterion) This lattice $G_{\mathbb{Z}}$ is cocompact if and only if $\mathfrak{g}_{\mathbb{Z}}$ does not contain non-zero nilpotent elements.

This fact allowed Borel to prove that any non-compact semisimple Lie group G contains both cocompact lattices and non-cocompact lattices (see [6]). In Theorem 1.1, we also used the following condition.

Definition 1.13. A discrete subgroup Δ of G is *irreducible* if for all infinite-index algebraic normal subgroups G' of G the intersection $\Delta \cap G'$ is finite.

Definition 1.13 is classical when Δ is a lattice in G. Note that in Theorem 1.1, we use this definition for a discrete subgroup Δ of G which is not a lattice in G but a lattice in U (see [3, Lemma 4.3] for more insight on this notion). Note also that this irreducibility condition is always satisfied when G is quasisimple.

The following fact, due to Borel, will also be useful.

Fact 1.14. (Borel density theorem) Let G be a Zariski connected semisimple algebraic Lie group with no compact factor. Then any lattice Γ of G is Zariski dense in G.

The assumption of *no compact factor* means that there does not exist an algebraic proper normal subgroup H of G for which the quotient G/H is compact.

We define the *real rank* $\mathrm{rank}_{\mathbb{R}} G \geq 0$ of G as the maximal dimension of a $\mathbb{R}$-split torus A of G, i.e. a commutative algebraic subgroup of G all of whose elements are diagonalizable

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over $\mathbb{R}$. For instance, the real rank of $\mathrm{SL}(d,\mathbb{R})$ is $d-1$. Indeed, one can choose $A := \{\mathrm{diag}(a_1,\dots,a_d) \mid a_1\cdots a_d = 1\}$. Similarly, the real rank of $\mathrm{SO}(p,q)$ is $\min(p,q)$.

One of the nicest surprises is that in higher rank, i.e. when $\mathrm{rank}_{\mathbb{R}} G \geq 2$, all lattices come from an arithmetic construction. This is the celebrated Margulis arithmeticity theorem (see [14, 29]). The statement of this theorem is slightly more involved than in Theorem 1.1, since one has to take into account the construction of cocompact lattices as in Example 1.9.

Fact 1.15. (Margulis arithmeticity theorem) Let G be an adjoint semisimple algebraic Lie group of higher rank and Γ be an irreducible lattice of G.

Then there exist a semisimple algebraic Lie group H, a Lie group morphism $p : H \to G$ with compact kernel and compact cokernel and a $\mathbb{Q}$-form $\mathfrak{h}_{\mathbb{Q}}$ of $\mathfrak{h}$ such that the groups Γ and $p(H_{\mathbb{Z}})$ are commensurable.

More on arithmetic lattices can be found in [7, 14, 17].

1.3. *Horospherical groups.* The only notion used in Theorem 1.1 that we have not yet defined is the notion of horospherical subgroup. For a general semisimple group G, the definition might look at first glance a little bit artificial. We will see below that these groups are very concrete. These subgroups are often defined using algebra. Here is a short equivalent definition with a dynamical flavor.

Definition 1.16. Let G be a Zariski connected semisimple algebraic Lie group and $e \in G$ the identity.

An element u in G is unipotent if all its eigenvalues are equal to one or equivalently if there exists g in G such that $\lim_{n\to\infty} g^{-n}ug^n = e$.

A unipotent subgroup of G is an algebraic subgroup all of whose elements are unipotent.

A horospherical subgroup U of G is the *unstable group* U_g of an element g in G, i.e. $U = U_g := \{u \in G \mid \lim_{n\to\infty} g^{-n}ug^n = e\}$.

The normalizer P of such a group U is called a *parabolic subgroup* and the group U is the unipotent radical of P, i.e. the largest normal unipotent subgroup of P. See [9] for more details.

The Lie algebra $\mathfrak{u}$ of U is called a *horospherical subalgebra* and the Lie algebra $\mathfrak{p}$ of P is called a *parabolic subalgebra*.

A horospherical subgroup U^- is said to be *opposite* to U if one has the direct sum decomposition $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^-$. Such an opposite subgroup U^- always exists: one can choose U^- to be the *stable group* U_g^- of the element g in G, i.e. $U^- = U_g^- := \{u \in G \mid \lim_{n\to\infty} g^n u g^{-n} = e\}$. The normalizer $P^- := N_G(U^-)$ is said to be *opposite* to P. The intersection $L := P^- \cap P$ is a reductive group, i.e. its unipotent radical is trivial. Moreover, one has both equalities $P = LU$ and $P^- = LU^-$.

The horospherical group U or the parabolic group P is said to be *reflexive* if there exists an element h in G such that hUh^{-1} is opposite to U. The set of such elements h is Zariski open in G.

There are exactly 2^{d-1} horospherical subgroups in the group $G = \mathrm{SL}(d,\mathbb{R})$ up to conjugacy. They are parametrized by the finite sequences of positive integers $\Theta =$

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$(d_1, \dots, d_\ell)$ such that $d_1 + \cdots + d_\ell = d$. For each sequence Θ, one chooses the element g to be diagonal, $g := \mathrm{diag}(t_1, t_2, \dots, t_d)$, with non-increasing positive coefficients $t_1 \geq t_2 \geq \cdots \geq t_d$ such that the d_i are the successive multiplicities of the eigenvalues of g. The parabolic group $P = P_\Theta$ is the group of upper triangular block matrices with diagonal blocks of size $d_1, \dots, d_\ell$. The horospherical group $U = U_\Theta$ is the subgroup of P_Θ for which the diagonal blocks are the identity. For instance, when $\Theta = (d_1, d_2, d_3)$, one has

$$U = \left\{ \left(\begin{array}{c|c|c} 1 & * & * \\ \hline 0 & 1 & * \\ \hline 0 & 0 & 1 \end{array} \right) \right\}, \quad P = \left\{ \left(\begin{array}{c|c|c} * & * & * \\ \hline 0 & * & * \\ \hline 0 & 0 & * \end{array} \right) \right\} \quad \text{and}$$

$$\mathfrak{u} = \left\{ \left(\begin{array}{c|c|c} 0 & * & * \\ \hline 0 & 0 & * \\ \hline 0 & 0 & 0 \end{array} \right) \right\}, \quad \mathfrak{l} = \left\{ \left(\begin{array}{c|c|c} * & 0 & 0 \\ \hline 0 & * & 0 \\ \hline 0 & 0 & * \end{array} \right) \right\}, \quad \mathfrak{u}^- = \left\{ \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline * & 0 & 0 \\ \hline * & * & 0 \end{array} \right) \right\}.$$

This horospherical subalgebra $\mathfrak{u}$ is reflexive when $d_1 = d_3$.

The horospherical subgroups are important when one focuses on lattices of semisimple Lie groups. Indeed, they are useful to understand the structure of the quotient G/Γ near infinity. For instance, one has the following result in [13].

Fact 1.17. (Kazhdan and Margulis) Let G be an algebraic semisimple Lie group with no compact factor and Γ be a lattice in G. The following are equivalent: Γ is not cocompact $\iff$ Γ contains a non-trivial unipotent element $\iff$ there exists a non-trivial horospherical subgroup U of G such that $\Gamma \cap U$ is cocompact in U.

Margulis' first approach in [13] to prove his arithmeticity theorem (Fact 1.15) for non-cocompact lattices was to focus on this subgroup $\Gamma \cap U$. The main content of our Theorem 1.1 is that, once we know that Γ intersects cocompactly U, one no longer needs to know that Γ is a lattice to conclude the arithmeticity.

In this first lecture, we have completely explained the statement of the main theorem (Theorem 1.1). In the next two lectures, we will prove it for the group $G = \mathrm{SL}(4, \mathbb{R})$ and the horospherical subgroup $U = U_\Theta$ with $\Theta = (2, 2)$. In the last lecture, we will discuss other examples.

2. *Second lecture. Closedness of the L-orbits*

In this second lecture and the next one, we plan to prove the following special case of our main theorem (Theorem 1.1).

THEOREM 2.1. *Let* $G := \mathrm{SL}(2p, \mathbb{R})$ *with* $p \geq 2$ *and* $U := \left\{ \left(\begin{array}{c|c} 1 & B \\ \hline 0 & 1 \end{array} \right) \right\}$, *where* B *is in* $\mathcal{M}(p, \mathbb{R})$. *Let* Γ *be a discrete Zariski dense subgroup of* G *such that* $\Gamma \cap U$ *is cocompact in* U. *Then* Γ *is commensurable to an arithmetic lattice* $G_\mathbb{Z}$ *of* G.

Recall that this special case, and more generally the case where G is simple and $\mathbb{R}$-split, is due to Hee Oh in [19]. We will explain, in this lecture and the next one, a strategy for

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this reflexive commutative case that can be extended to all cases as we will see in the last lecture.

Remark 2.2. The conclusion is not true for $p = 1$. Indeed, in the group $G = \mathrm{SL}(2, \mathbb{R})$, the subgroup generated by the matrices $\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ has infinite index in $\mathrm{SL}(2, \mathbb{Z})$ and therefore is not a lattice in G; see [1, Figure 5.3.2] for details.

In geometric language this means that 'hyperbolic surfaces might have cusps even when they have infinite area'.

2.1. *The main example.* Before entering the proof, let us give a few examples of arithmetic lattices $G_{\mathbb{Z}}$ that can occur in the conclusion of Theorem 2.1. Here are three of them.

Example 2.3. The group $G_{\mathbb{Z}} = \mathrm{SL}(2p, \mathbb{Z})$.

Example 2.4. The group $G_{\mathbb{Z}} = \mathrm{SL}(2, D_{\mathbb{Z}})$, where $D_{\mathbb{Z}}$ is the ring of integers of a division algebra $D_{\mathbb{Q}}$ of dimension p^2 over $\mathbb{Q}$ such that $\mathbb{R} \otimes_{\mathbb{Q}} D_{\mathbb{Q}}$ is isomorphic to the algebra of matrices $\mathcal{M}(p, \mathbb{R})$.

We recall that such a division algebra $D_{\mathbb{Q}}$ exists for all $p \geq 1$, and that $\mathrm{SL}(2, D_{\mathbb{Q}})$ is the group of elements of norm one in the central simple algebra $\mathcal{M}(2, D_{\mathbb{Q}})$.

Example 2.5. The unitary group associated to a real quadratic field, like $G_{\mathbb{Z}} = \{g \in \mathrm{SL}(2p, \mathbb{Z}[\sqrt{2}]) \mid g^{\sigma} = {}^t g^{-1}\}$, where σ is the Galois automorphism.

We need one word of explanation for this last example. We recall that, for $g = a + b\sqrt{2}$ with a, b integral matrices, one has $g^{\sigma} := a - b\sqrt{2}$. As in Example 1.8, it is an exercise to check that Γ is discrete in G. By the Borel and Harish-Chandra theorem, $G_{\mathbb{Z}}$ is a lattice in G. This group $G_{\mathbb{Z}}$ does not intersect cocompactly the horospherical group U but it intersects cocompactly a conjugate of U. The reason is that the hermitian form h on $\mathbb{Q}[\sqrt{2}]^{2p}$ given by $h(z_1, \ldots, z_{2p}) = \sum_i z_i z_i^{\sigma}$ admits a p-dimensional isotropic subspace W, for instance the one spanned by the vectors $f_i := e_i + (\sqrt{2}-1)e_{p+i}$ for $1 \leq i \leq p$.

The two main features of this horospherical group U are that U is commutative and reflexive. This means that U is conjugate to the opposite horospherical group $U^- := \left\{\left(\begin{array}{c|c} 1 & 0 \\ \hline C & 1 \end{array}\right)\right\}$, where C is in $\mathcal{M}(p, \mathbb{R})$.

We now begin the proof of Theorem 2.1 following [3, §3]. We first introduce some notation and explain some preliminary reduction. We choose notation that will make it easier to extend the proof to other cases.

We write $V = \mathbb{R}^{2p} = W \oplus W^-$, where $W = \mathbb{R}^p \times 0$ and $W^- = 0 \times \mathbb{R}^p$, so that the normalizer P of U is the stabilizer of W and the normalizer P^- of U^- is the stabilizer of W^-. We set $L := P \cap P^-$, so that $P = LU$ and $P^- = LU^-$. In concrete terms, one has

$$P = \left\{\left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right)\right\}, \quad L = \left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)\right\}, \quad P^- = \left\{\left(\begin{array}{c|c} A & 0 \\ \hline C & D \end{array}\right)\right\},$$

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where $\det A \ \det D = 1$. Let $\mathfrak{g}$, $\mathfrak{p}$, $\mathfrak{p}^-$, $\mathfrak{u}$, $\mathfrak{u}^-$ and $\mathfrak{l}$ be the corresponding Lie algebras. For instance, one has

$$\mathfrak{u} = \left\{\left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)\right\}, \quad \mathfrak{l} = \left\{\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)\right\}, \quad \mathfrak{u}^- = \left\{\left(\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right)\right\},$$

where $\operatorname{trace}(A) + \operatorname{trace}(D) = 0$. We claim that one can assume that

$$\Gamma \cap U^- \text{ is cocompact in } U^-.$$

Indeed, since Γ is Zariski dense in G, there exists $\gamma_0 = \left(\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right) \in \Gamma$ such that $W \cap \gamma_0(W) = \{0\}$ or equivalently such that $\det(C_0) \neq 0$. Changing the basis of $\mathbb{R}^{2p}$, we can assume that

$$\gamma_0 = \left(\begin{array}{c|c} 0 & B_0 \\ \hline 1 & D_0 \end{array}\right) \text{ so that } U^- = \gamma_0 U \gamma_0^{-1}.$$

We need more notation. Since U is a commutative unipotent group, the exponential map $\exp : \mathfrak{u} \to U$ is a group isomorphism: for all X, X' in $\mathfrak{u}$, one has

$$\exp(X + X') = \exp(X) \exp(X').$$

We denote by $\log : U \to \mathfrak{u}$ the inverse map, so that the set $\Lambda := \log(\Gamma \cap U)$ is a lattice in the vector space $\mathfrak{u}$. In term of matrices, one has

$$\Lambda = \left\{ X = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right) \text{ such that } \left(\begin{array}{c|c} 1 & B \\ \hline 0 & 1 \end{array}\right) \in \Gamma \right\}.$$

Similarly, the set $\Lambda^- := \log(\Gamma \cap U^-)$ is a lattice in the vector space $\mathfrak{u}^-$. We will normalize the Lebesgue measure on $\mathfrak{u}$ and $\mathfrak{u}^-$ so that the lattices Λ and Λ^- have covolume one.

The group L acts by the adjoint action on the Lie algebras $\mathfrak{u}$ and $\mathfrak{u}^-$. This action is given by the left- and right-multiplication in $\mathcal{M}(p, \mathbb{R})$:

$$\operatorname{Ad}\left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right)\left(\left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)\right) = \left(\begin{array}{c|c} 0 & ABD^{-1} \\ \hline 0 & 0 \end{array}\right).$$

We introduce the subgroup of volume-preserving elements

$$L_0 = \{\ell \in L \mid \det_{\mathfrak{u}}(\operatorname{Ad}\ell) = 1\} \simeq \operatorname{SL}(p, \mathbb{R}) \times \operatorname{SL}(p, \mathbb{R})$$

up to finite index.

The first result of this lecture is the closedness of the *single orbit*, i.e. the orbit $L_0\Lambda$ of Λ in the space $X_{\mathfrak{u}}$ of covolume-one lattices in $\mathfrak{u}$ under the adjoint action of L_0.

The true aim of this lecture is the closedness of the *double orbit*, i.e. the orbit of (Λ, Λ^-) in the product space $X_{\mathfrak{u}} \times X_{\mathfrak{u}^-}$ of covolume-one lattices in $\mathfrak{u}$ and $\mathfrak{u}^-$ under the diagonal action of L_0.

PROPOSITION 2.6.

(a) *The single L_0-orbit $L_0\,\Lambda$ is closed in $X_{\mathfrak{u}}$.*

(b) *The double L_0-orbit $L_0(\Lambda, \Lambda^-)$ is closed in $X_{\mathfrak{u}} \times X_{\mathfrak{u}^-}$.*

We first focus on the closedness of the single orbit. The idea will be to introduce a L_0-invariant polynomial F on $\mathfrak{u}$ and to check that the values it takes on Λ form a closed and discrete subset of $\mathbb{R}$. The proof for the double orbit will be very similar.

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2.2. *Using the Bruhat decomposition.* We will use the following decomposition:

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{l} \oplus \mathfrak{u}^-.$$

The first idea is to think of the elements $\mathrm{Ad}g$ of $\mathrm{Ad}G$ as 3×3 block matrices and to extract the upper-left block $M(g)$. Indeed, we denote by the same letter π the injection $\pi : \mathfrak{u} \to \mathfrak{g}$ and the projection $\pi : \mathfrak{g} \to \mathfrak{u}$, and set

$$M(g) := \pi \ \mathrm{Ad}g \ \pi \in \mathrm{End}(\mathfrak{u}).$$

There is a very simple formula for $M(g)$ when g belongs to the set

$$\Omega = U^- L U = \left\{ g = \left(\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array} \right) \in G \mid \det(A_g) \neq 0 \right\}.$$

This set Ω is a Zariski open subset of G called the open Bruhat cell. Every element g of Ω admits a decomposition $g = v\ell u$ with $v \in U^-$, $\ell \in L$ and $u \in U$, which is called the Bruhat decomposition of g. For such an element g, one has the formula

$$M(g)X = \mathrm{Ad}\ell \ X \quad \text{for all } X \text{ in } \mathfrak{u}\,.$$

Remark 2.7. More generally, for all $g = \left(\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array} \right)$ and $X = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array} \right)$, one can check that $M(g)X = \left(\begin{array}{c|c} 0 & A_g B D_{g^{-1}} \\ \hline 0 & 0 \end{array} \right)$.

LEMMA 2.8. *The set* $\{M(g)X \mid g \in \Gamma \cap \Omega\,,\ X \in \Lambda\}$ *is a closed discrete subset of* $\mathfrak{u}$.

Proof. Assume that there exists a converging sequence of distinct elements

$$X'_n = M(g_n)X_n \longrightarrow X'_\infty \in \mathfrak{u}$$

with $g_n = v_n \ell_n u_n \in \Gamma$, where $v_n \in U^-\,,\ \ell_n \in L\,,\ u_n \in U$ and $X_n \in \Lambda$. Since $\Gamma \cap U^-$ is cocompact in U^-, after passing to a subsequence, we can write $v_n = \delta_n^{-1} v'_n$ with $\delta_n \in \Gamma \cap U^-$ and $v'_n \longrightarrow v'_\infty \in U^-$. The following sequence of elements of Γ:

$$\gamma_n := \delta_n g_n e^{X_n} g_n^{-1} \delta_n^{-1} = v'_n \ell_n e^{X_n} \ell_n^{-1} {v'_n}^{-1} = v'_n e^{X'_y n} {v'_n}^{-1}$$

converges to $\gamma_\infty := v'_\infty e^{X'_\infty} {v'_\infty}^{-1}$. Since Γ is discrete, one must have $\gamma_n = \gamma_\infty$ for n large. Since X'_n and X'_∞ are nilpotent matrices and since the exponential map is injective on nilpotent matrices, one deduces that $\mathrm{Ad}(v'_n)\ X'_n = \mathrm{Ad}(v'_\infty)\ X'_\infty$ for n large. Comparing the $\mathfrak{u}$-components of these two vectors, one gets $X'_n = X'_\infty$ for n large, which is a contradiction. □

2.3. *Using the Mahler criterion.* For g in G, we set $\Phi(g) := \det_{\mathfrak{u}}(M(g))$.

Remark 2.9. If $g = \left(\begin{array}{c|c} A_g & B_g \\ \hline C_g & D_g \end{array} \right)$, one can check that $\Phi(g) = \det(A_g)^{2p}$.

LEMMA 2.10. *The set* $\Phi(\Gamma)$ *is a closed discrete subset of* $\mathbb{R}$.

Proof. Let $g_n \in \Gamma$ be a sequence such that $\Phi(g_n) \longrightarrow \Phi_\infty \in \mathbb{R}$. We want to prove that $\Phi(g_n) = \Phi_\infty$ for n large. If this is not the case, the matrices A_{g_n} are invertible for n large, and the lattices $M(g_n)\Lambda$ have uniformly bounded covolume in $\mathfrak{u}$. By Lemma 2.8, the union

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of these lattices does not contain small non-zero vectors. Hence, by the Mahler criterion 1.5, one can assume, after extraction, that this sequence $M(g_n)\Lambda$ converges to a lattice Λ_∞ of $\mathfrak{u}$. By the same Lemma 2.8, the union of these lattices stays in a closed discrete subset of $\mathfrak{u}$. Hence, this sequence must be eventually constant and the sequence of their covolumes $\Phi(g_n)$ too, which is a contradiction. □

2.4. *Closedness of the single and double L-orbits.* For X in $\mathfrak{u}$ and Y in $\mathfrak{u}^-$, we set

$$F(X) := \Phi(e^X\gamma_0) \text{ and } G(X, Y) = \Phi(e^X e^Y).$$

If $X = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right)$ and $Y = \left(\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right)$, a small computation gives

$$F(X) = \det(B)^{2p},$$
$$G(X, Y) = \det(1 + BC)^{2p}.$$

COROLLARY 2.11. *The sets $F(\Lambda)$ and $G(\Lambda, \Lambda^-)$ are closed and discrete in $\mathbb{R}$.*

Remark 2.12. The polynomials F and G satisfy the following equivariance properties. For ℓ in L_0, X in $\mathfrak{u}$ and Y in $\mathfrak{u}^-$, one has

$$F(\mathrm{Ad}\ell\, X) = F(X),$$
$$G(\mathrm{Ad}\ell\, X, \mathrm{Ad}\ell\, Y) = G(X, Y).$$

We will need to know that these transformations $\mathrm{Ad}\ell$ are, up to finite index, the only linear transformations of $\mathfrak{u}$ that preserve F. This is what the following elementary remark tells us.

LEMMA 2.13.

- *The groups $\mathrm{Ad}_{\mathfrak{u}}(L_0)$ and $H := \{\varphi \in \mathrm{SL}(\mathfrak{u}) \mid F \circ \varphi = F\}$ have the same connected component.*
- *Moreover, if ℓ, ℓ' are in L_0 such that $G \circ (\mathrm{Ad}\ell, \mathrm{Ad}\ell') = G$, then $\ell' = \pm\ell$.*

Proof. One way to check the first point is to notice that there are no connected subgroups between $\mathrm{SL}(\mathbb{R}^p) \times \mathrm{SL}(\mathbb{R}^p)$ and $\mathrm{SL}(\mathbb{R}^p \otimes \mathbb{R}^p)$. This follows, looking at the Lie algebra level, from the following decomposition of $\mathfrak{sl}(\mathbb{R}^p \otimes \mathbb{R}^p)$ as a sum of irreducible representations of $\mathrm{SL}(\mathbb{R}^p) \times \mathrm{SL}(\mathbb{R}^p)$, where 1 denotes the identity matrix:

$$\mathfrak{sl}(\mathbb{R}^p \otimes \mathbb{R}^p) = \mathfrak{sl}(\mathbb{R}^p) \otimes \mathfrak{sl}(\mathbb{R}^p) \oplus \mathbb{R}1 \otimes \mathfrak{sl}(\mathbb{R}^p) \oplus \mathfrak{sl}(\mathbb{R}^p) \otimes \mathbb{R}1.$$

The second point can be checked with a similar argument. □

Proof of Proposition 2.6. (a) Let $\ell_n \in L_0$ be such that the sequence of lattices $\mathrm{Ad}\ell_n(\Lambda)$ converges to a lattice Λ_∞ of $\mathfrak{u}$. We write $\mathrm{Ad}\ell_n(\Lambda) = \varphi_n(\Lambda_\infty)$ with $\varphi_n \in \mathrm{SL}(\mathfrak{u})$ converging to e. We want to prove that $\varphi_n \in \mathrm{Ad}L_0$ for n large. For every X in Λ_∞, the sequence $F(\varphi_n(X))$ is in $F(\Lambda)$ and converges to $F(X)$. Since the set $F(\Lambda)$ is closed and discrete, this sequence is eventually constant: for all X in Λ, there exists an integer n_X such that $F(\varphi_n(X)) = F(X)$ for $n \geq n_X$. The time n_X starting from which this equality always holds might depend on X. However, notice that the degrees of these polynomials are uniformly bounded by an integer d_0. Notice also that there exists a finite subset of Λ on

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which the restriction of any non-zero polynomial function on $\mathfrak{u}$ of degree at most d_0 is non-zero. Therefore, one deduces that $F \circ \varphi_n = F$ for n large. Then, since φ_n is near e, Lemma 2.13 tells us that φ_n belongs to $\mathrm{Ad}L_0$ for n large, as required.

(b) The proof is similar using the discreteness of the set $G(\Lambda, \Lambda^-)$. Let $\ell_n \in L_0$ be such that the sequence of lattices $\mathrm{Ad}\ell_n(\Lambda, \Lambda^-)$ converges to a pair $(\Lambda_\infty, \Lambda_\infty^-)$ of lattices. We write $\mathrm{Ad}\ell_n(\Lambda, \Lambda^-) = (\mathrm{Ad}\varepsilon_n\Lambda_\infty, \mathrm{Ad}\varepsilon'_n\Lambda_\infty^-)$ with both ε_n and ε'_n in L_0 converging to e. We want to prove that $\varepsilon_n = \varepsilon'_n$ for n large. For every (X, Y) in $\Lambda_\infty \times \Lambda_\infty^-$, the sequence $G(\mathrm{Ad}\varepsilon_n(X), \mathrm{Ad}\varepsilon'_n(Y))$ is in $G(\Lambda, \Lambda^-)$ and converges to $G(X, Y)$. Since the set $G(\Lambda, \Lambda^-)$ is closed and discrete, this sequence is eventually constant. Since the degrees of these polynomials are uniformly bounded, one deduces that $G \circ (\mathrm{Ad}\varepsilon_n, \mathrm{Ad}\varepsilon'_n) = G$ for n large. Then Lemma 2.13 tells us that $\varepsilon_n = \varepsilon'_n$ for n large, as required. □

In the next lecture we will explain why the closedness of the double orbit implies the arithmeticity of Γ.

3. *Third lecture. From closedness to arithmeticity*

In this third lecture we go on to the proof of Hee Oh's Theorem 2.1, which is the special case of our main theorem (Theorem 1.1) when $G = \mathrm{SL}(2p, \mathbb{R})$ and the normalizer P of U is the stabilizer of the p-plane $\mathbb{R}^p \subset \mathbb{R}^{2p}$. For this second part of the proof, as in Hee Oh's PhD thesis [**19**], we will fit together previously known results: the Dani–Margulis finiteness theorem, the Raghunathan–Venkataramana finite-index subgroup theorem and Margulis' construction of $\mathbb{Q}$-forms. Most of this lecture is devoted to the proof of these classical facts, which are often useful when one studies discrete subgroups of semisimple Lie groups.

We keep the notation $G, U, U^-, P, P^-, L, L_0, \Gamma, \Lambda := \log(\Gamma \cap U)$ and $\Lambda^- := \log(\Gamma \cap U^-)$ of Lecture 2. We know that Λ and Λ^- are lattices in $\mathfrak{u}$ and $\mathfrak{u}^-$. We proved in the previous lecture that the double orbit $L_0(\Lambda, \Lambda^-)$ is closed in the product of the spaces of lattices of $\mathfrak{u}$ and $\mathfrak{u}^-$. We want now to deduce from this closedness that Γ is arithmetic.

3.1. *From closed orbit to infinite stabilizer.*

The first step is a direct application of the following result of Dani and Margulis.

PROPOSITION 3.1. (Dani and Margulis) *Let $d_0 \geq 2$, let $G_0 := \mathrm{SL}(d_0, \mathbb{R})$, let $\Gamma_0 := \mathrm{SL}(d_0, \mathbb{Z})$ and let $X_0 = G_0/\Gamma_0$. Let $S_0 \subset G_0$ be a semisimple subgroup and $x_0 \in X_0$ be the base point. If the S_0-orbit S_0x_0 is closed, then the stabilizer $\Gamma_0 \cap S_0$ is a lattice in S_0.*

COROLLARY 3.2. *The stabilizer of (Λ, Λ^-) in L_0 is a lattice in L_0.*

Proof of Corollary 3.2. The space $X_\mathfrak{u} \times X_{\mathfrak{u}^-}$ is is included in the space of lattices of $\mathfrak{u} \oplus \mathfrak{u}^-$, and the subgroup $\mathrm{Ad}_{\mathfrak{u}\oplus\mathfrak{u}^-}(L_0)$ which is locally isomorphic to $\mathrm{SL}(p, \mathbb{R}) \times \mathrm{SL}(p, \mathbb{R})$ is semisimple. We just combine Propositions 2.6 and 3.1. □

Since Γ_0 is a lattice in G_0, Proposition 3.1 looks like a special case of Ratner's topological theorem in [**24**]. It is! It was known before and it is indeed a key ingredient in the proof of Ratner's theorem.

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The conclusion of Proposition 3.1 is false when the subgroup S_0 is not semisimple. Indeed, when $X = \mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$ is the space of covolume-one lattices in $\mathbb{R}^2$ and when x_0 is the lattice $\mathbb{Z}^2$, the orbit A_0x_0 under the group A_0 of diagonal matrices is closed but not compact.

The proof of Proposition 3.1 relies on the following two classical facts.

Fact 3.3. (Howe–Moore mixing theorem) Let $S_0 = S_1 \cdots S_\ell$ be a connected semisimple Lie group with finite center written as a product of quasisimple factors S_i. Let $(\mathcal{H}_0, \pi_0)$ be a unitary representation of S_0 in a Hilbert space which admits no non-zero S_0-invariant vector. Then one has, for all v_0, w_0 in $\mathcal{H}_0$,

$$\lim_{s \to \infty \text{ in all } S_i} \langle \pi_0(s)v_0, w_0 \rangle = 0.$$

The condition on s means that all the 'components' s_i of s in the quasisimple factors S_i go to infinity. This condition is useful since $\mathcal{H}_0$ might contain non-zero vectors which are invariant under some quasisimple factor S_i. For a proof, see [2, 29].

We will combine this mixing theorem with the following result, which says that in a finite-volume homogeneous space, all unipotent trajectories are 'recurrent in law', i.e. they spend most of their time in a compact set (see [11] for a proof).

Fact 3.4. (Dani–Margulis recurrence theorem) Let $d_0 \geq 2$, $G_0 := \mathrm{SL}(d_0, \mathbb{R})$, $\Gamma_0 := \mathrm{SL}(d_0, \mathbb{Z})$ and $X_0 = G_0/\Gamma_0$. Let $(u_t)_{t \in \mathbb{R}}$ be a one-parameter unipotent subgroup of G_0. For all $\varepsilon > 0$ and x in X_0, there exists a compact subset $K \subset X_0$ such that, for all $T > 0$,

$$\frac{1}{T} |\{t \in [0, T] | u_t x \in K\}| \geq 1 - \varepsilon.$$

Here the symbol $|\cdot|$ means the Lebesgue measure.

Proof of Proposition 3.1. See [15, Theorem 11.5]. We write S_0 as a product $S_0 = S_1 \cdots S_\ell$ of quasisimple factors S_i.

We first assume that none of the factors S_i is compact. This will be the case in our application where $S_1 = S_2 = \mathrm{SL}(p, \mathbb{R})$.

We choose $(u_t)_{t \in \mathbb{R}}$ to be a one-parameter unipotent subgroup of S_0 which is not included in a proper normal subgroup of S_0.

Since the orbit S_0x_0 is closed, there exist a S_0-invariant Radon measure λ_0 on X_0 which is supported by this orbit S_0x_0. We want to prove that λ_0 has finite volume. We choose π_0 to be the regular representation in the Hilbert space of square integrable functions $\mathcal{H}_0 := L^2(X_0, \lambda_0)$.

By the Dani–Margulis recurrence theorem, one can find a compact subset $M \subset X_0$ of positive measure $\lambda_0(M) > 0$ and a compact subset $K \subset X_0$ such that, for all x in M and all $T > 0$,

$$\frac{1}{T} |\{t \leq T \mid u_t x \in K\}| \geq \frac{1}{2}.$$

One can then estimate, using the Fubini theorem, for all $T > 0$, the time average along this flow of the coefficients of the two vectors $\mathbf{1}_M$ and $\mathbf{1}_K$ of $\mathcal{H}_0$:

$$\frac{1}{T} \int_0^T \langle \pi_0(u_t)\mathbf{1}_M, \mathbf{1}_K \rangle \, dt \;=\; \frac{1}{T} \int_M |\{t \leq T \mid u_t x \in K\}| \, d\lambda_0(x) \geq \frac{1}{2} \lambda_0(M) > 0.$$

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This average does not converge to 0. Therefore, the Howe–Moore mixing theorem tells us that the Hilbert space $L^2(X_0, \lambda_0)$ contains a non-zero S_0-invariant function φ. This function φ must be λ_0-almost surely constant. Since this function is square integrable, λ_0 has finite volume. □

Remark 3.5. We need to discuss also the general case where S_0 is a product $S_0 = S_c S_{nc}$ of a compact normal Lie subgroup S_c and a normal Lie subgroup S_{nc} with non-compact factors. Indeed, this is useful for the cases in §4.6. The closure H of the group $(\Gamma_0 \cap S_0)S_{nc}$ is an intermediate subgroup $S_{nc} \subset H \subset S_0$. The H-orbit Hx_0 is closed and the H-invariant measure λ_H on Hx_0 is S_{nc}-ergodic. Therefore, we can repeat the above argument with the representation of S_{nc} in $L^2(X_0, \lambda_H)$.

Here is another corollary of Proposition 3.1.

COROLLARY 3.6. *The group* $\mathrm{SL}(d, \mathbb{Z})$ *is a lattice in* $\mathrm{SL}(d, \mathbb{R})$.

Proof. The proof of Fact 3.4 relies on the Mahler compactness theorem and does not use the fact that $\mathrm{SL}(d, \mathbb{Z})$ is a lattice in $\mathrm{SL}(d, \mathbb{R})$. Therefore, Proposition 3.1 applied with $S_0 = G_0$ gives a dynamical proof of the finiteness of the volume of the quotient space $\mathrm{SL}(d, \mathbb{R})/\mathrm{SL}(d, \mathbb{Z})$, which was already claimed with no proof in Example 1.4. □

Proposition 3.1 gives also a proof of Example 1.6.

COROLLARY 3.7. *Let* $d \geq 3$ *and* Q *be a non-degenerate integral quadratic form on* $\mathbb{R}^d$. *Then the group* $\mathrm{SO}(Q, \mathbb{Z})$ *is a lattice in* $\mathrm{SO}(Q, \mathbb{R})$.

Proof. We use the notation of Proposition 3.1 with $d_0 = d$ and we set $S_0 := \mathrm{SO}(Q, \mathbb{R})$. The homogeneous space $Y_0 := S_0 \backslash G_0$ can be seen as a set of quadratic forms in $\mathbb{R}^d$. We notice that the S_0-orbit $S_0 x_0$ is closed in the quotient space X_0 because the corresponding Γ_0-orbit in Y_0 is closed and discrete in Y_0. Indeed this Γ_0-orbit consists of integral quadratic forms. Therefore, Proposition 3.1 tells us that $S_0 \cap \Gamma_0$ is a lattice in G. □

Using similar ideas, Proposition 3.1 gives also a proof of the Borel and Harish-Chandra theorem 1.12.

3.2. *From infinite stabilizer to infinite intersection.* In the next part 3.3 of this lecture we will prove the following proposition.

PROPOSITION 3.8. (Margulis construction of $\mathbb{Q}$-forms) *Same assumption as in Theorem 2.1. If the intersection* $\Gamma \cap L_0$ *is a lattice in* L_0, *then there exists a* $\mathbb{Q}$*-form of* G *such that* $\Gamma \subset G_{\mathbb{Q}}$.

This proposition is due to Margulis in [**13**, §7 and §8] and to Hee Oh in [**19**, Proposition 2.4.2] in this case at hand where the subgroup L_0 has no compact factor. In the next section, we will sketch a proof of Proposition 3.8 that can be adapted to all semisimple groups.

Here is the general version of Proposition 3.8 that applies to all of the groups in Theorem 1.1 and whose proof can be found in [**3**, Proposition 4.11].

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PROPOSITION 3.9. *Let G be an adjoint semisimple real algebraic Lie group, U, U^- be two opposite unipotent subgroups and L be the intersection of their normalizers in G. Let Γ be a discrete Zariski dense subgroup of G such that $\Gamma \cap U$ is an irreducible lattice of U and $\Gamma \cap U^-$ is an irreducible lattice of U^-. Assume that the group $\Gamma \cap L$ is infinite.*

Then there exists a $\mathbb{Q}$-form $G_{\mathbb{Q}}$ of G such that $\Gamma \subset G_{\mathbb{Q}}$

First, we explain how to combine Proposition 3.8 with the following fact to complete the proof of Theorem 2.1. This fact, which is due to Raghunathan and Venkataramana in [**23**, **28**] (see also [**19**, Proposition 2.2.2]), says that Theorem 1.1 is true when U is reflexive and when there exists a $\mathbb{Q}$-structure on G such that $\Gamma \subset G_{\mathbb{Q}}$.

We say that a $\mathbb{Q}$-structure on G is $\mathbb{Q}$-simple if G does not contain a non-trivial normal subgroup defined over $\mathbb{Q}$.

Fact 3.10. (Raghunathan–Venkataramana finite-index subgroup theorem) Let G be a semisimple real algebraic Lie group of real rank at least two which is defined over $\mathbb{Q}$ and is $\mathbb{Q}$-simple. Let U and U^- be opposite horospherical subgroups which are defined over $\mathbb{Q}$ and $\Delta \subset U_{\mathbb{Z}}$ and $\Delta^- \subset U^-_{\mathbb{Z}}$ be finite-index subgroups. Then the subgroup Γ generated by Δ and Δ^- has finite index in $G_{\mathbb{Z}}$.

The proof of this fact is related to the solution of the congruence subgroup problem in [**22**]. We will prove it only for $G_{\mathbb{Q}} = \mathrm{SL}(d, \mathbb{Q})$ in §3.4.

Proof of Theorem 2.1 using Proposition 3.8. We introduce the subgroup Γ' of Γ generated by the lattices $\Gamma \cap U$ and $\Gamma \cap U^-$. We also introduce the group $\Gamma'' \supset \Gamma'$, the normalizer of Γ' in G. By Remark 3.11, the group Γ'' is discrete. By construction, this group contains the stabilizer of (Λ, Λ^-) in L_0. Hence, by Corollary 3.2, the intersection $\Gamma'' \cap L_0$ is a lattice in L_0. By Proposition 3.8, there exists a $\mathbb{Q}$-form $G_{\mathbb{Q}}$ of G such that $\Gamma'' \subset G_{\mathbb{Q}}$. In particular, one has $\Gamma' \subset G_{\mathbb{Q}}$. Since the lattice $\Gamma \cap U$ is irreducible in G, this $\mathbb{Q}$-form is $\mathbb{Q}$-simple. Since $\Gamma \cap U$ contains a finite-index subgroup of $U_{\mathbb{Z}}$ and $\Gamma \cap U^-$ contains a finite-index subgroup of $U^-_{\mathbb{Z}}$, by Fact 3.10, the group Γ' contains a subgroup commensurable to $G_{\mathbb{Z}}$. Since Γ is discrete and $G_{\mathbb{Z}}$ is a lattice in G, Γ itself is commensurable to $G_{\mathbb{Z}}$. □

In the above proof, we used the following lemma, whose proof is a simple exercise.

LEMMA 3.11. *Let G be an algebraic group with finite center and $\Gamma' \subset G$ a discrete Zariski dense subgroup. Then the normalizer $\Gamma'' := N_G(\Gamma')$ is a discrete Zariski dense subgroup of G.*

Remark 3.12. Let us now explain how the higher rank assumption on G in Theorem 1.1 is used. This assumption is equivalent to the non-compactness of the group $L_0 := \{\ell \in L \mid \det_{\mathfrak{u}}(\mathrm{Ad}\ell) = 1\}$. When L_0 is non-compact, the output of an analog of Corollary 3.2 will tell us that the stabilizer of (Λ, Λ^-) in L is infinite. This is exactly this information that is needed in order to apply Proposition 3.9.

When the group L_0 is non-compact but the semisimple group $S_0 := [L, L]$ is compact, one cannot use the Dani–Margulis proposition (Proposition 3.1) any more to prove that this stabilizer is infinite. We will explain how to deal with this issue in §4.4 and §4.5.

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3.3. *Construction of $\mathfrak{g}_{\mathbb{Q}}$ when $\Gamma \cap L_0$ is a lattice.* The aim of this section is to prove Proposition 3.8.

We recall that $G = \mathrm{SL}(2p, \mathbb{R})$ with $p \geq 2$ and that L_0 is the block diagonal subgroup $L_0 \simeq \mathrm{SL}(p, \mathbb{R}) \times \mathrm{SL}(p, \mathbb{R})$. Let $A := \left\{ \left(\begin{array}{c|c} \lambda I & 0 \\ \hline 0 & \lambda^{-1} I \end{array} \right) \right\}$ be the centralizer of L_0 in G, so that one has $L = L_0 A$. At the level of Lie algebras, one has the decompositions

$$\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{l} \oplus \mathfrak{u}^- \quad \text{and} \quad \mathfrak{l} = \mathfrak{l}_0 \oplus \mathfrak{a}.$$

The first part of the proof is the remark that we have no choice on what should be the vector subspace $\mathfrak{g}_{\mathbb{Q}}$. Let us explain why. Assume for a while that $\mathfrak{g}_{\mathbb{Q}}$ exists.

- The group $U \cap G_{\mathbb{Q}}$ must contain the Zariski dense subgroup $\Gamma \cap U$ of U. Therefore, the subgroup U of G must be defined over $\mathbb{Q}$ and
$$\mathfrak{u}_{\mathbb{Q}} \text{ must be the } \mathbb{Q}\text{-span of } \Lambda. \tag{1}$$
- Similarly, the subgroup U^- must be defined over $\mathbb{Q}$ and
$$\mathfrak{u}^-_{\mathbb{Q}} \text{ must be the } \mathbb{Q}\text{-span of } \Lambda^-. \tag{2}$$
- The normalizers P and P^- must also be defined over $\mathbb{Q}$, their intersection L too. The adjoint action of L on $\mathfrak{u}$ must be defined over $\mathbb{Q}$. Therefore,
$$\mathfrak{l}_{\mathbb{Q}} \text{ must be } \{E \in \mathfrak{l} \mid [E, \mathfrak{u}_{\mathbb{Q}}] \subset \mathfrak{u}_{\mathbb{Q}}\}. \tag{3}$$

In the second part of the proof, we define $\mathfrak{g}_{\mathbb{Q}}$ as the $\mathbb{Q}$-vector subspace

$$\mathfrak{g}_{\mathbb{Q}} = \mathfrak{u}_{\mathbb{Q}} \oplus \mathfrak{l}_{\mathbb{Q}} \oplus \mathfrak{u}^-_{\mathbb{Q}},$$

the direct sum of the above spaces, and we check in the following lemma that it gives a Γ-invariant $\mathbb{Q}$-form of $\mathfrak{g}$.

We will say that a horospherical subgroup U_1 is Γ-*compact* if $\Gamma \cap U_1$ is cocompact in U_1.

For each pair (U_1, U_1^-) of opposite Γ-compact horospherical subgroups we also define, by the same formulas as in (1) and (3), a $\mathbb{Q}$-form $\mathfrak{p}_{1,\mathbb{Q}} = \mathfrak{l}_{1,\mathbb{Q}} \oplus \mathfrak{u}_{1,\mathbb{Q}}$ on $\mathfrak{p}_1 := \mathfrak{l}_1 \oplus \mathfrak{u}_1$. The key point of the following lemma will be to check, for all such pairs (U_1, U_1^-), that the subspace $\mathfrak{p}_1$ of $\mathfrak{g}$ is defined over $\mathbb{Q}$ and that $\mathfrak{p}_{1,\mathbb{Q}} = \mathfrak{p}_1 \cap \mathfrak{g}_{\mathbb{Q}}$.

Remark 3.13. This lemma is delicate since, at this stage of the construction, we do not know that $\mathfrak{g}_{\mathbb{Q}}$ is a Lie algebra. However, by construction, we know that the subspace $\mathfrak{p}_{\mathbb{Q}} := \mathfrak{l}_{\mathbb{Q}} \oplus \mathfrak{u}_{\mathbb{Q}}$ is a Lie algebra.

LEMMA 3.14.

(a) *One has $\mathfrak{g} = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{g}_{\mathbb{Q}}$ and $\mathfrak{p}_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of $\mathfrak{p}$.*
(b) *The $\mathbb{Q}$-structure $\mathfrak{l}_{\mathbb{Q}}$ on $\mathfrak{l}$ does not change if we swap the roles of U and U^-.*
(c) *The $\mathbb{Q}$-structure $\mathfrak{p}_{\mathbb{Q}}$ on $\mathfrak{p}$ does not depend on the choice of the Γ-compact opposite horospherical subgroup U^-.*
(d) *For any Γ-compact horospherical subgroup U_1, the $\mathbb{Q}$-structure on its normalizer $\mathfrak{p}_1$ is equal to $\mathfrak{p}_{1,\mathbb{Q}} = \mathfrak{p}_1 \cap \mathfrak{g}_{\mathbb{Q}}$.*
(e) *The $\mathbb{Q}$-vector space $\mathfrak{g}_{\mathbb{Q}}$ does not depend on the choice of U and U^-.*

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(f) *The $\mathbb{Q}$-vector space $\mathfrak{g}_{\mathbb{Q}}$ is $\mathrm{Ad}\Gamma$-invariant.*

(g) *The $\mathbb{Q}$-vector space $\mathfrak{g}_{\mathbb{Q}}$ is a $\mathbb{Q}$-form of the Lie algebra $\mathfrak{g}$.*

Proof. (a) and (b) will be proven simultaneously. By construction, one has $\mathfrak{u} = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{u}_{\mathbb{Q}}$, one has $\mathfrak{u}^- = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{u}^-_{\mathbb{Q}}$ and also $[\mathfrak{p}_{\mathbb{Q}}, \mathfrak{p}_{\mathbb{Q}}] \subset \mathfrak{p}_{\mathbb{Q}}$. We only need to check that $\mathfrak{l} = \mathbb{R} \otimes_{\mathbb{Q}} \widetilde{\mathfrak{l}}_{\mathbb{Q}}$, where

$$\widetilde{\mathfrak{l}}_{\mathbb{Q}} := \{E \in \mathfrak{l} \mid [E, \mathfrak{u}_{\mathbb{Q}}] \subset \mathfrak{u}_{\mathbb{Q}} \text{ and } [E, \mathfrak{u}^-_{\mathbb{Q}}] \subset \mathfrak{u}^-_{\mathbb{Q}}\}.$$

We know that the group $\Gamma \cap L_0$ is a lattice in L_0. By the Borel density Theorem 1.14, this group $\Gamma \cap L_0$ is Zariski dense in L_0. The adjoint action of this group on $\mathfrak{u} \oplus \mathfrak{u}^-$ preserves $\mathfrak{u}_{\mathbb{Q}} \oplus \mathfrak{u}^-_{\mathbb{Q}}$. Therefore, the Zariski closure H of the group $\mathrm{Ad}_{\mathfrak{u}\oplus\mathfrak{u}^-}(\Gamma \cap L_0)$ in $\mathrm{GL}(\mathfrak{u} \oplus \mathfrak{u}^-)$ is defined over $\mathbb{Q}$. The Lie algebra of H, which is equal to $\mathrm{ad}_{\mathfrak{u}\oplus\mathfrak{u}^-}(\mathfrak{l}_0)$, is also defined over $\mathbb{Q}$. This proves the equality $\mathfrak{l}_0 = \mathbb{R} \otimes_{\mathbb{Q}} \widetilde{\mathfrak{l}}_{0,\mathbb{Q}}$, where $\widetilde{\mathfrak{l}}_{0,\mathbb{Q}} := \widetilde{\mathfrak{l}}_{\mathbb{Q}} \cap \mathfrak{l}_0$.

Since the group A acts on $\mathfrak{u}$ and $\mathfrak{u}^-$ by scalar matrices with inverse ratios, one also has $\mathfrak{a} = \mathbb{R} \otimes_{\mathbb{Q}} \widetilde{\mathfrak{a}}_{\mathbb{Q}}$, where $\widetilde{\mathfrak{a}}_{\mathbb{Q}} := \mathfrak{l}_{\mathbb{Q}} \cap \mathfrak{a}$. Therefore, one has $\mathfrak{l} = \mathbb{R} \otimes_{\mathbb{Q}} \mathfrak{l}_{\mathbb{Q}}$.

(c) For any Γ-compact horospherical subgroup U' opposite to U, let P' be the normalizer of U' and $L' := P \cap P'$ and

$$\mathfrak{l}'_{\mathbb{Q}} := \{E \in \mathfrak{l}' \mid [E, \mathfrak{u}_{\mathbb{Q}}] \subset \mathfrak{u}_{\mathbb{Q}}\}.$$

By point (a), this is a $\mathbb{Q}$-form of $\mathfrak{l}'$. We want to check that the inclusion $\mathfrak{l}' \cap \mathfrak{p}_{\mathbb{Q}} \subset \mathfrak{l}'_{\mathbb{Q}}$ is an equality. It is enough to check that the subalgebra $\mathfrak{l}'$ of $\mathfrak{p}$ is defined over $\mathbb{Q}$. As above, we write $L' = L'_0 A'$. By Lemma 3.15 below, the group $\Gamma \cap L'_0$ is a lattice in L'_0; this will allow us to use a similar argument as in (a) and (b). Indeed, by the Borel density Theorem 1.14, this group $\Gamma \cap L'_0$ is Zariski dense in L'_0. The adjoint action of this group on $\mathfrak{p}$ preserves $\mathfrak{p}_{\mathbb{Q}}$. Therefore, the Zariski closure of the subgroup $\mathrm{Ad}_{\mathfrak{p}}(L'_0)$ of $\mathrm{GL}(\mathfrak{p})$ is defined over $\mathbb{Q}$, and the Lie subalgebra $\mathfrak{l}'_0$ of $\mathfrak{p}$ is also defined over $\mathbb{Q}$.

Since the Lie algebra $\mathfrak{a}'$ is the centralizer of $\mathfrak{l}'_0$ in $\mathfrak{p}$, it is also defined over $\mathbb{Q}$ and the Lie subalgebra $\mathfrak{l}'$ of $\mathfrak{p}$ is defined over $\mathbb{Q}$, as required.

(d) We want to check that the Lie subalgebra $\mathfrak{p}_1$ of $\mathfrak{g}$ is defined over $\mathbb{Q}$, and that the $\mathbb{Q}$-structure $\mathfrak{p}_{1,\mathbb{Q}}$ introduced above is the one induced by $\mathfrak{g}_{\mathbb{Q}}$.

We first assume that U_1 is opposite to both U and U^-. In this case, one checks, by explicit dimension estimates, the equality $\mathfrak{p}_1 = (\mathfrak{p}_1 \cap \mathfrak{p}) + (\mathfrak{p}_1 \cap \mathfrak{p}^-)$. By point (c), the $\mathbb{Q}$-structure on both spaces is the one induced by $\mathfrak{g}_{\mathbb{Q}}$.

In general, we just use a pair of opposite horospherical subgroups U_2, U_2^- such that both of them are simultaneously opposite to U, U^- and U_1 and, using three times the first case, we prove successively our claim for U_2, U_2^- and U_1.

(e) This follows from point (d).

(f) This follows from point (e).

(g) By point (f), for all $X \in \Lambda$ and all integers n, the matrix $e^{n\,\mathrm{ad}X}$ preserves $\mathfrak{g}_{\mathbb{Q}}$. Therefore, the nilpotent matrix $\mathrm{ad}X$ also preserves $\mathfrak{g}_{\mathbb{Q}}$. This proves that $[\mathfrak{u}_{\mathbb{Q}}, \mathfrak{g}_{\mathbb{Q}}] \subset \mathfrak{g}_{\mathbb{Q}}$. Similarly, one has $[\mathfrak{u}^-_{\mathbb{Q}}, \mathfrak{g}_{\mathbb{Q}}] \subset \mathfrak{g}_{\mathbb{Q}}$ and hence $[\mathfrak{g}_{\mathbb{Q}}, \mathfrak{g}_{\mathbb{Q}}] \subset \mathfrak{g}_{\mathbb{Q}}$. □

In this proof we used the following lemma, which involves the subgroup $P_0 := L_0 U$ of P. This group is called the *unimodular normalizer* of U. This lemma focuses on various orbits of the basis point $x_0 := \Gamma/\Gamma$ on $X := G/\Gamma$.

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LEMMA 3.15. *Let G be a semisimple algebraic Lie group, P and P' be opposite parabolic subgroups, U and U' their unipotent radicals and $L' := P \cap P'$ so that $P = L'U$ and $P' = L'U'$. Let $P_0 := \{p \in P \mid \det_u \mathrm{Ad}\, p = 1\}$ and $L'_0 := L' \cap P_0$. Let Γ be a discrete subgroup of G.*

(a) *If the orbit Ux_0 is compact, then the orbit P_0x_0 is closed.*

(b) *If both orbits Ux_0 and $U'x_0$ are compact, then the orbit L'_0x_0 is also closed.*

(c) *In this case, the group $(\Gamma \cap U)(\Gamma \cap L'_0)$ has finite index in $\Gamma \cap P_0$.*

Remark 3.16. Since the group $\Gamma \cap P$ normalizes the lattice $\Delta := \Gamma \cap U \subset U$, one has $\Gamma \cap P \subset P_0$. Moreover, when $\Gamma \cap P_0$ is a lattice in P_0, Lemma 3.15 tells us that $\Gamma \cap L'_0$ is a lattice in L'_0.

Proof. (a) Let $p_n \in P_0$ be a sequence such that the sequence $x_n := p_nx_0$ converges to a point $x_\infty \in X$. We want to prove that x_∞ is still on the orbit P_0x_0.

We can find $\gamma_n \in \Gamma$ such that the sequence $g_n := p_n\gamma_n$ converges to an element $g_\infty \in G$. Since the sequence g_n converges, there exists a neighborhood Ω_0 of e in G such that the groups $\Delta'_n := p_n\Delta p_n^{-1} \subset g_n\Gamma g_n^{-1}$ intersect Ω_0 trivially. Since these lattices of U have the same covolume, by the Mahler compactness criterion, after extraction, these lattices Δ'_n converge to a lattice Δ'_∞ of U. Therefore, the subgroups $\Delta_n = g_n^{-1}\Delta'_n g_n = \gamma_n^{-1}\Delta\gamma_n$ of Γ converge to the subgroup $\Delta_\infty := g_\infty^{-1}\Delta'_\infty g_\infty$.

Since the group Δ_∞ is a lattice in a unipotent group, it is finitely generated (see [**21**, Theorem 2.10]). For instance, in the case of Theorem 2.1, the group Δ_∞ is isomorphic to $\mathbb{Z}^{p^2}$. Since Γ is discrete and Δ_∞ is finitely generated, there exists $n_0 \geq 1$ such that $\Delta_n = \Delta_\infty$ for all $n \geq n_0$. Therefore, the elements $\delta_n := \gamma_n\gamma_{n_0}^{-1}$ belong to $\Gamma \cap P_0$ for all $n \geq n_0$, the sequence $p_n\delta_n \in P_0$ converges to $p_\infty := g_\infty\gamma_{n_0}^{-1} \in P_0$ and one has $x_\infty = p_\infty x_0$.

(b) By point (a), the orbits P_0x_0 and $P_0^-x_0$ are closed. Since Γ is discrete, all the L_0-orbits in the intersection $P_0x_0 \cap P_0^-x_0$ are open. Hence, they are also closed.

(c) It is a classical fact that the topology induced on a closed orbit Hx_0 coincides with the quotient topology on $H/(\Gamma \cap H)$.

Assume by contradiction that the group $(\Gamma \cap U)(\Gamma \cap L'_0)$ has infinite index in $\Gamma \cap P_0$. Let $\gamma_n = u_n\ell_n$ with $\ell_n \in L'_0$, $u_n \in U$ be a sequence of elements of $\Gamma \cap P_0$ whose images in $(\Gamma \cap U)\backslash G/(\Gamma \cap L'_0)$ are distinct. Since $\Gamma \cap U$ is cocompact in U, one can assume that the sequence u_n converges. Then the sequence $\ell_nx_0 = u_n^{-1}x_0$ converges in X. Since the orbit L'_0x_0 is closed, one can find δ_n in $\Gamma \cap L_0$ such that the sequence $m_n := \ell_n\delta_n$ converges in L_0. Then the sequence $\gamma_n\delta_n = u_nm_n \in \Gamma \cap P_0$ also converges. Since Γ is discrete, this sequence is constant for n large, which is a contradiction. □

3.4. *Finite-index subgroups in* $\mathrm{SL}(d, \mathbb{Z})$. The aim of this section is to give a flavor of the meaning of Fact 3.10 by providing a complete proof on the simplest example, i.e. when $G_\mathbb{Z} = \mathrm{SL}(d, \mathbb{Z})$. In this case, Fact 3.10 is due to Tits and Vaserstein in [**26**, **27**]. The proof is very algebraic. Indeed, some of the arguments in the proof come from the congruence subgroup problem.

In this section, which uses very algebraic techniques, we will use different notation to avoid confusion. For all $d \geq 2$, we denote $F := \mathrm{SL}(d, \mathbb{Z})$. For $i \neq j$, we denote by $u_{i,j}$ the

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unipotent matrix $u_{i,j} = 1 + e_i \otimes e_j^*$ whose non-diagonal coefficients are null except at the spot (i, j) where it equals 1. For all $n \in \mathbb{Z}$, one has

$$u_{i,j}^n = 1 + n\, e_i \otimes e_j^*.$$

We first recall the following basic lemma.

LEMMA 3.17. *The group F is generated by the matrices $u_{i,j}$ with $i \neq j$.*

Proof. Right multiplying a matrix g by an element $u_{i,j}^n$ is a 'move' that consists in adding n times the ith column to the jth column. Using the euclidean algorithm, we can use these moves to replace the upper-left coefficient of g by a 1. Using again these moves, we can then replace all the other coefficients of the first row by 0. Going on, we can replace g by a lower triangular matrix with 1 on the diagonal. Using once more these moves, we replace all but the last coefficients of the last row by 0. Going on, we can replace g by the identity matrix. □

We will prove the following result.

PROPOSITION 3.18. (Tits and Vaserstein) *Let $d \geq 3$ and $n \geq 1$. Let F^n be the subgroup of $\mathrm{SL}(d, \mathbb{Z})$ generated by the unipotent matrices $u_{i,j}^n$ with $i \neq j$. This group F^n has finite index in $\mathrm{SL}(d, \mathbb{Z})$.*

For f, g in F, we denote by $[f, g] := fgf^{-1}g^{-1}$ their commutator. In the proofs below, we will use implicitly the following formulas for the commutators of elementary matrices when $(k, \ell) \neq (j, i)$ and $m, n \in \mathbb{Z}$:

$$[u_{i,j}^m, u_{k,\ell}^n] = \begin{cases} u_{i,\ell}^{mn} & \text{when } j = k, \\ u_{k,j}^{-mn} & \text{when } i = \ell, \\ e & \text{otherwise.} \end{cases}$$

We split Proposition 3.18 into two lemmas (Lemmas 3.19 and 3.20). We first introduce the smallest normal subgroup E^n of F containing F^n. The first lemma tells us that F^n contains a normal subgroup of F.

LEMMA 3.19. *One has the inclusion $E^{n^2} \subset F^n$.*

Proof. For $i \neq j$, we introduce:

- the subgroup $F_{i,j} \simeq \mathrm{SL}(2, \mathbb{Z})$ generated by $u_{i,j}$ and $u_{j,i}$;
- the subgroup $F_{i,j}^n \subset F_{i,j}$ generated by $u_{i,j}^n$ and $u_{j,i}^n$;
- the smallest normal subgroup $E_{i,j}^n$ of $F_{i,j}$ containing $F_{i,j}^n$;
- the group $\widetilde{E}^n \subset E^n$ generated by all the subgroups $E_{i,j}^n$.

First claim. One has $\widetilde{E}^n = E^n$.

It is enough to check that $\widetilde{E}^n$ is normal in F. For that, we check that for all $i \neq j$ and $k \neq \ell$, the conjugate $u_{k,\ell} u_{i,j}^n u_{k,\ell}^{-1}$ or, equivalently, the commutator $[u_{k,\ell}, u_{i,j}^n]$ belongs to $\widetilde{E}^n$. When $(k, \ell) = (i, j)$ or (j, i), by definition, this commutator belongs to $E_{i,j}^n$. When

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$\ell = i$, this commutator belongs to $F^n_{k,j}$. Similarly, when $j = k$, this commutator belongs to $F^n_{i,\ell}$. In the other cases, this commutator is trivial.

Second claim. One has $\widetilde{E}^{\,n^2} \subset F^n$.

We fix $i \neq j$. It is enough to check that $E^{n^2}_{i,j} \subset F^n$. Let $\widetilde{F}^n_{i,j}$ be the subgroup of F^n generated by all the $u^n_{k,\ell}$ with $(k, \ell) \neq (i, j)$ and (j, i). We notice, using again the commutator formulas, that, for all f in $F_{i,j}$, one has

$$f\,\widetilde{F}^n_{i,j}\,f^{-1} \subset \widetilde{F}^n_{i,j}.$$

As $d \geq 3$, one can find $k \neq i, j$. The equality $[u^n_{i,k}, u^n_{k,j}] = u^{n^2}_{i,j}$ tells us that

$$F^{n^2}_{i,j} \subset \widetilde{F}^n_{i,j}.$$

Therefore, one has $f\,F^{n^2}_{i,j}\,f^{-1} \subset \widetilde{F}^n_{i,j}$ and $E^{n^2}_{i,j} \subset F^n$, as required. □

We could stop the proof of Proposition 3.18 here by invoking the Margulis normal subgroup theorem, which says that an infinite normal subgroup in a lattice Γ of a higher rank simple Lie group has finite index in Γ. In our special case, where $\Gamma = \mathrm{SL}(d, \mathbb{Z})$, it is simpler to give a direct elementary proof. This is our second lemma.

LEMMA 3.20. *The normal subgroup $E^{\,n}$ has finite index in $F = \mathrm{SL}(d, \mathbb{Z})$.*

Proof. An element $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ is said to be *primitive* if $\gcd(x_1, \ldots, x_d) = 1$.

First claim. For all x, x' primitive in $\mathbb{Z}^d$ with $x' \equiv x \mod n$, there exists $g \in E^n$ such that $x' = gx$.

We begin the proof by the following subclaim.

Subclaim. The first claim is true if there exists a subset $I \subset [1, d]$ such that,

- for all $i \in I$, one has $x'_i = x_i$ and,
- for all $j \notin I$, one has $x'_j \equiv x_j \mod nq$, where $q := \gcd(x_i \mid i \in I)$.

Indeed, for $j \notin I$, one can write $x'_j = x_j + \sum_{i \in I} nt_{i,j}x_i$ with $t_{i,j} \in \mathbb{Z}$. One then chooses g to be the commuting product of the elements $u^{\,n\,t_{i,j}}_{j,i}$ with $i \in I$ and $j \notin I$. This proves the subclaim.

We can now prove the first claim. We may assume that $x' = (1, 0, \ldots, 0)$. Let $r := 1 - x_1$. By assumption, the integers $x_2, rx_3, \ldots, rx_d$ generate the ring $\mathbb{Z}/x_1\mathbb{Z}$. Therefore, one can find an integer $b = x_2 + \sum_{i \geq 3} t_i r x_i$, with $t_i \in \mathbb{Z}$, which is invertible in $\mathbb{Z}/x_1\mathbb{Z}$: this is clear when x_1 is a prime power and the general case follows by the Chinese remainder theorem.

The subclaim with $I = [3, d]$ proves that the vector $y = (x_1, b, x_3, \ldots, x_d)$ belongs to $E^n x$.

Since x_1 and b are coprime and $d \geq 3$, the same subclaim with $I = [1, 2]$ proves that the vector $y' = (x_1, b, r, 0, \ldots, 0)$ belongs to $E^n y$. A direct calculation tells us that the vector $y'' := u_{1,3}\, y'$ is equal to $y'' = (1, b, r, 0, \ldots, 0)$.

The same subclaim with $I = \{1\}$ proves that the vector x' belongs to $E^n y''$. Therefore, the same vector $x' = u^{-1}_{1,3}\, x'$ also belongs to $E^n y' = E^n x$. This proves the first claim.

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We now introduce the *principal congruence subgroup* K^n of the group $F = \mathrm{SL}(d, \mathbb{Z})$, which is

$$K^n = \{g \in F \mid g \equiv 1 \bmod n\}.$$

This subgroup K^n is the kernel of the natural morphism

$$\mathrm{SL}(d, \mathbb{Z}) \longrightarrow \mathrm{SL}(d, \mathbb{Z}/n\mathbb{Z}).$$

Hence, K^n is a normal subgroup of finite index in F. This group contains the normal subgroup E^n.

Second claim. The quotient group $C^n := K^n/E^n$ is finite.

We begin the proof by the following subclaim.

Subclaim. The group C^n is a central subgroup of F/E^n.

We want to check that, for all $i \neq j$ and all g in K^n, the commutator $[u_{i,j}, g]$ belongs to E^n. After conjugating both $u_{i,j}$ and g, we can assume that $(i, j) = (1, 2)$. Then, using the first claim with $x = e_1$ and $x' = ge_1$, we can assume that g fixes the first basis vector e_1; therefore, the commutator $[u_{1,2}, g]$ is a upper unipotent matrix belonging to K^n. Such a matrix is in E^n. This proves the subclaim.

We can now prove the second claim. The group C^n is a finite-index subgroup in the finitely generated group F/E^n. Therefore, this group C^n is finitely generated. If this abelian group C^n is not finite, it admits a quotient isomorphic to $\mathbb{Z}$. Let p be a prime number which does not divide the cardinality of F/K^n. Using the exact sequence

$$1 \longrightarrow C^n \longrightarrow F/E^n \longrightarrow F/K^n \longrightarrow 1,$$

one can construct a central extension

$$1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow H \longrightarrow F/K^n \longrightarrow 1,$$

where the group H is a quotient of F. Since the group F/K^n has order prime to p, by Fact 3.21, this extension splits and one has $F \neq [F, F]$. However, the commutator relations between the $u_{i,j}$ prove that $F = [F, F]$. This contradiction finishes the proof of the second claim. □

Fact 3.21. Let G_0 be a finite group of order d and C_0 be a finite abelian group of order p prime to d. Then any central extension

$$1 \longrightarrow C_0 \longrightarrow H_0 \longrightarrow G_0 \longrightarrow 1 \tag{4}$$

splits.

Proof of Fact 3.21. This follows from the cohomological interpretation of central extensions as (4) and from the vanishing of the cohomology group $H^2(G_0, C) = 0$ (see [**12**, Proposition 5.3]).

Here is a short direct proof. We write H_0 as the product set $H_0 = G_0 \times C_0$ with product law

$$(g_1, c_1)\,(g_2, c_2) = (g_1 g_2, c_1 + c_2 + \sigma(g_1, g_2).$$

By associativity, the map $\sigma : G_0 \times G_0 \to C_0$ satisfies

$$\sigma(g_1 g_2, g_3) + \sigma(g_1, g_3) = \sigma(g_1, g_2 g_3) + \sigma(g_2, g_3). \tag{5}$$

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Since d is prime to p, the map $s : G_0 \to H_0$

$$s(g) := \left(g, -\frac{1}{d} \sum_{x \in G} \sigma(g, x) \right) \tag{6}$$

is well defined. Summing equation (5) for g_3 in G_0, one proves that this map s is a group morphism from G_0 to H_0 that splits the extension (4). □

Remark 3.22. For this arithmetic group $F = \mathrm{SL}(d, \mathbb{Z})$ with $d \geq 3$, one can check that one has indeed the equality $E^n = K^n$. This equality tells us that *every finite-index subgroup of* $\mathrm{SL}(d, \mathbb{Z})$ *contains a principal congruence subgroup*. This is the famous *congruence subgroup property for* $\mathrm{SL}(d, \mathbb{Z})$. We will not use it here.

4. *Fourth lecture. Five other examples*

In this fourth lecture we will discuss more examples of the arithmeticity properties of Zariski dense discrete subgroups of a semisimple group when they intersect irreducibly and cocompactly a horospherical subgroup. We first restate once more Theorem 1.1.

THEOREM 1.1. *Let G be a semisimple algebraic Lie group of real rank at least two and U be a non-trivial horospherical subgroup of G. Let Γ be a discrete Zariski dense subgroup of G that contains an irreducible lattice Δ of U. Then Γ is commensurable to an arithmetic lattice $G_{\mathbb{Z}}$ of G.*

We have proven in Lectures 2 and 3 this theorem on a first example, namely when $G = \mathrm{SL}(4, \mathbb{R})$ and U is the unipotent radical of the stabilizer of a 2-plane in $\mathbb{R}^4$. The proof given there uses the following three assumptions:

- the horospherical group U is reflexive;
- the horospherical group U is commutative;
- the group $[L, L]$ is not compact.

See Remark 3.12 for comments on the use of this last assumption.

In this lecture we explain how to get rid of these assumptions. More precisely, we will prove Theorem 1.1 for five other concrete examples.

- Example 4.1 with U not reflexive.
- Example 4.2 with U Heisenberg and $[L, L]$ not compact.
- Example 4.3 with U not commutative and not Heisenberg.
- Example 4.4 with U commutative and $[L, L]$ compact.
- Example 4.5 with U Heisenberg and $[L, L]$ compact.

The proofs for these five examples are representative of the general ideas used to prove Theorem 1.1 in [**3**].

4.1. *Second example.*

$$\text{When } G = \mathrm{SL}(3, \mathbb{R}) \text{ and } U = \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This example is due to Hee Oh in [**20**]. In this case, the horospherical group U is commutative but it is not reflexive. Indeed, the normalizer $P = N_G(U)$ is the stabilizer

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in G of the line $\mathbb{R}e_1$, and a parabolic subgroup P^- opposite to P is the stabilizer of a 2-plane of $\mathbb{R}^3$ which is transverse to $\mathbb{R}e_1$. It is not conjugate to P. As we will see now, the proof in the non-reflexive case is much easier thanks to the Auslander projection theorem.

PROPOSITION 4.1. *A conjugate of the group* Γ *is commensurable to* $\mathrm{SL}(3,\mathbb{Z})$.

Proof. Since Γ is Zariski dense in G, one can find g, h in Γ such that (e_1, ge_1, he_1) is a basis of $\mathbb{R}^3$. We can assume that this basis is the canonical basis (e_1, e_2, e_3) of $\mathbb{R}^3$. Therefore, the group Γ also intersects cocompactly the horospherical groups

$$U' := gUg^{-1} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ \star & 1 & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} \text{ and } U'' := hUh^{-1} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \star & \star & 1 \end{pmatrix} \right\}.$$

Let Γ_1 be the subgroup of Γ generated by $\Gamma \cap U$ and $\Gamma \cap U'$. This group Γ_1 is Zariski dense in the group H generated by U and U'. One computes easily that

$$H = \left\{ \begin{pmatrix} \star & \star & \star \\ \star & \star & \star \\ 0 & 0 & 1 \end{pmatrix} \right\} \simeq S \ltimes V$$

with $S = \mathrm{SL}(2,\mathbb{R})$ and $V = \mathbb{R}^2$.

For $i \neq j$ between one and three, we denote by $U_{i,j}$ the one-parameter unipotent subgroup of G whose Lie algebra is spanned by the matrix $e_i \otimes e_j^\star$. According to the Auslander theorem (Fact 4.2), the image $p(\Gamma_1)$ of Γ_1 by the projection on S is a discrete subgroup of S. The restriction of this projection p to U has image $U_{1,2}$ and kernel $U_{1,3}$. Since $\Gamma_1 \cap U$ has rank two, and since discrete subgroups of $U_{1,2}$ have rank at most one, one has $\Gamma \cap U_{1,3} \neq 1$. Focusing similarly on $p(\Gamma_1 \cap U')$, one proves that $\Gamma \cap U_{2,3} \neq 1$.

Repeating this argument with the pairs (U, U'') and (U', U''), one proves that

$$\Gamma \cap U_{i,j} \neq 1 \quad \text{for all } i \neq j.$$

Then, conjugating Γ by a suitable diagonal element, one can assume that the group Γ contains the two matrices $1 + e_1 \otimes e_2^*$ and $1 + e_2 \otimes e_3^*$. Then it has to contain the six matrices $1 + Ne_i \otimes e_j^*$ for some integer $N \geq 1$. Therefore, by Fact 3.10, and more precisely by its special case (Proposition 3.18), the group Γ is commensurable to $\mathrm{SL}(3,\mathbb{Z})$. □

In this proof we have used the following classical result of Auslander, which can be found in [**21**, Theorem 8.24].

Fact 4.2. (Auslander) Let H be an algebraic Lie group which is a semidirect product $H := S \ltimes V$ of a semisimple Lie group S and of a normal solvable subgroup V. Let $p : H \to S$ be the projection and Γ_1 be a Zariski dense discrete subgroup of H. Then the group $p(\Gamma_1)$ is a discrete subgroup of S.

Sketch of proof for S quasisimple and V abelian. The first main idea is the existence of a Zassenhauss neighborhood Ω_H of e in (any Lie group) H, i.e. a neighborhood such that any discrete group generated by elements of Ω_H is nilpotent. The second main idea is the existence of a one-parameter family of automorphisms $\varphi_t : (s, v) \mapsto (s, e^{-t}v)$ of H that contracts V for t large.

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The closure C of $p(\Gamma_1)$ is a Zariski dense Lie subgroup of S. Therefore, its Lie algebra $\mathfrak{c}$ is an ideal of $\mathfrak{s}$. Since $\mathfrak{s}$ is simple, it is either 0 or $\mathfrak{s}$. This means that $p(\Gamma_1)$ is either discrete or dense. Assume by contradiction that it is dense. One can choose finitely many elements in a small neighborhood of e in S that generate a non-nilpotent subgroup of S. We choose lifts $g_1, \dots, g_\ell$ of these elements in Γ_1. These lifts generate a discrete non-nilpotent subgroup of Γ_1. But, for t large, their images $\varphi_t(g_i)$ belong to the Zassenhauss neighborhood Ω_H, which is a contradiction. □

In the next four examples, the horospherical subgroup U will be reflexive and we will use similar notation as in the main example 2.1. We fix an element $\gamma_0 \in \Gamma$ such that $U^- = \gamma_0 U \gamma_0^{-1}$ is opposite to U. Both horospherical groups U and U^- intersect Γ cocompactly. We denote by $P := N_G(U)$ and $P^- := N_G(U^-)$ their normalizers, by L the intersection $L := P \cap P^-$ and by

$$L_0 := \{\ell \in L \mid \det_{\mathfrak{u}}(\mathrm{Ad}\ell) = 1\}.$$

One always has the inclusion $[L, L] \subset L_0$. Note that the higher rank assumption on G is equivalent to the non-compactness of L_0. We introduce a lattice $\Lambda \subset \mathfrak{u}$ contained in $\log(\Gamma \cap U)$ and a lattice $\Lambda^- \subset \mathfrak{u}^-$ contained in $\log(\Gamma \cap U^-)$. Such lattices do exist. As in Lecture 1, we will focus our attention on the double orbit $L_0(\Lambda, \Lambda^-)$.

4.2. *Third example.*

$$\text{When } G = \mathrm{SL}(4, \mathbb{R}) \text{ and } U = \left\{ \begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

This example is also due to Hee Oh in [19]. In this case the derived group $[L, L]$ is not compact and the horospherical group U is Heisenberg. When proving Theorem 1.1, one has to define 'Heisenberg' as a two-step horospherical group U such that the group L acts on the center $\mathfrak{z}$ of the Lie algebra $\mathfrak{u}$ of U by similarities for a suitable euclidean norm. We will not need this precise definition. Indeed, in our example the group U is also 'Heisenberg' in the classical sense since the center is one dimensional.

In this example, one has

$$L = \left\{ \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & \star & 0 \\ 0 & \star & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix} \right\}$$

and the derived group $S := [L, L]$ is isomorphic to $\mathrm{SL}(2, \mathbb{R})$.

PROPOSITION 4.3. *The double L_0-orbit $L_0(\Lambda, \Lambda^-)$ is closed in $X_{\mathfrak{u}} \times X_{\mathfrak{u}^-}$.*

Therefore, either the S-orbit $S(\Lambda, \Lambda^-)$ is closed or it is dense in $L_0(\Lambda, \Lambda^-)$. As in Lecture 3, this implies that either the stabilizer of (Λ, Λ^-) in S is a lattice in S, or the stabilizer of (Λ, Λ^-) in L_0 is a lattice in L_0. In both cases, Theorem 1.1 follows by the same argument as in Lecture 3.

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Proof of Proposition 4.3. The strategy is the same as in Lecture 1. We will not repeat it. The only modification is in the definition of the projection π. We introduce the diagonal element $h = \mathrm{diag}(1, 0, 0, -1)$ and diagonalize $\mathfrak{g}$ under the adjoint action of h. One gets the decomposition

$$\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-2}.$$

In this decomposition, the subspace $\mathfrak{g}_2$ is the one-dimensional center $\mathfrak{z}$ of $\mathfrak{u} = \mathfrak{g}_2 \oplus \mathfrak{g}_1$. One chooses $\pi : \mathfrak{g} \to \mathfrak{g}$ to be the projection on this first factor $\mathfrak{z} = \mathfrak{g}_2$. For $g \in G$, one defines $M(g) = \pi \,\mathrm{Ad} g\, \pi$ and $\Phi(g) := \det_{\mathfrak{z}} M(g)$. For $X \in \mathfrak{u}$ and $Y \in \mathfrak{u}^-$, one sets $F(X) = \Phi(e^X \gamma_0)$ and $G(X, Y) = \Phi(e^X e^Y)$.

For instance, when

$$X = \begin{pmatrix} 0 & x_1 & x_2 & z \\ 0 & 0 & 0 & y_1 \\ 0 & 0 & 0 & y_2 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

one computes explicitly this polynomial:

$$F(X) = \tfrac{1}{4}(x_1 y_1 + x_2 y_2)^2 - z^2.$$

The key point in the proof is again the fact that the sets $F(\Lambda)$ and $G(\Lambda, \Lambda^-)$ are closed and discrete in $\mathbb{R}$. □

4.3. *Fourth example.*

$$\text{When } G = \mathrm{SL}(4, \mathbb{R}) \text{ and } U = \left\{ \begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

This example is still due to Oh in [19]. In this case, the horospherical group U is not commutative and not Heisenberg.

We introduce the two unipotent subgroups: the derived subgroup

$$U' := [U, U] = \left\{ \begin{pmatrix} 1 & 0 & \star & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \text{ and } U_0 := C_U(U') = \left\{ \begin{pmatrix} 1 & 0 & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

its centralizer in U.

PROPOSITION 4.4. *The group $\Gamma \cap U_0$ is cocompact in the horospherical group U_0.*

Proposition 4.4 proves Theorem 1.1 in this case since the horospherical subgroup U_0 is the one we focused on in Lectures 2 and 3.

The proof of Proposition 4.4 relies on the following remark [21, Ch. 2].

Remark 4.5. Let U be a unipotent algebraic Lie group. For any lattice Δ of U, there exists a unique $\mathbb{Q}$-form of U such that $\Delta \subset U_{\mathbb{Q}}$. Conversely, for any $\mathbb{Q}$-form of U, the group $U_{\mathbb{Z}}$ is a cocompact lattice of U.

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Proof of Proposition 4.4. Since $\Gamma \cap U$ is a lattice in U, it is included in a group $U_{\mathbb{Q}}$ for a $\mathbb{Q}$-form of U. The derived group $U' := [U, U]$ and its centralizer $U_0 := C_U(U')$ are then defined over $\mathbb{Q}$. Therefore, $\Gamma \cap U'$ is a lattice in U' and $\Gamma \cap U_0$ is a lattice in U_0. □

Remark 4.6. Up to automorphism, there remains only two horospherical subgroups U of the group $G = \mathrm{SL}(4, \mathbb{R})$ that we have not discussed so far:

$$U = \left\{ \begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \quad \text{and} \quad U = \left\{ \begin{pmatrix} 1 & \star & \star & \star \\ 0 & 1 & \star & \star \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

They are not reflexive and can be dealt with through the method of §4.1.

4.4. *Fifth example.*

$$\text{When } G = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R}) \text{ and } U = \left\{ \left(\begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \right) \right\}.$$

This example is due to Selberg (cf. [5, 16, 25]). In this case the horospherical group U is commutative and reflexive but the derived group $[L, L]$ is compact.

In this example, one has

$$L_0 = \left\{ \left(\pm \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \pm \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \right) \mid \lambda > 0 \right\}.$$

PROPOSITION 4.7. *The double L_0-orbit $L_0(\Lambda, \Lambda^-)$ is compact.*

Proposition 4.7 proves Theorem 1.1 by the same argument as in Lecture 3.

The assumption that $\Gamma \cap U$ is irreducible in G means that, for all elements $X = \left(\begin{pmatrix} 0 & x_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_2 \\ 0 & 0 \end{pmatrix} \right)$ in Λ, one has the equivalence $x_1 = 0 \iff x_2 = 0$.

Proof of Proposition 4.7. The constructions and the results of Lecture 2 are valid for this example. We already know that the double orbit $L_0(\Lambda, \Lambda^-)$ is closed. We only need to check that it is relatively compact. For instance, let us check that the single orbit $L_0\Lambda$ is relatively compact in $X_{\mathfrak{u}}$. For that, we will use Mahler's criterion Fact 1.5.

We remember also the polynomial F on $\mathfrak{u}$. We know that the set $F(\Lambda)$ is closed and discrete in $\mathbb{R}$. We can compute explicitly this L_0-invariant polynomial F on $\mathfrak{u}$. For X as above, one has

$$F(x) = x_1^2 x_2^2.$$

Since the subgroup $\Gamma \cap U$ is irreducible in G, the polynomial F does not vanish on $\Lambda \setminus 0$ and the quantity $m := \inf_{X \in \Lambda \setminus 0} F(X)$ is non-zero. One computes, for all ℓ in L_0,

$$\inf_{X \in \Lambda \setminus 0} \|\mathrm{Ad}\ell(X)\|^4 \geq \inf_{X \in \Lambda \setminus 0} F(\mathrm{Ad}\ell(X)) = m > 0.$$

This proves that the single L_0-orbit $L_0\Lambda$ is relatively compact. □

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For more insight on the case where G is not simple, the reader should consult [3, §4.6, 5, 16] when G is a product of rank-one Lie groups. In general, one has to follow reduction steps, see [3, Ch. 5], where one keeps track of the assumption that Δ is irreducible. We would like to mention that this irreducibility assumption is also used in Margulis' construction of $\mathbb{Q}$-forms and in the Raghunathan–Venkataramana finite-index subgroup theorem.

4.5. *Sixth example.*

$$\text{When } G = \mathrm{SL}(3,\mathbb{R}) \text{ and } U = \left\{ \begin{pmatrix} 1 & \star & \star \\ 0 & 1 & \star \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

This example is due to Benoist and Hee Oh in [4]. In this case the horospherical group U is Heisenberg but the derived group $[L, L]$ is compact.

In this example, one has

$$L_0 = \left\{ \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \mid \lambda \in \mathbb{R}^* \right\}.$$

PROPOSITION 4.8. *If we are not in Case 4.1, the L_0-orbit $L_0(\Lambda, \Lambda^-)$ is compact.*

Proposition 4.8 proves Theorem 1.1 by the same argument as in Lecture 3.

The assumption that we are not in Case 4.1 means that, for all elements

$$X = \begin{pmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

in the lattice Λ, one has the equivalence $x = 0 \Longleftrightarrow y = 0$.

Proof of Proposition 4.8. It is very similar to the proof of Lemma 4.7. We remark that the results of Example 4.2 are valid for Example 4.5. We already know that the double orbit $L_0(\Lambda, \Lambda^-)$ is closed. We only need to check that the single orbit $L_0\Lambda$ is relatively compact in $X_{\mathfrak{u}}$. For that, we will again use Mahler's criterion 1.5.

We remember also the L_0-invariant polynomial F on $\mathfrak{u}$. We can compute it explicitly. For X as above, one has

$$F(X) = \tfrac{1}{4}x^2y^2 - z^2.$$

We know that the set $F(\Lambda)$ is closed and discrete in $\mathbb{R}$. Since we are not in Case 4.1, the function $H(X) := \max(|F(X)|, \frac{1}{4}|F(2X)|)$ does not vanish on $\Lambda \setminus 0$ and the quantity $m := \inf_{X \in \Lambda \setminus 0} H(X)$ is non-zero. One computes, for all ℓ in L_0,

$$\inf_{X \in \Lambda \setminus 0} \|\mathrm{Ad}\ell(X)\|^4 + \|\mathrm{Ad}\ell(X)\|^2 \geq \inf_{X \in \Lambda \setminus 0} H(\mathrm{Ad}\ell(X)) = m > 0.$$

This proves that the single L_0-orbit $L_0\Lambda$ is relatively compact. □

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4.6. *Conclusion.* In this series of lectures we have presented our proof of Theorem 1.1 on six examples. None of these examples were new. Most of them were already in Hee Oh's PhD thesis. However, the new strategy that we followed works for all cases (see [3]). For instance, it works in the following new cases.

- When G is a complex simple Lie group.
- When $G = \mathrm{SL}(n, \mathbb{H})$ with $n \geq 4$ and P is the stabilizer of a quaternionic line and a quaternionic hyperplane containing it.
- When $G = \mathrm{SO}(2, n)$ with $n \geq 4$ and P is the stabilizer of an isotropic 2-plane: in this case U is Heisenberg and the strategy of Example 4.2 applies.
- When $G = \mathrm{SO}(2, n) \times \mathrm{SO}(1, N)$ with $n, N \geq 2$ and P is the stabilizer of isotropic lines: in this case U is commutative and reflexive and the strategy of Lectures 2 and 3 applies.

In these last two examples, the group $[L, L]$ can be written as a product $S_{nc} \times S_c$ of a non-compact simple factor S_{nc} and a compact simple factor S_c:

$$[L, L] = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(n-2) \text{ or } [L, L] = \mathrm{SO}(1, n-1) \times \mathrm{SO}(N-1).$$

Since the compact factor S_c is large, it is difficult to describe the closure of the orbit $S_{nc}\Lambda$ using the Ratner theorem as in [19]. Indeed, there exist too many intermediate groups in between the two groups $\mathrm{Ad}_{\mathfrak{u}}(S_{nc})$ and $\mathrm{Aut}(\mathfrak{u})$. This is why a new strategy was needed.

We end this course by open questions that have a flavor similar to our main theorem (Theorem 1.1) where the group U is replaced by a simple Lie group H. The first question, where H is higher rank, is in [10] and is partially solved there.

Is every discrete Zariski dense subgroup Γ *of the group* $G := \mathrm{SL}(4, \mathbb{R})$ *that contains a lattice of the subgroup* $H := \mathrm{SL}(3, \mathbb{R})$ *always a lattice in* G?

The second question where H has rank one is also open.

Is every discrete Zariski dense subgroup Γ *of the group* $G := \mathrm{SL}(3, \mathbb{R})$ *that contains a lattice of the subgroup* $H := \mathrm{SO}(2, 1)$ *always a lattice in* G?

When both H and G have rank one, one knows that the answer to the analogous question is no (see [10]):

There exist infinite-covolume discrete Zariski dense subgroups Γ *of the group* $G := \mathrm{SO}(3, 1)$ *that contain a lattice of the subgroup* $H := \mathrm{SO}(2, 1)$.

Deforming these examples by a bending process, one also gets the following:

There exist infinite-covolume discrete Zariski dense subgroups Γ *of the group* $G := \mathrm{SO}(3, 2)$ *that contain a lattice of the subgroup* $H := \mathrm{SO}(2, 1)$.

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