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On the computability of rotation sets and their entropies

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Abstract. Let $f : X \to X$ be a continuous dynamical system on a compact metric space X and let $\Phi : X \to \mathbb{R}^m$ be an m-dimensional continuous potential. The (generalized) rotation set $\mathrm{Rot}(\Phi)$ is defined as the set of all μ-integrals of Φ, where μ runs over all invariant probability measures. Analogous to the classical topological entropy, one can associate the localized entropy $\mathcal{H}(w)$ to each $w \in \mathrm{Rot}(\Phi)$. In this paper, we study the computability of rotation sets and localized entropy functions by deriving conditions that imply their computability. Then we apply our results to study the case where f is a subshift of finite type. We prove that $\mathrm{Rot}(\Phi)$ is computable and that $\mathcal{H}(w)$ is computable in the interior of the rotation set. Finally, we construct an explicit example that shows that, in general, $\mathcal{H}$ is not continuous on the boundary of the rotation set when considered as a function of Φ and w. In particular, $\mathcal{H}$ is, in general, not computable at the boundary of $\mathrm{Rot}(\Phi)$.

1. *Introduction*

1.1. *Motivation.* Frequently, the trajectory of a particular orbit of a dynamical system is difficult, if not impossible, to determine. For instance, computations may be sensitive to the accuracy of the initial conditions. This difficulty motivates the study of statistical properties of the system. In this approach, one typically considers averages or similar statistical computations of measurements performed at different times. The mathematical theory supplies several objects and invariants, such as the entropy, pressure, and characteristic exponents, that give insight into the statistical behavior of a system. In this paper, we study integrals of (vector-valued) potential functions with respect to

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measures invariant under the dynamics. In particular, we prove computability results for the set of integrals of these potential functions as well as their localized entropies.

To illustrate the computational challenges of this approach, we consider the dynamical system given by the doubling map, i.e., $f : [0, 1) \to [0, 1)$, where $f(x) = 2x \pmod 1$. Since computers use binary arithmetic, the standard number types on a computer (such as floats or doubles) represent dyadic rational numbers, i.e., elements of $\mathbb{Z}[\frac{1}{2}]$. Since these numbers have finite binary expansions, a straight-forward calculation shows that each dyadic rational number in $[0, 1)$ is eventually mapped to 0 under iteration. Therefore, computational experiments with dyadic integers might lead to the incorrect hypothesis that 0 is an attracting fixed point that attracts all $x \in [0, 1)$. Alternately, if one were to symbolically represent rational numbers, such experiments might lead to the incorrect hypothesis that every point in $[0, 1)$ is preperiodic. On the other hand, with computability theory, we study the behavior of $x \in [0, 1) \setminus \mathbb{Q}$ even though we may only compute the behavior of preperiodic points. In this simple example, it is possible to use symbolic tools to study the behavior of more points, such as the roots of polynomials with integral coefficients. Since this may not be possible in more sophisticated systems, we do not address such computations here.

The main idea behind computability theory is to represent mathematical objects, e.g., points, sets, and functions, by convergent sequences produced by Turing machines (computer algorithms for our purposes). We say that a point, set, or function is computable if there exists a Turing machine that outputs an approximation to any prescribed accuracy; for additional details, see §2.1 and [**54**]. Using convergent sequences of points instead of single points allows one to study the behavior of a larger class of objects by increasing the precision of an approximation to the initial conditions, as needed, to adjust for the sensitivity to the initial conditions.

In this paper, we study the rotation set, i.e., the set of integrals of potential functions with respect to all invariant measures, and the localized entropy function on the rotation set. We provide conditions so that the rotation set and the localized entropy function are computable, i.e., can be algorithmically approximated to any prescribed accuracy. Rotation sets are natural extensions of Poincaré's rotation number for circle homeomorphisms and, more generally, of pointwise rotation sets for homeomorphisms on the n-torus; see [**41**]. Rotation sets play a role in several parts of ergodic theory and dynamical systems, and they have been studied recently by several authors; see, e.g., [**5**, **6**, **21**, **24**, **25**, **33**, **38**, **39**, **41**, **60**]. These studies include applications to higher-dimensional multi-fractal analysis; see, e.g., [**1**] and the references therein, ergodic optimization [**21**, **34**], and the study of ground states and zero-temperature measures [**40**].

Our results apply directly to subshifts of finite type (SFTs). For these systems, we prove computability of the rotation set $\mathrm{Rot}(\Phi)$ for a continuous potential Φ and the localized entropy $\mathcal{H}(w)$ for all $w \in \operatorname{int} \mathrm{Rot}(\Phi)$. Our results extend, immediately, to systems that can be modeled (via a computable conjugacy) by a symbolic system. Prime candidates of systems for which one might be able to establish the computability of the conjugacy are uniformly hyperbolic systems with a computable Markov partition and certain parabolic systems; see, e.g., [**2**, **9**, **55**]. Other potential applications include systems that can be exhausted by sufficiently large sets on which they are conjugate

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to symbolic systems. Prime candidates of systems with this property include certain non-uniformly hyperbolic systems, e.g., [**23**], systems with shadowing [**26**], and systems with discontinuous potentials, e.g., the geometric potential in the presence of critical points [**45**].

In the literature, there are several recent papers that study invariant sets, topological entropy, and other invariants from the computability point of view. The computability of Julia sets has been particularly popular; see, e.g., [**3**, **4**, **9**–**13**, **16**, **17**]. There are several results about the computability of certain specific measures; see [**3**, **18**] and the references therein. These papers include computability results for maximal entropy or physical measures, and the numerical computation of entropy and dimension for hyperbolic systems; see, e.g., [**35**] and [**36**] and the references therein. Furthermore, there are studies in the literature concerning the computation of the topological entropy or pressure for one- and multi-dimensional shift maps; see, e.g., [**27**, **28**, **43**, **44**, **52**, **53**]. To the best of our knowledge, our work is the first to establish the computability of an entire entropy spectrum within the space of all invariant measures.

1.2. *Background material from dynamical systems.* In this section, we introduce the relevant material from the theory of dynamical systems. Our main objects of study are rotation sets and their associated entropies.

Let $f : X \to X$ be a continuous map on a compact metric space X. Let $\mathcal{M}$ denote the space of all f-invariant Borel probability measures on X endowed with the weak* topology. This makes $\mathcal{M}$ into a compact, convex, and metrizable topological space. Recall that $\mu \in \mathcal{M}$ is ergodic if every f-invariant set has measure either zero or one. We denote the subset of ergodic measures by $\mathcal{M}_E \subset \mathcal{M}$.

We denote the *set of all periodic points of f with smallest period n* by $\mathrm{Per}_n(f)$. We also call n the *prime period* of $x \in \mathrm{Per}_n(f)$. Moreover, $\mathrm{Per}(f) = \bigcup_{n\geq 1} \mathrm{Per}_n(f)$ denotes the *set of periodic points* of f. The elements of $\mathrm{Per}_1(f)$ are the *fixed points* of f. For $x \in \mathrm{Per}_n(f)$, we denote the unique invariant measure supported on the orbit of x by $\mu_x = (1/n)(\delta_x + \cdots + \delta_{f^{n-1}(x)})$. We also call μ_x the periodic point measure of x. Moreover, we write $\mathcal{M}_{\mathrm{Per}} = \{\mu_x : x \in \mathrm{Per}(f)\}$. We observe that $\mathcal{M}_{\mathrm{Per}} \subset \mathcal{M}_E$.

Throughout this paper, we assume that f has finite topological entropy (see, e.g., [**56**] for the definition of topological entropy). Given an m-dimensional potential $\Phi = (\Phi_1, \ldots, \Phi_m) \in C(X, \mathbb{R}^m)$, we denote the *generalized rotation set* of Φ with respect to f by $\mathrm{Rot}(\Phi) = \mathrm{Rot}(f, \Phi)$ defined by

$$\mathrm{Rot}(\Phi) = \{\mathrm{rv}(\mu) : \mu \in \mathcal{M}\},$$

where

$$\mathrm{rv}(\mu) = \left(\int \Phi_1 \, d\mu, \ldots, \int \Phi_m \, d\mu\right)$$

denotes the *rotation vector* of the measure μ. Given $w \in \mathrm{Rot}(\Phi)$, we call the convex set $\mathcal{M}_\Phi(w) = \{\mu \in \mathcal{M} : \mathrm{rv}(\mu) = w\}$ the *rotation class* of w. It follows from the definition that the rotation set is a compact and convex subset of $\mathbb{R}^m$.

The relevance of rotation sets for understanding the behavior of dynamical systems can be seen by considering a sequence of potentials $(\Phi_k)_k$ that is dense in $C(X, \mathbb{R})$. Let R_m

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be the rotation set of the initial m-segment of potentials, i.e., $R_m = \mathrm{Rot}(\Phi_1, \ldots, \Phi_m)$. It follows, from the Riesz representation theorem, that the rotation classes of the rotation sets R_m form a decreasing sequence of partitions of $\mathcal{M}$ whose intersections contain a unique invariant measure. Therefore, for large m, the set R_m provides a fine partition of $\mathcal{M}$ and acts as a finite-dimensional approximation to the set of all invariant probability measures. We say that $(R_m)_m$ is a *filtration* of $\mathcal{M}$.

We say that $(\Phi_{\varepsilon_n})_n$, where $\Phi_{\varepsilon_n} \in C(X, \mathbb{R}^m)$, is an *approximating sequence* of Φ if ε_n decreases to 0 as $n \to \infty$ and $\|\Phi_{\varepsilon_n} - \Phi\|_\infty < \varepsilon_n$ for all $n \in \mathbb{N}$. Here, $\|\cdot\|_\infty$ denotes the supremum norm on $C(X, \mathbb{R}^m)$. One may, instead, work with a sequence $\varepsilon_n \to 0$ and pass to a subsequence of strictly decreasing ε_n when necessary. We leave the details to the reader.

Next, we define the localized entropy of rotation vectors. Following [**33**, **38**], we define the *localized entropy* of $w \in \mathrm{Rot}(\Phi)$ by

$$\mathcal{H}(w) = \mathcal{H}_\Phi(w) \stackrel{\text{def}}{=} \sup\{h_\mu(f) : \mu \in \mathcal{M}_\Phi(w)\}.$$

Here, $h_\mu(f)$ denotes the measure-theoretic entropy of f with respect to μ (see [**56**] for details). We consider systems for which the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous on $\mathcal{M}$; thus, there exists at least one $\mu \in \mathcal{M}_\Phi(w)$ with

$$h_\mu(f) = \mathcal{H}(w).$$

In this case, we say that μ is a *localized measure of maximal entropy* at w. Next, we discuss some regularity properties of $\mathcal{H}$. Clearly, the upper semi-continuity of the entropy map implies the upper semi-continuity of the localized entropy function $\mathcal{H}$. Furthermore, since $\mu \mapsto h_\mu(f)$ is affine, $w \mapsto \mathcal{H}(w)$ is concave and, thus, continuous on the (relative) interior of $\mathrm{Rot}(\Phi)$, i.e., the interior of $\mathrm{Rot}(\Phi)$ as a subset of the smallest affine space containing $\mathrm{Rot}(\Phi)$. Many of our results and theorems can be extended to the case of the relative interior of $\mathrm{Rot}(\Phi)$. We leave the details to the interested reader, but note that the results of Lemma 3.7 would need to be assumed. We refer the interested reader to [**48**] for details. The function $\mathcal{H}$ is continuous on any line segment in $\mathrm{Rot}(\Phi)$; in particular, $\mathcal{H}$ is continuous if $m = 1$. We refer the interested reader to a recent result [**59**] by the third author that establishes the possibility of $w \mapsto \mathcal{H}(w)$ having discontinuities on the boundary of $\mathrm{Rot}(\Phi)$ for $m \geq 2$. On the other hand, a classical result of Gale, Klee and Rockafellar [**19**] guarantees the continuity of the localized entropy function on the entire rotation set provided $\mathrm{Rot}(\Phi)$ is a polyhedron. We note that the case of polyhedral rotation sets actually occurs in relevant situations, e.g., when f is a subshift of finite type and Φ is locally constant; see §5.

1.3. *Statement of the results.* We continue to use the notation from §1.2. Let $f : X \to X$ be a continuous map on a compact metric space X and let $\Phi : X \to \mathbb{R}^m$ be a continuous potential. We assume that $\mu \mapsto h_\mu(f)$ is upper semi-continuous, which guarantees that the localized entropy function $w \mapsto \mathcal{H}(w)$ is continuous on the (relative) interior of $\mathrm{Rot}(\Phi)$; see §1.2. It follows, from the definitions, that $\mathrm{Rot}(\Phi)$ is a compact and convex subset of $\mathbb{R}^m$. Conversely, for symbolic systems, every compact and convex subset of $\mathbb{R}^m$ can be realized as the rotation set of an appropriate potential [**38**]. We establish, in Theorem 3.1, a general criterion for the computability of $\mathrm{Rot}(\Phi)$. We then prove in §5 that this criterion is satisfied for subshifts of finite type.

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THEOREM 5.9. *Let $f : X \to X$ be a transitive SFT with computable distance d_θ. If $\Phi \in C(X, \mathbb{R}^m)$ is computable, then* $\mathrm{Rot}(\Phi)$ *is computable.*

Here, the computability of a set S means that there exists an algorithm that approximates S in the Hausdorff metric by a finite union of computable closed balls. We refer the interested reader to §2.1 for the precise definition. As a consequence of the computability of $\mathrm{Rot}(\Phi)$, we obtain the computability of the distance function from a point $w \in \operatorname{int}\mathrm{Rot}(\Phi)$ to $\partial\mathrm{Rot}(\Phi)$; see Corollary 3.8.

We expect Theorem 5.9 to have several applications: for example, Theorem 5.9 is applicable for computing maximizing integrals of one-dimensional potentials that are of interest in the area ergodic optimization; see [**34**] for an overview of the subject. Furthermore, our result can be applied to obtain computability results for certain optimizing functions that were studied in the context of relative optimization by Garibaldi and Lopes [**21**]. Theorem 5.9 also applies to the computation of classical rotation sets for certain toral homeomorphisms homotopic to the identity. We refer the interested reader to [**60**] to make the connections between these rotation sets and a symbolic system. Finally, Theorem 5.9 can be applied to the computation of barycenter sets; see, e.g., [**6**, **30**–**32**].

Next, we discuss the computability of the localized entropy function. One of the difficulties when attempting to compute $\mathcal{H}(w)$ is that, at any given time, a Turing machine has only access to a finite amount of data associated with an approximation Φ_ε rather than the precise data of the actual potential Φ. To overcome this problem, we consider the minimal and maximal local entropy functions of Φ_ε in the closed ball centered at w and radius r, which we denote by $h^l_{\Phi_\varepsilon}(w, r)$ and $h^u_{\Phi_\varepsilon}(w, r)$, respectively; see equations (3) and (4) in §4. We show, in Proposition 4.3, that if $w \in \operatorname{int}\mathrm{Rot}(\Phi)$, then

$$\lim_{n\to\infty} h^l_{\Phi_{\varepsilon_n}}(w, \alpha\varepsilon_n) = \mathcal{H}_\Phi(w) = \lim_{n\to\infty} h^u_{\Phi_{\varepsilon_n}}(w, \alpha\varepsilon_n) \tag{1}$$

for all $\alpha \geq 1$. Moreover, if the ε_n decrease sufficiently rapidly, then $h^l_{\Phi_{\varepsilon_n}}(w, \alpha\varepsilon_n)$ is increasing and $h^u_{\Phi_{\varepsilon_n}}(w, \alpha\varepsilon_n)$ is decreasing. As a consequence, we obtain the following result.

THEOREM 4.5. *Let $f : X \to X$ be a continuous map on a compact metric space such that $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Then the global entropy function $(\Phi, w) \mapsto \mathcal{H}_\Phi(w)$ is continuous on $\bigcup_{\Phi\in C(X,\mathbb{R}^m)}\{\Phi\} \times \operatorname{int}\mathrm{Rot}(\Phi)$.*

This result indicates that, for points in the interior of the rotation set, it is sufficient for the computation of $\mathcal{H}_\Phi$ to compute the localized entropy of an approximation Φ_ε. We then use this approach, applying methods from the thermodynamic formalism, to compute the localized entropy of Φ_ε. In particular, we consider potentials Φ_ε for which the corresponding one-dimensional potential $v \cdot \Phi_\varepsilon$ has a unique equilibrium state $\mu_{v\cdot\Phi_\varepsilon}$ for all $v \in \mathbb{R}^m$; see §2.2 for the definitions and details. It is important to observe that we only require Φ to be continuous. However, we have some flexibility in the construction of the approximating potentials and, in particular, can require Hölder or Lipschitz continuity, for which there exists a well-developed theory of equilibrium states. We prove the following general result.

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THEOREM 4.9. *Let $f : X \to X$ be a continuous map on a computable compact metric space X such that $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Let $\Phi : X \to \mathbb{R}^m$ and* $\mathrm{Rot}(\Phi)$ *be computable with* $\operatorname{int}\mathrm{Rot}(\Phi) \neq \varnothing$. *Suppose that there exists an approximating sequence $(\Phi_{\varepsilon_n})_n$ of Φ such that for all $n \in \mathbb{N}$ and all $v \in \mathbb{R}^m$, the potential $v \cdot \Phi_{\varepsilon_n}$ has a unique equilibrium state $\mu_{v\cdot\Phi_{\varepsilon_n}}$. Moreover, assume that there are oracles approximating the functions $n \mapsto \varepsilon_n$, $(v, n) \mapsto h_{\mu_{v\cdot\Phi_{\varepsilon_n}}}(f)$ and $(v, n) \mapsto \mathrm{rv}(\mu_{v\cdot\Phi_{\varepsilon_n}})$ to arbitrary precision. Then there is a Turing machine whose inputs include these oracles which computes $\mathcal{H}_\Phi$ on* $\operatorname{int}\mathrm{Rot}(\Phi)$.

We note that the condition of the uniqueness of the equilibrium states is known to hold for several classes of systems and potentials including Axiom A systems, SFTs, and expansive homeomorphisms with specification and Hölder continuous potentials. Recently, there has been significant progress in generalizing uniqueness results for equilibrium states to wider classes of shift transformations, non-uniformly hyperbolic maps, and flows. We refer the interested reader to the survey article [**14**] for further references and details.

Theorem 4.9 is applicable to SFTs with computable potentials Φ. One advantage when dealing with SFTs is that we can work with locally constant computable approximations Φ_ε. For these potentials, we are able to establish the assumptions in Theorem 4.9. We conclude that the localized entropy $\mathcal{H}(w)$ is computable in the interior of $\mathrm{Rot}(\Phi)$; see Theorem 6.4. To the best of our knowledge, Theorems 4.9 and 6.4 represent the first results that establish computability of the entropy beyond computing the topological entropy or measure-theoretic entropy of certain specific invariant measures. Our proof of Theorem 4.9 relies on equation (1) in a crucial way. It turns out that the right-hand-side identity in equation (1) remains true for boundary points of the rotation set. Our proof, however, of the left-hand-side identity does not carry over to the boundary. Obviously, this does not imply that the left-hand-side identity of equation (1) does not hold. However, we are able to prove the following.

THEOREM 7.3. *Let $f : X \to X$ be a one-sided full shift over an alphabet with four symbols. Then there exist a potential $\Phi \in C(X, \mathbb{R}^2)$ and a sequence of locally constant potentials $\Phi_{\varepsilon_n} : X \to \mathbb{R}^2$ with $\lim_{n\to\infty} \|\Phi - \Phi_{\varepsilon_n}\|_\infty = 0$ such that the following holds:*

- $\partial\mathrm{Rot}(\Phi)$ *is an infinite polygon with a smooth exposed point w_∞; and*
- $0 = \lim_{n\to\infty} h^l_{\Phi_{\varepsilon_n}}(w_\infty, \varepsilon_n) < \mathcal{H}(w_\infty) = \lim_{n\to\infty} h^u_{\Phi_{\varepsilon_n}}(w_\infty, \varepsilon_n) = \log 2$.

One consequence of Theorem 7.3 is that, in general, one cannot extend the continuity of the global entropy function $(\Phi, w) \mapsto \mathcal{H}_\Phi(w)$ to the boundary of the rotation set; see Theorem 4.5. We also refer the interested reader to a recent preprint [**59**], where it is shown that, for a fixed potential Φ, the localized entropy function $w \mapsto \mathcal{H}_\Phi(w)$ can have discontinuities at the boundary of $\mathrm{Rot}(\Phi)$.

1.4. *Outline of paper.* The remainder of this paper is organized as follows: in §2, we review some basic concepts from computational analysis and the thermodynamic formalism. In §3, we discuss the computability of rotation sets for computable maps on compact computable metric spaces. Section 4 is devoted to the study of the localized entropy function for continuous maps on compact metric spaces. In §5, we apply the

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results from §3 to SFTs and establish the computability of their rotation sets. Section 6 establishes, in Theorem 6.4, the computability of $\mathcal{H}$ on the interior of $\mathrm{Rot}(\Phi)$ for shift maps. Finally, in §7, we construct an example which shows that the global entropy function is, in general, discontinuous at the boundary of the rotation set.

2. *Preliminaries*

In this section, we discuss the relevant background material. We continue to use the notation from §1.2.

2.1. *Basics from computability theory.* We are interested in the feasibility of computational experiments on rotation sets and entropies. Computability theory allows us to guarantee the correctness and accuracy of our computational experiments. We recall that a computer can approximate only countably many real numbers. Thus, without an accuracy guarantee, a computer experiment might miss interesting behaviors away from this collection of approximable numbers.

For a more thorough discussion of these topics, see, e.g., [**8**, **12**, **18**, **47**, **49**, **57**]. We use different, but closely related, definitions to those in [**12**] and [**18**] as well as mirroring their notation in order to allow for cross referencing. Throughout this discussion, we use a bit-based computation model, such as a Turing machine, as opposed to a real RAM model [**51**] (where these questions are trivial). One can think of the set of Turing machines as a particular, countable set of functions; we denote $\phi(x)$ as the output of the Turing machine ϕ on input x. We exclusively use the upper-case Φ for potentials and lower-case ϕ for Turing machines. Both of these uses are fairly common in the respective literature.

We begin by defining the spaces that we study as well as the computable points in these spaces. Throughout this section, we fix a triple $(X, d_X, \mathcal{S}_X)$, where (X, d_X) is a separable metric space and $\mathcal{S}_X = (s_1, s_2, \ldots)$ is a dense sequence with $s_i \in X$, i.e., $\mathcal{S}_X$ corresponds to a fixed injective function $\mathcal{S}_X : \mathbb{N} \to X$ with dense image. Similarly, we fix another separable metric space $(Y, d_Y, \mathcal{S}_Y)$ with $\mathcal{S}_Y = (t_1, t_2, \ldots)$ and $t_i \in Y$. If the metric or dense subset is clear from context, we may drop those from the notation. Since the specifics of the map are often unnecessary, we suppress the notation of the map and, instead, equate $\mathcal{S}_X$ with its image.

Definition 2.1. (Cf. [**12**, Definition 1.2.1]) Let $(X, d_X, \mathcal{S}_X)$ be a separable metric space. An *oracle* for $\alpha \in X$ is a function ϕ such that on input n, $\phi(n)$ is a natural number so that $d_X(\alpha, s_{\phi(n)}) < 2^{-n}$. Moreover, we say that α is *computable* if there is a Turing machine ϕ which is an oracle for α.

To make this definition more explicit, we begin with an example for real spaces. For computational purposes, we often set $\mathcal{S}_{\mathbb{R}}$ to be the rationals or the dyadic rational numbers $\mathbb{Z}[\frac{1}{2}]$ since both can be represented exactly on a computer. In our discussions, we use the rational points $\mathbb{Q}^m$ in $\mathbb{R}^m$, as in the following one-dimensional example.

Example 2.2. Consider the triple $(\mathbb{R}, d_{\mathbb{R}}, \mathcal{S}_{\mathbb{R}})$ with $d_{\mathbb{R}}$ the Euclidean distance and $\mathcal{S}_{\mathbb{R}} = \mathbb{Q}$. For a real number α, an oracle for α is a function ϕ such that on input n, $\phi(n)$ is a rational number so that $|\alpha - \phi(n)| < 2^{-n}$.

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Since there are only countably many Turing machines, there are only countably many computable points in any X. In the case of real numbers, the computable numbers include the rational and algebraic numbers as well as some transcendental numbers, such as e and π. We extend the notion of computability to functions as follows.

Definition 2.3. (Cf. [**12**, Definition 1.2.5]) Let $(X, d_X, \mathcal{S}_X)$ and $(Y, d_Y, \mathcal{S}_Y)$ be separable metric spaces. Suppose that $S \subset X$. A function $f : S \to Y$ is *computable* if there is a Turing machine ψ such that for any $\alpha \in S$ and any oracle ϕ of α, $d_Y(t_{\psi(\phi,n)}, f(\alpha)) < 2^{-n}$.

For example, if $S \subset \mathbb{R}^n$ and $g = (g_1, \ldots, g_m) : S \to \mathbb{R}^m$, then g is computable if and only if each g_i is computable. We observe that, in this definition, α does not need to be computable, i.e., the oracle ϕ does not need to be a Turing machine. In the case where α is computable, however, $f(\alpha)$ is computable because $\psi(\phi, n)$ is an oracle Turing machine for $f(\alpha)$.

The composition of computable functions is computable because the output of one Turing machine can be used as the input approximation for subsequent machines. In addition, basic operations, such as the arithmetic operations and the minimum and maximum functions, are computable; see [**8**] for more details on these and related properties.

There are weaker notions of computability, e.g., where $s_{\phi(n)} \to \alpha$ without a guarantee on the speed of convergence. Many of our theorems can be stated with weaker hypotheses to allow for this and other types of computability. We leave the details to the interested reader.

In this paper, we often consider computable sequences of points or functions. Such sequences can be written as functions where one input is the index.

Definition 2.4. (Cf. [**3**, §3.5]) Let $(X, d_X, \mathcal{S}_X)$ be a separable metric space. A sequence $(\alpha_n)_n$ of points in X is *uniformly computable* if it is computable as a function $\mathbb{N} \to X$, i.e., there exists a Turing machine ψ such that $s_{\psi(n,k)}$ is an approximation to α_n, so that $d_X(\alpha_n, s_{\psi(n,k)}) < 2^{-k}$.

Let $(Y, d_Y, \mathcal{S}_Y)$ be a separable metric space. A sequence of computable functions $(\Phi_n)_n$ is *uniformly computable* if it is computable as a function $\mathbb{N} \times X \to Y$, i.e., if there exists a Turing machine ψ such that for any $\alpha \in X$ and any oracle ϕ of α, $t_{\psi(\phi,n,k)}$ is an approximation to $\Phi_n(\alpha)$, so that $d_Y(\Phi_n(\alpha), t_{\psi(\phi,n,k)}) < 2^{-k}$.

We call a uniformly computable sequence of points or functions which is convergent a *uniformly computable, convergent sequence*.

We observe that in Definition 2.4, each α_n is computable because each α_n has an oracle Turing machine $\psi_n(k) = \psi(n, k)$. Similarly, each Φ_n in Definition 2.4 is a computable function. We observe that the existence of a uniformly computable, convergent sequence converging to $\alpha \in X$ is a different notion from the existence of a convergent sequence of computable points $(\alpha_n)_n$ converging to α. In particular, since the computable points are dense in X, every $\alpha \in X$ is a limit of a convergent sequence of computable points. In this sequence, however, there is a, possibly distinct, Turing machine ϕ_n for each $\alpha_n \in X$. Suppose that the α_n are not generated in a uniform way via a single Turing machine, as in the definition above. In this case, one would need an infinite amount of information, i.e.,

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Turing machines for all of the α_n, to be able to work with the sequence. We note also that the limit point $\alpha \in X$ need not be computable even if the α_n form a uniformly computable, convergent sequence. If there are some computable estimates on the rate of convergence of the α_n, however, then α is computable and some computable subsequence of $\psi(n, n)$ forms an oracle Turing machine for α.

Since the definition for a computable function uses any oracle for α and applies even when α is not computable, we can conclude that for any sufficiently close approximation x to α, $f(x)$ approximates the value of $f(\alpha)$, i.e., f is continuous.

LEMMA 2.5. (Cf. [**12**, Theorem 1.5]) *Let $(X, d_X, \mathcal{S}_X)$ and $(Y, d_Y, \mathcal{S}_Y)$ be separable metric spaces, $S \subset X$, and $f : S \to Y$. If f is computable, then f is continuous.*

We can now define computable metric spaces, which are metric spaces whose distance function is computable.

Definition 2.6. (Cf. [**18**, Definition 2.2]) Suppose that $(X, d_X, \mathcal{S}_X)$ is a separable metric space and consider the separable metric space $(\mathbb{R}, d_{\mathbb{R}}, \mathbb{Q})$ with $d_{\mathbb{R}}$ the Euclidean distance. Then X is a *computable metric space* if the map $\mathbb{N}^2 \to \mathbb{R}$ given by $(i, j) \mapsto d_X(s_i, s_j)$ is a computable function. In other words, the distances between s_i and s_j are uniformly computable in i and j.

The continuity of f in Lemma 2.5 can be made more precise in the case of computable metric spaces as follows: fix $n \in \mathbb{N}$. Since, in the definition of a computable function, ψ can be applied to any oracle ϕ for α, the correctness of the output is dependent only on the accuracy to which ϕ is queried within the Turing machine ψ. The accuracy to which ϕ is computed is finite since the algorithm terminates. Hence, if β is sufficiently close to α, then there is an oracle ϕ' for β which agrees with ϕ up to the computed accuracy, and the output of $\psi(\phi, n)$ equals the output for $\psi(\phi', n)$. We can use the maximum precision to which the oracle ϕ is queried for the following result.

LEMMA 2.7. (Cf. [**12**, Theorem 1.6]) *Let $(X, d_X, \mathcal{S}_X)$ be a computable metric space, $(Y, d_Y, \mathcal{S}_Y)$ be a separable metric space, $S \subset X$, and $f : S \to Y$. If f is computable, then there is a computable function $g : S \times \mathbb{N} \to \mathbb{N}$ such that if $\alpha, \beta \in S$ and $d_X(\alpha, \beta) < 2^{-g(\alpha,k)}$, then $d_Y(f(\alpha), f(\beta)) < 2^{-k}$. In this case, we say that f has a* computable local modulus of continuity. *Since g is computable, there exists a Turing machine μ such that for any oracle ϕ for α, $g(\alpha, k) = \mu(\phi, k)$.*

In some cases, we can extend the local modulus of continuity in Lemma 2.7 to a global modulus of continuity. In order to do this, we need a notion of computability for subsets of X. We use the *Hausdorff distance* to determine the accuracy of an approximation to a set. The Hausdorff distance between two *compact* subsets A and B of a metric space X is

$$d_H(A, B) = \max\left\{\max_{a \in A} d_X(a, B), \max_{b \in B} d_X(b, A)\right\},$$

where $d_X(c, D)$ denotes the minimum distance from a point c to a point in a convex set D. In words, the Hausdorff distance is the largest distance of a point in one set to the other set.

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Definition 2.8. Let $(X, d_X, \mathcal{S}_X)$ be a separable metric space. Let $S \subset X$ be compact. We say that S is *approximated* by an oracle ψ if on input n, $\psi(n)$ is a finite collection of pairs $\{(n_{k_i}, n_{l_i})\}$, where n_{k_i} is a natural number and n_{l_i} is an integer, representing closed balls $\overline{B}(s_{n_{k_i}}, 2^{-n_{l_i}})$ centered at $s_{n_{k_i}}$ and of radius $2^{-n_{l_i}}$ such that

$$d_H\left(\bigcup_i \overline{B}(s_{n_{k_i}}, 2^{-n_{l_i}}), S\right) < 2^{-n}.$$

Moreover, we say that S is *computable* if there exists a Turing machine ψ which gives an approximation for S.

We observe that in the definition above, the union of balls of the form $\overline{B}(s_{n_{k_i}}, 2^{-n_{l_i}} + 2^{-n})$ covers S and has Hausdorff distance at most 2^{-n+1} to S.

In order to turn the local modulus of continuity in Lemma 2.7 into a global modulus of continuity, the idea is to cover a compact and computable subset $S \subset X$ by finitely many small balls and compute a local modulus of continuity on each ball. By taking the minimum of the local moduli of continuity (and taking smaller balls if necessary), we derive a global modulus of continuity. We formalize the process of covering S by finitely many small balls in the following definition.

Definition 2.9. (Cf. [**18**, Definition 2.10]) Suppose that $(X, d_X, \mathcal{S}_X)$ is a separable metric space. We say that X is *recursively precompact* if there exists a Turing machine ϕ such that for any n, $\phi(n)$ is a finite collection of natural numbers $\{n_{\ell_j}\}$ such that

$$\bigcup_j B(s_{n_{\ell_j}}, 2^{-n}) = X.$$

We observe that many of the sets considered in this paper, such as compact, convex subsets of $\mathbb{R}^m$ with non-empty interior, are recursively precompact. When X is recursively precompact and $S \subset X$ is compact and computable, then we can cover S by arbitrarily small balls as in the following construction: since S is computable, we can find a finite union of balls of the form $\overline{B}(s_{n_{k_i}}, 2^{-n_{l_j}})$ whose Hausdorff distance to S is at most 2^{-n}. We take all open balls from the recursively precompact cover of X of the form $B(s_{n_{\ell_j}}, 2^{-n+1})$, where $d_X(s_{n_{\ell_j}}, s_{n_{k_i}}) < 2^{-n+2} + 2^{-n_{\ell_j}}$. In this case, these open balls form an open cover for S and the closure of their union has Hausdorff distance at most 2^{-n+3} to S.

Additionally, if X is a vector space and S is convex, then we define C as the convex hull of the points $s_{n_{\ell_j}}$ selected above. In this case, the Hausdorff distance between C and S is at most 2^{-n+3}. Furthermore, we observe that the boundary of S lies in a tubular neighborhood of radius 2^{-n+3} of the boundary of C. Any point not in this tubular neighborhood is guaranteed to be either interior to both S and C or exterior to both.

LEMMA 2.10. (Cf. [**12**, Section 1.2]) *Let $(X, d_X, \mathcal{S}_X)$ be a recursively precompact computable metric space, $(Y, d_Y, \mathcal{S}_Y)$ be a separable metric space, $S \subset X$, and $f : S \to Y$. Suppose that S is compact and computable. If f is computable, then there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that if $\alpha, \beta \in S$ and $d_X(\alpha, \beta) < 2^{-g(k)}$, then $d_Y(f(\alpha), f(\beta)) < 2^{-k}$. Since g is computable, there exists a Turing machine μ such that for any k, $g(k) = \mu(k)$.*

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We call the g constructed in the lemma above a global modulus of continuity for f and say that f has a computable global modulus of continuity.

These concepts can be extended to the space $\mathrm{PM}(X)$ of Borel probability measures on X (with the weak* topology) by taking the dense subset to be rational convex combinations of Dirac measures, i.e., finite sums of the form $\sum_i \lambda_i \mu_{s_i}$ for $s_i \in S_X$, $\lambda_i \in \mathbb{Q} \cap [0, 1]$ with $\sum_i \lambda_i = 1$. In this set-up, we use the Wasserstein–Kantorovich distance defined by

$$W_1(\mu_1, \mu_2) = \sup_{f \in 1\text{-Lip(X)}} \left| \int f d\mu_1 - \int f d\mu_2 \right|$$

for all $\mu_1, \mu_2 \in \mathrm{PM}(X)$ (where 1-Lip(X) denotes the space of Lipschitz continuous functions on X with Lipschitz constant 1). We observe that the Wasserstein–Kantorovich distance induces the weak* topology on $\mathrm{PM}(X)$. In [**18**, Lemma 2.12], it is shown that if X is a recursively precompact computable metric space, then so is $\mathrm{PM}(X)$; see also [**29**] for additional details.

2.2. *The thermodynamic formalism.* A detailed discussion of the thermodynamic formalism can be found in [**7**, **42**, **50**, **56**]. Here, we briefly recall some of the relevant facts. Let $f : X \to X$ be a continuous map on a compact metric space X. Given a continuous one-dimensional potential $\Phi : X \to \mathbb{R}$, we denote the topological pressure of Φ (with respect to f) by $P_{\mathrm{top}}(\Phi)$. Additionally, we denote the topological entropy of f by $h_{\mathrm{top}}(f)$; see [**56**] for the definition and further details. We recall that $h_{\mathrm{top}}(f) = P_{\mathrm{top}}(0)$. The topological pressure satisfies the variational principle, namely,

$$P_{\mathrm{top}}(\Phi) = \sup_{\mu \in \mathcal{M}} \left(h_\mu(f) + \int_X \Phi \, d\mu \right). \tag{2}$$

A measure $\mu \in \mathcal{M}$ that attains the supremum in equation (2) is called an equilibrium state (or an equilibrium measure) of the potential Φ. We denote the set of all equilibrium states of Φ by $ES(\Phi)$. We note that the upper semi-continuity of the map $\mu \mapsto h_\mu(f)$ implies that $ES(\Phi)$ is a compact, convex, and non-empty subset of $\mathcal{M}$. Moreover, $ES(\Phi)$ contains at least one ergodic equilibrium state.

Given $\gamma > 0$, we say that Φ is *Hölder continuous with exponent* γ if there exists a $C > 0$ such that $\|\Phi(x) - \Phi(y)\| \leq C d(x, y)^\gamma$ for all $x, y \in X$. We denote the space of all Hölder continuous potentials with exponent γ by $C^\gamma(X, \mathbb{R})$. Analogously, we denote the space of Hölder continuous functions from X to $\mathbb{R}^m$ by $C^\gamma(X, \mathbb{R}^m)$.

In [**38**], the authors discuss the class of systems with strong thermodynamic properties (STP). Roughly speaking, STP systems are those systems for which the topological pressure has strong regularity properties. The class of STP systems includes SFTs, uniformly hyperbolic systems, and expansive homeomorphisms with specification. We refer the interested reader to [**38**] for details. In the following list, we highlight some properties of the topological pressure that hold for many classes of systems including STP systems:

(1) $h_{\mathrm{top}}(f) < \infty$;
(2) the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous;
(3) the map $\Phi \mapsto P_{\mathrm{top}}(\Phi)$ is real-analytic on $C^\gamma(X, \mathbb{R})$;

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(4) each potential $\Phi \in C^\gamma(X, \mathbb{R})$ has a unique equilibrium state μ_Φ. Furthermore, μ_Φ is ergodic and, given $\Psi \in C^\gamma(X, \mathbb{R})$, we have

$$\frac{d}{dt} P_{\text{top}}(\Phi + t\Psi)\Big|_{t=0} = \int_X \Psi \, d\mu_\Phi.$$

Next, we discuss an application of the thermodynamic formalism to the theory of rotation sets of STP systems. Let $f : X \to X$ be an STP system and let $\Phi \in C^\gamma(X, \mathbb{R}^m)$. Given $v \in \mathbb{R}^m$, let $\mu_{v\cdot\Phi}$ denote the unique equilibrium state of the potential $v \cdot \Phi = v_1\Phi_1 + \cdots + v_m\Phi_m$. We have the following result.

THEOREM 2.11. ([**38**], see also [**25**]) *Let $\Phi \in C^\gamma(X, \mathbb{R}^m)$ with* int Rot$(\Phi) \neq \varnothing$. *Then:*

(i) *the map $T_\Phi : \mathbb{R}^m \to$* int Rot$(\Phi)$, *where $v \mapsto \text{rv}(\mu_{v\cdot\Phi})$, is a real-analytic diffeomorphism;*

(ii) *for all $v \in \mathbb{R}^m$, the measure $\mu_{v\cdot\Phi}$ is the unique localized measure of maximal entropy at $T_\Phi(v)$; and*

(iii) *the map $w \mapsto \mathcal{H}(w)$ is real-analytic on* int Rot(Φ).

3. *Computability of rotation sets*

In this section, we describe a class of computable dynamical systems for which we are able to establish the computability of the rotation set of a computable potential. Throughout this section, we assume that $f : X \to X$ is a computable map on a compact, recursively precompact, and computable metric space X. In addition, in order to approximate rotation vectors of invariant measures, we assume the existence of a uniformly computable sequence of invariant measures which is dense in the set of invariant measures. The first goal of this section is to prove the following result.

THEOREM 3.1. *Suppose that X is a compact, recursively precompact, and computable metric space, $f : X \to X$ is a computable map, and $\Phi \in C(X, \mathbb{R}^m)$ is a computable potential. Moreover, suppose that there exists a uniformly computable sequence of invariant measures which is dense in the set of invariant measures. Then* Rot(Φ) *is computable.*

We break the proof of this theorem into the following two lemmas. Some of these lemmas use techniques which are well known to experts. We include more details than are strictly necessary for completeness and clarity.

LEMMA 3.2. *Suppose that X is a compact, recursively precompact, and computable metric space and $f : X \to X$ is a computable map. Moreover, suppose that there exists a uniformly computable sequence of invariant measures which is dense in the set of invariant measures. Then $\mathcal{M}$ is a computable subset of* PM(X).

Proof. By following the ideas in [**18**, Section 3.1], we see that a measure $\mu \in PM(X)$ is invariant if and only if $W_1(\mu, f_*\mu) = 0$, where $f_*\mu$ denotes the pushforward of μ defined by $f_*\mu(A) = \mu(f^{-1}(A))$. Since X is recursively precompact, PM(X) is also recursively precompact. We let ϕ be the corresponding Turing machine, where $\phi(n) = \{n_{\ell_j}\}$ and the balls $B(\mu_{n_{\ell_j}}, 2^{-n})$ cover PM(X), as in Definition 2.9. Let ψ be the Turing machine generating the uniformly computable sequence of invariant measures. In particular,

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$\psi(n, k) = \{n_{k_i}\}$, where $\nu_{n_{k_i}}$ is a rational convex combination of point measures supported on $\mathcal{S}_X$ that approximates the invariant measure ν_{n_i} with $W_1(\nu_{n_i}, \nu_{n_{k_i}}) < 2^{-k}$.

Our approach is to construct an open cover of $\mathrm{PM}(X)$ by open balls of the following two forms: $B(\mu_{n_{\ell_j}}, 2^{-m_{\ell_j}})$, which are disjoint from the set of invariant measures, and $B(\nu_{n_{k_i}}, 2^{-k})$, for k sufficiently large. In this case, the sets $\overline{B}(\nu_{n_{k_i}}, 2^{-k})$ form a computable cover for the set of invariant measures with Hausdorff distance bounded by 2^{-k+1}. We can, computationally, detect when a collection of open balls is an open cover by showing that every ball $B(\mu_{p_{\ell_j}}, 2^{-p})$ returned by $\phi(p)$ is contained within one of the open balls of the potential open cover. This approach succeeds when p is sufficiently large, e.g., when 2^{-p} is less than half of the Lebesgue number of the cover.

Finally, we construct the balls $B(\mu_{n_{\ell_j}}, 2^{-m_{\ell_j}})$ which are disjoint from the set of invariant measures as follows. In [**18**, Section 4.1], it is shown that the map $\mu \mapsto W_1(\mu, f_*\mu)$ is computable. If $W_1(\mu_{n_{\ell_j}}, f_*\mu_{n_{\ell_j}})$ is bounded away from zero, we use the modulus of continuity of the map $\mu \mapsto W_1(\mu, f_*\mu)$ to find a radius of a ball containing only measures with $W_1(\mu, f_*\mu) \neq 0$. □

LEMMA 3.3. *Suppose that X is a compact, recursively precompact, and computable metric space and $\Phi : X \to \mathbb{R}^m$ is a computable map. Suppose that the set $\mathcal{M}$ of invariant measures is a computable subset of all Borel probability measures. Then* $\mathrm{Rot}(\Phi)$ *is computable.*

Proof. We observe that it is enough to show that the map $\mu \mapsto \mathrm{rv}(\mu)$ is computable. We recall that since $\mathcal{M}$ is a computable subset of a recursively precompact and computable metric space, we can cover $\mathcal{M}$ by finitely many small balls where the closure of their union has small Hausdorff distance to $\mathcal{M}$. Our goal is to show that by choosing these balls to be sufficiently small, any two measures μ and ν in the same ball have $\|\mathrm{rv}(\mu) - \mathrm{rv}(\nu)\|$ sufficiently small. The main challenge in showing computability is that Φ might not be Lipschitz continuous.

To show computability of the map $\mu \mapsto \mathrm{rv}(\mu)$, we first recall that Φ is computable and measures are approximated by rational convex combinations of point measures supported in $\mathcal{S}_X$. Consider the continuous function $\Phi_k : X \to \mathbb{R}^m$, where, in each coordinate, $(\Phi_k)_i(x) = \max_{y \in X}(\Phi_i(y) - k d_X(x, y))$. We observe that in each coordinate, $(\Phi_k)_i$ is k-Lipschitz (i.e., Lipschitz continuous with Lipschitz constant k). We also observe that if $k > 2^{\ell+1}\|\Phi_i\|_\infty$, then

$$\Phi_i(x) \leq (\Phi_k)_i(x) \leq \sup_{y \in B(x, 2^{-\ell})} \Phi_i(y).$$

Since X is recursively precompact, for any n, we can find an ℓ and corresponding k so that $\|(\Phi_k)_i - \Phi_i\|_\infty < 2^{-n}$ for all i. Moreover, since $(1/k)(\Phi_k)_i$ is 1-Lipschitz,

$$\left| \int_X (\Phi_k)_i(x) d\mu - \int_X (\Phi_k)_i(x) d\nu \right| \leq k W_1(\mu, \nu).$$

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Putting all of this together, we have that

$$\begin{aligned}\|\mathrm{rv}_\Phi(\nu) - \mathrm{rv}_\Phi(\mu)\| &\\ &\le \|\mathrm{rv}_\Phi(\mu) - \mathrm{rv}_{\Phi_k}(\mu)\| + \|\mathrm{rv}_{\Phi_k}(\mu) - \mathrm{rv}_{\Phi_k}(\nu)\| + \|\mathrm{rv}_{\Phi_k}(\nu) - \mathrm{rv}_\Phi(\nu)\| \\ &\le 2^{-n+1}\sqrt{m} + k\sqrt{m}W_1(\mu, \nu).\end{aligned}$$

By allowing n and $W_1(\mu, \nu)$ to decrease, i.e., by using smaller balls in the cover of $\mathcal{M}$, we obtain the computability of $\mu \mapsto \mathrm{rv}(\mu)$ and that the convex hull of the rotation vectors for the centers of the balls of the covering of $\mathcal{M}$ has small Hausdorff distance to $\mathrm{Rot}(\Phi)$. □

We also observe that when the computable sequence of invariant measures includes stronger error estimates, then $\mathrm{Rot}(\Phi)$ may be computable even if X is not recursively precompact or the sequence of invariant measures is not dense. A particularly nice choice of invariant measures consists of the periodic point measures $\mathcal{M}_{\mathrm{Per}}$. Moreover, there are several classes of dynamical systems whose periodic point measures are dense; see, e.g., [**22**] and the references therein. In particular, we use a computable sequence of periodic point measures in §5 to apply the following result to SFTs.

Observation 3.4. Suppose that there exists a Turing machine such that for each n, $\phi(n)$ is a finite collection of computable invariant measures $\{\nu_{n_i}\}$ such that for all $\nu \in \mathcal{M}$, there exists $\widetilde{\nu} \in \mathcal{M}$ which is a rational convex combination of the measures $\{\nu_{n_i}\}$ such that $\|\mathrm{rv}(\nu) - \mathrm{rv}(\widetilde{\nu})\| < 2^{-n}$. Then $\mathrm{Rot}(\Phi)$ is computable. In particular, the convex hull of $\{\mathrm{rv}(\nu_{n_i})\}$ is an approximation to $\mathrm{Rot}(\Phi)$ with Hausdorff distance at most 2^{-n}.

We note that the convex hull of $\{\mathrm{rv}(\mu_i)\}$ can be approximated using any standard convex hull algorithm; see, for example, [**15**]. In computational geometry it is common to use the real RAM (random access machine) model of computation; see [**15**]. This model is considered to be unrealistic because it assumes that all real numbers can be represented explicitly. In our case, because the input can be represented exactly, the convex hull algorithms can be implemented on a Turing machine.

Next, we observe that the results of Theorem 3.1 can be applied to conjugate systems.

COROLLARY 3.5. *Let X and Y be computable metric spaces. Suppose that (X, f) and (Y, g) are dynamical systems which are conjugate via the homeomorphism $h : X \to Y$, i.e., $h \circ f = g \circ h$. Suppose that h, h^{-1}, and $\Phi \in C(X, \mathbb{R}^m)$ are computable. Then:*

(1) *the conjugate potential $\Phi' = \Phi \circ h^{-1} \in C(Y, \mathbb{R}^m)$ is computable;*
(2) *for all $\mu \in \mathcal{M}_f$, the map $h_* : \mathcal{M}_f \to \mathcal{M}_g$ defined by $(h_*\mu)(B) = \mu(h^{-1}(B))$ is a bijection, where $\mathcal{M}_f$ and $\mathcal{M}_g$ are the f- and g-invariant probability measures on the corresponding spaces, respectively. Moreover, (X, μ, f) and $(Y, \mathrm{h}_*\mu, g)$ are measure-theoretically isomorphic;*
(3) *we have $\mathrm{rv}_\Phi(\mu) = \mathrm{rv}_{\Phi'}(h_*\mu)$ and $\mathrm{Rot}(f, \Phi) = \mathrm{Rot}(g, \Phi')$.*

Moreover, suppose that X satisfies the conditions of Theorem 3.1; then Y also satisfies the conditions of Theorem 3.1.

Remark 3.6. In §5, we show that $\mathrm{Rot}(\Phi)$ is computable for SFTs. Thus, Corollary 3.5 establishes the computability of rotation sets for systems that are computably conjugate to a SFT.

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Next, we establish a criterion for the computability of the function for the distance from a point in the rotation set to its boundary. In general, the distance to the boundary function is not computable, even when the underlying set is computable; see, e.g., [**11**, Theorem 4.1 and Corollary 5.17]. In the following lemma, however, we show that in the special case of a computable convex set in $\mathbb{R}^m$ with non-empty interior, the distance to the boundary function is computable.

LEMMA 3.7. *Let $C \subset \mathbb{R}^m$ be a computable and convex set with non-empty interior. Let $r : \operatorname{int} C \to \mathbb{R}$ be the distance to the boundary function. Then r is computable.*

Proof. We first observe that computable sets are necessarily bounded. Let ψ be an oracle Turing machine which approximates C. For all n, we can compute a polytope P_n whose Hausdorff distance to C is at most 2^{-n}. In particular, we construct a polytope whose Hausdorff distance to the union of balls constructed by $\psi(n+1)$ is at most 2^{-n-1}. To do this, we enumerate all convex hulls of finitely many points with rational coefficients. We note that for each of these finite sets, we are able to compute their convex hull using a standard convex hull algorithm. Then the Hausdorff distance between a union of closed balls and a polytope can be computed by appropriate maxima and minima of distances between centers of balls and faces of the polytope. Since every bounded convex set can be approximated by a convex hull constructed in this way, if these calculations are done to sufficiently high precision, an appropriate P_n can be found in finite time.

We now observe that since C and P_n are convex sets which are close in Hausdorff distance, their boundaries lie in tubular neighborhoods of radius 2^{-n} of each other. Let $r' = r'_n : \operatorname{int} C \to \mathbb{R}$ be the function that computes the distance to the boundary of P_n for points interior to P_n and 0 otherwise. This function is computable because the distance between a point and a face of a polytope with rational vertices has an elementary formula.

Fix $w \in \operatorname{int} C$ and ϕ an oracle for w. Let v be a closest point to w on the boundary of P_n. Then, by the tubular neighborhood observation, there is some u on the boundary of C whose distance to v is at most 2^{-n}. Then, by the reverse triangle inequality, $r'(w) = \|w - v\| \geq \|w - u\| - \|u - v\|$. Since u is on the boundary of C, $\|w - u\| \geq r(w)$, so $r'(w) \geq r(w) - 2^{-n}$. By repeating the argument with the roles of P_n and $\operatorname{Rot}(\Phi)$ reversed, we conclude that $|r(w) - r'(w)| \leq 2^{-n}$. Since $r'(w)$ can be computed to any precision, we can compute $r(w)$ to any precision. □

Since $\operatorname{Rot}(\Phi)$ is a convex set, Lemma 3.7 leads directly to the following corollary.

COROLLARY 3.8. *Suppose that* $\operatorname{Rot}(\Phi)$ *is computable with* $\operatorname{int}\operatorname{Rot}(\Phi) \neq \varnothing$. *Let* $r : \operatorname{int}\operatorname{Rot}(\Phi) \to \mathbb{R}$ *be the function such that* $r(w)$ *is the distance from* w *to the boundary of* $\operatorname{Rot}(\Phi)$. *Then* r *is a computable function.*

We observe that, when the conditions of Theorem 3.1 are satisfied and $\operatorname{int}\operatorname{Rot}(\Phi) \neq \varnothing$, the conditions of Corollary 3.8 are satisfied as well.

4. *Computability of localized entropy*

Our goals in this section are twofold: first, we develop a general theory for the localized entropy function of approximations. Second, we apply this theory to study the

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computability of the localized entropy at points in the interior of the rotation set. In §7, we establish that there are fundamental differences between interior and boundary points of the rotation set. Throughout the remainder of this section, we assume that $f : X \to X$ is a continuous map on a compact metric space X with $h_{\text{top}}(f) < \infty$. Moreover, we assume that the map $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Recall that, under these assumptions, for any $\Phi \in C(X, \mathbb{R}^m)$, the localized entropy function $\mathcal{H}_\Phi$ is continuous on $\operatorname{int}\operatorname{Rot}(\Phi)$ as well as on any line segment in $\operatorname{Rot}(\Phi)$. Moreover, for each $w \in \operatorname{Rot}(\Phi)$, there exists at least one $\mu \in \mathcal{M}_\Phi(w)$ with $h_\mu(f) = \mathcal{H}(w)$, i.e., μ is a localized measure of maximal entropy at w.

4.1. *Localized entropies of approximations.* Given $\Phi \in C(X, \mathbb{R}^m)$, $w_0 \in \mathbb{R}^m$, and $r > 0$, we define the maximum and minimum local entropies as follows: the maximum local entropy on $\overline{B}(w_0, r)$ is defined by

$$h^u_\Phi(w_0, r) = \sup\{\mathcal{H}_\Phi(w) : w \in \overline{B}(w_0, r) \cap \operatorname{Rot}(\Phi)\} \tag{3}$$

and the minimum local entropy on $\overline{B}(w_0, r)$ is defined by

$$h^l_\Phi(w_0, r) = \inf\{\mathcal{H}_\Phi(w) : w \in \overline{B}(w_0, r) \cap \operatorname{Rot}(\Phi)\}. \tag{4}$$

Here, we use the conventions that $\sup \varnothing = +\infty$ and $\inf \varnothing = -\infty$.

Since $w \mapsto \mathcal{H}_\Phi(w)$ is continuous on line segments in $\operatorname{Rot}(\Phi)$, we obtain that $h^{u/l}_\Phi(w_0, r)$ is continuous with respect to r on $\{r > 0 : \overline{B}(w_0, r) \cap \operatorname{Rot}(\Phi) \neq \varnothing\}$. Furthermore, if $\overline{B}(w_0, r) \cap \operatorname{Rot}(\Phi) \neq \varnothing$, then the upper semi-continuity of $\mathcal{H}$ yields that the supremum in the definition of $h^u_\Phi(w_0, r)$ is actually a maximum. Moreover, by the continuity of $\mathcal{H}$ on $\operatorname{int}\operatorname{Rot}(\Phi)$, when $\overline{B}(w_0, r) \subset \operatorname{int}\operatorname{Rot}(\Phi)$, the infimum in the definition of $h^l_\Phi(w_0, r)$ is actually a minimum.

The following result provides a tool to compute the localized entropy of a given potential in terms of the limit of the maximal local entropies of an approximating sequence.

PROPOSITION 4.1. *Let $\Phi \in C(X, \mathbb{R}^m)$ and let $(\Phi_{\varepsilon_n})_n$ be an approximating sequence for Φ. Let $w_0 \in \operatorname{Rot}(\Phi)$ and let $\alpha \geq 1$. Then:*

(i) $h^u_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n) \to \mathcal{H}_\Phi(w_0)$ *as* $n \to \infty$*; and*

(ii) *if $\alpha > 1$ and $\varepsilon_{n+1} < ((\alpha - 1)/(\alpha + 1))\varepsilon_n$ for all $n \in \mathbb{N}$, then $(h^u_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n))_n$ is a decreasing sequence.*

Proof. To prove statement (i), we observe that since the map $\nu \mapsto h_\nu(f)$ is upper semi-continuous on $\mathcal{M}$, there exists $\mu \in \mathcal{M}_\Phi(w_0)$ with $h_\mu(f) = \mathcal{H}_\Phi(w_0)$. It now follows from $\|\Phi - \Phi_{\varepsilon_n}\|_\infty < \varepsilon_n$ that $\operatorname{rv}_{\Phi_{\varepsilon_n}}(\mu) \in B(w_0, \varepsilon_n)$. Since $\alpha\varepsilon_n \geq \varepsilon_n$,

$$\mathcal{H}_\Phi(w_0) = h_\mu(f) \leq h^u_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n) \tag{5}$$

for all $n \in \mathbb{N}$. It follows from the upper semi-continuity of $\nu \mapsto h_\nu(f)$ that we can pick, for each $n \in \mathbb{N}$, an invariant measure μ_n with $\operatorname{rv}_{\Phi_{\varepsilon_n}}(\mu_n) \in \overline{B}(w_0, \alpha\varepsilon_n)$ and

$$h_{\mu_n}(f) = h^u_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n). \tag{6}$$

We claim that $\limsup_{n\to\infty} h_{\mu_n}(f) \leq \mathcal{H}_\Phi(w_0)$. To prove the claim, we consider a subsequence $(n_i)_i$ such that

$$\lim_{i\to\infty} h_{\mu_{n_i}}(f) = \limsup_{n\to\infty} h_{\mu_n}(f) \quad \text{and} \quad \lim_{i\to\infty} \mu_{n_i} = \nu \tag{7}$$

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for some $\nu \in \mathcal{M}$. The existence of ν follows from the compactness of $\mathcal{M}$. We obtain

$$\begin{aligned}
&\|\mathrm{rv}_\Phi(\nu) - w_0\| \\
&\quad \le \|\mathrm{rv}_\Phi(\nu) - \mathrm{rv}_\Phi(\mu_{n_i})\| + \|\mathrm{rv}_\Phi(\mu_{n_i}) - \mathrm{rv}_{\Phi_{\varepsilon_{n_i}}}(\mu_{n_i})\| + \|\mathrm{rv}_{\Phi_{\varepsilon_{n_i}}}(\mu_{n_i}) - w_0\| \\
&\quad < \|\mathrm{rv}_\Phi(\nu) - \mathrm{rv}_\Phi(\mu_{n_i})\| + \varepsilon_{n_i} + \alpha\varepsilon_{n_i} \to 0 \text{ as } i \to \infty.
\end{aligned}$$

We conclude that $\mathrm{rv}_\Phi(\nu) = w_0$, which implies that $h_\nu(f) \le \mathcal{H}_\Phi(w_0)$. On the other hand, the definition of ν in equation (7), in combination with the upper semi-continuity of $\nu \mapsto h_\nu(f)$, implies that $h_\nu(f) \ge \limsup_{n\to\infty} h_{\mu_n}(f)$. We conclude that

$$\limsup_{n\to\infty} h_{\mu_n}(f) \le \mathcal{H}_\Phi(w_0) \tag{8}$$

and the claim is proven. Finally, combining equation (8) with equations (5) and (6) completes the proof of statement (i).

Next, we prove statement (ii): let μ_n be as in equation (6). We claim that $\mathrm{rv}_{\Phi_{\varepsilon_n}}(\mu_{n+1}) \in \overline{B}(w_0, \alpha\varepsilon_n)$. We have

$$\begin{aligned}
&\|\mathrm{rv}_{\Phi_{\varepsilon_n}}(\mu_{n+1}) - w_0\| \\
&\quad \le \|\mathrm{rv}_{\Phi_{\varepsilon_n}}(\mu_{n+1}) - \mathrm{rv}_\Phi(\mu_{n+1})\| + \|\mathrm{rv}_\Phi(\mu_{n+1}) - \mathrm{rv}_{\Phi_{\varepsilon_{n+1}}}(\mu_{n+1})\| \\
&\qquad + \|\mathrm{rv}_{\Phi_{\varepsilon_{n+1}}}(\mu_{n+1}) - w_0\| \\
&\quad < \varepsilon_n + \varepsilon_{n+1} + \alpha\varepsilon_{n+1} < \alpha\varepsilon_n.
\end{aligned}$$

The final inequality comes from the assumed relationship between ε_n and ε_{n+1} in the statement of the theorem. Finally, statement (ii) follows from the definition of $h^u_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n)$ and equation (6). □

Next, we consider rotation vectors in the interior of the rotation set. Our goal is to strengthen Proposition 4.1 for interior points. We need the following elementary lemma.

LEMMA 4.2. *Let $m \in \mathbb{N}$ and $w_0 \in \mathbb{R}^m$. For all $\varepsilon > 0$, there exist $w_1, \ldots, w_{2^m} \in \overline{B}(w_0, 2\sqrt{m}\varepsilon)$ such that for all $\widetilde{w}_1, \ldots, \widetilde{w}_{2^m}$ with $\|w_i - \widetilde{w}_i\| < \varepsilon$ for $i \in \{1, \ldots, 2^m\}$, we have $B(w_0, \varepsilon) \subset \mathrm{conv}(\widetilde{w}_1, \ldots, \widetilde{w}_{2^m})$.*

Proof. By translation invariance, we may assume, without loss of generality, that $w_0 = 0$. Let the w_i be the points with coordinates $\pm 2\varepsilon$. We prove, by induction, that $[-\varepsilon, \varepsilon]^m \subset \mathrm{conv}(\widetilde{w}_1, \ldots, \widetilde{w}_{2^m})$. When $m = 1$, $w_1 = 2\varepsilon$ and $w_2 = -2\varepsilon$. By assumption, we know that $\widetilde{w}_1 > \varepsilon$ and $\widetilde{w}_2 < -\varepsilon$. Therefore, $\mathrm{conv}(\widetilde{w}_1, \widetilde{w}_2) = [\widetilde{w}_2, \widetilde{w}_1]$, which contains $[-\varepsilon, \varepsilon]$.

When $m > 1$, let $p \in [-\varepsilon, \varepsilon]^m$. For each vector $j \in \{\pm 1\}^{m-1}$, let $w_{j,+}$ and $w_{j,-}$ be the w_i whose first $m-1$ coordinates are given by $(w_{j,*})_k = 2j_k\varepsilon$, but whose last coordinate differs, i.e., $(w_{j,+})_m = 2\varepsilon$ and $(w_{j,-})_m = -2\varepsilon$. By the base case, we know that there is a convex combination v_j of $\widetilde{w}_{j,+}$ and $\widetilde{w}_{j,-}$ such that $(v_j)_m = p_m$. Let π_m be the projection that ignores the last coordinate. We observe that $\pi_m(v_j)$ is within ε of the vector $2\varepsilon j$. Then, by applying the inductive hypothesis to the $\pi_m(v_j)$, we get that $p \in \mathrm{conv}(v_j)$. Since each v_j is a convex combination of the w_i, it follows that $p \in \mathrm{conv}(\widetilde{w}_1, \ldots, \widetilde{w}_{2^m})$. Since p is arbitrary, the claim holds. The desired result holds since $B(w_0, \varepsilon) \subset [-\varepsilon, \varepsilon]^m$. □

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THEOREM 4.3. *Let $\Phi \in C(X, \mathbb{R}^m)$ and let $(\Phi_{\varepsilon_n})_n$ be an approximating sequence of Φ. Let $w_0 \in \operatorname{int}\operatorname{Rot}(\Phi)$, $\alpha \geq 1$, and $r = 2\sqrt{m}$. Then:*

(i) $\lim_{n\to\infty} h^l_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n) = \mathcal{H}_\Phi(w_0) = \lim_{n\to\infty} h^u_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n)$*; and*

(ii) *if $\alpha > r$ and $\varepsilon_{n+1} < ((\alpha - r)/\alpha r)\varepsilon_n$ for all $n \in \mathbb{N}$, then $(h^l_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n))_n$ is an increasing sequence for n sufficiently large.*

Proof. We first prove statement (i): the second equality in statement (i) was shown in Proposition 4.1. To prove the first equality, fix $n \in \mathbb{N}$ so that $\overline{B}(w_0, r\alpha\varepsilon_n) \subset \operatorname{int}\operatorname{Rot}(\Phi)$. Let $w_1, \ldots, w_{2^m} \in \overline{B}(w_0, r\alpha\varepsilon_n)$ be the points constructed in Lemma 4.2. Since $\nu \mapsto h_\nu(f)$ is upper semi-continuous, there exist $\mu_1, \ldots, \mu_{2^m} \in \mathcal{M}$ with $\operatorname{rv}_\Phi(\mu_i) = w_i$ and $h_{\mu_i}(f) = \mathcal{H}_\Phi(w_i)$ for all $i \in \{1, \ldots, 2^m\}$. Let $\widetilde{w}_i \stackrel{\text{def}}{=} \operatorname{rv}_{\Phi_{\varepsilon_n}}(\mu_i)$ for $i \in \{1, \ldots, 2^m\}$.

First, we observe that $\widetilde{w}_i \in B(w_i, \alpha\varepsilon_n)$ for $i \in \{1, \ldots, 2^m\}$ since $\|\Phi - \Phi_{\varepsilon_n}\|_\infty < \varepsilon_n \leq \alpha\varepsilon_n$. It follows from the definition of $h^l_\Phi(w_0, r\alpha\varepsilon_n)$ and from $h_{\mu_i}(f) \leq \mathcal{H}_{\Phi_{\varepsilon_n}}(\widetilde{w}_i)$ that

$$\begin{aligned} h^l_\Phi(w_0, r\alpha\varepsilon_n) &\leq \min\{h_{\mu_1}(f), \ldots, h_{\mu_{2^m}}(f)\} \\ &\leq \min\{\mathcal{H}_{\Phi_{\varepsilon_n}}(\widetilde{w}_1), \ldots, \mathcal{H}_{\Phi_{\varepsilon_n}}(\widetilde{w}_{2^m})\}. \end{aligned} \tag{9}$$

By Lemma 4.2, it follows that $B(w_0, \alpha\varepsilon_n) \subset \operatorname{conv}(\widetilde{w}_1, \ldots, \widetilde{w}_{2^m})$. The convexity of $\nu \mapsto h_\nu(f)$ implies that

$$\min\{\mathcal{H}_{\Phi_{\varepsilon_n}}(\widetilde{w}_1), \ldots, \mathcal{H}_{\Phi_{\varepsilon_n}}(\widetilde{w}_{2^m})\} \leq h^l_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n). \tag{10}$$

By combining inequalities (9) and (10), we obtain

$$h^l_\Phi(w_0, r\alpha\varepsilon_n) \leq h^l_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n).$$

Now, taking the limit as $n \to \infty$ and using the fact that $\mathcal{H}_\Phi$ is continuous on $\operatorname{int}\operatorname{Rot}(\Phi)$, results in

$$\mathcal{H}_\Phi(w_0) = \lim_{n\to\infty} h^l_\Phi(w_0, r\alpha\varepsilon_n) \leq \lim_{n\to\infty} h^l_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n).$$

We observe that

$$\lim_{n\to\infty} h^l_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n) \leq \lim_{n\to\infty} h^u_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n). \tag{11}$$

Therefore, statement (i) follows from inequality (11) and Proposition 4.1.

We prove statement (ii) using a similar approach as in the proof of (i): suppose that n is large enough so that $\overline{B}(w_0, r(\alpha\varepsilon_{n+1} + \varepsilon_n)) \subset \operatorname{Rot}(\Phi_{\varepsilon_n})$. We may then choose $w_1, \ldots, w_{2^m} \in \overline{B}(w_0, r(\alpha\varepsilon_{n+1} + \varepsilon_n))$ as in Lemma 4.2. By upper semi-continuity, there exist $\mu_1, \ldots, \mu_{2^m} \in \mathcal{M}$ such that $\operatorname{rv}_{\Phi_{\varepsilon_n}}(\mu_i) = w_i$ and $h_{\mu_i}(f) = \mathcal{H}_{\Phi_{\varepsilon_n}}(w_i)$ for all $i \in \{1, \ldots, 2^m\}$. Define $\widetilde{w}_i = \operatorname{rv}_{\Phi_{\varepsilon_{n+1}}}(\mu_i)$ for $i \in \{1, \ldots, 2^m\}$. We observe that since $\|\Phi_{\varepsilon_n} - \Phi_{\varepsilon_{n+1}}\|_\infty \leq \varepsilon_n + \varepsilon_{n+1} \leq \varepsilon_n + \alpha\varepsilon_{n+1}$, $\widetilde{w}_i \in B(w_i, \alpha\varepsilon_{n+1} + \varepsilon_n)$.

Since $\widetilde{w}_i = \operatorname{rv}_{\Phi_{\varepsilon_{n+1}}}(\mu_i)$, it follows that $h_{\mu_i}(f) \leq \mathcal{H}_{\Phi_{\varepsilon_{n+1}}}(\widetilde{w}_i)$. Since $\operatorname{rv}_{\Phi_{\varepsilon_n}}(\mu_i) \in B(w_0, r(\alpha\varepsilon_{n+1} + \varepsilon_n))$, we know that

$$h^l_{\Phi_{\varepsilon_n}}(w_0, r(\alpha\varepsilon_{n+1} + \varepsilon_n)) \leq \min\{\mathcal{H}_{\Phi_{\varepsilon_{n+1}}}(\widetilde{w}_1), \ldots, \mathcal{H}_{\Phi_{\varepsilon_{n+1}}}(\widetilde{w}_{2^m})\}. \tag{12}$$

By Lemma 4.2, it follows that $B(w_0, \alpha\varepsilon_{n+1} + \varepsilon_n) \subset \operatorname{conv}(\widetilde{w}_1, \ldots, \widetilde{w}_{2^m})$. Since $w \mapsto \mathcal{H}_{\Phi_{\varepsilon_{n+1}}}(w)$ is concave, we conclude that

$$\min\{\mathcal{H}_{\Phi_{\varepsilon_{n+1}}}(\widetilde{w}_1), \ldots, \mathcal{H}_{\Phi_{\varepsilon_{n+1}}}(\widetilde{w}_{2^m})\} \leq h^l_{\Phi_{\varepsilon_{n+1}}}(w_0, \alpha\varepsilon_{n+1} + \varepsilon_n). \tag{13}$$

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Combining inequalities (12) and (13), we have

$$h^l_{\Phi_{\varepsilon_n}}(w_0, r(\alpha\varepsilon_{n+1} + \varepsilon_n)) \le h^l_{\Phi_{\varepsilon_{n+1}}}(w_0, \alpha\varepsilon_{n+1} + \varepsilon_n).$$

Since $\alpha\varepsilon_n > r(\alpha\varepsilon_{n+1} + \varepsilon_n)$ and $\alpha\varepsilon_{n+1} \le \alpha\varepsilon_{n+1} + \varepsilon_n$, by assumption, the result follows. □

Next, we extend the entropy function $\mathcal{H}$ by considering Φ as a variable.

Definition 4.4. Let $T \subset C(X, \mathbb{R}^m) \times \mathbb{R}^m$ be the *(total) parameter space* of the rotation sets. In other words, the fibers of the projection π_1 onto the first factor are the rotation sets, so, for $\Phi \in C(X, \mathbb{R}^m)$, $\pi_1^{-1}(\Phi) = \{\Phi\} \times \mathrm{Rot}(\Phi)$. Set theoretically,

$$T = \bigcup_{\Phi \in C(X, \mathbb{R}^m)} \{\Phi\} \times \mathrm{Rot}(\Phi).$$

As a consequence of Theorem 4.3, we obtain the following.

THEOREM 4.5. *Let $f : X \to X$ be a continuous map on a compact metric space such that $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Then the global entropy function is continuous on $\bigcup_{\Phi \in C(X, \mathbb{R}^m)} \{\Phi\} \times \mathrm{int}\,\mathrm{Rot}(\Phi)$ (cf. Definition 4.4).*

4.2. *Computability at interior points of the rotation set.* We now address the question concerning the computability of the localized entropy for points in the interior of the rotation set. Throughout the remainder of this section, we assume that $f : X \to X$ is a computable map on a compact computable metric space $(X, d_X, \mathcal{S}_X)$.

The following result provides a computability criterion for interior points.

THEOREM 4.6. *Let $f : X \to X$ be a continuous map on a compact computable metric space X such that $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Let $\Phi : X \to \mathbb{R}^m$ be computable and let $w_0 \in \mathrm{int}\,\mathrm{Rot}(\Phi)$. Suppose that a computable $r > 0$ is given such that $B(w_0, r) \subset \mathrm{int}\,\mathrm{Rot}(\Phi)$. Suppose that there exists an approximating sequence $(\Phi_{\varepsilon_n})_n$ of Φ such that $(\varepsilon_n)_n$ is computable. Suppose that there are oracles approximating the functions $(n, s) \mapsto h^l_{\Phi_{\varepsilon_n}}(w_0, 2^{-s})$ and $(n, s) \mapsto h^u_{\Phi_{\varepsilon_n}}(w_0, 2^{-s})$ to arbitrary precision, where $n \in \mathbb{N}$ and s is a real number given by an oracle. Then there is a Turing machine whose inputs include the oracles for $h^l_{\Phi_{\varepsilon_n}}$ and $h^u_{\Phi_{\varepsilon_n}}$ which computes $\mathcal{H}_\Phi(w_0)$.*

Proof. Since $r > 0$, we can find an integer $\alpha > 1$ so that $\alpha > r$. By passing to a subsequence of the ε_n (and using sufficiently good approximations for r, $\sqrt{m}$, and ε_n), we may assume that:

(1) $\varepsilon_n < r/\alpha$;
(2) $\varepsilon_{n+1} < ((\alpha - 2\sqrt{m})/2\alpha\sqrt{m})\varepsilon_n$; and
(3) $\varepsilon_{n+1} < ((\alpha - 1)/(\alpha + 1))\varepsilon_n$.

Therefore, by Proposition 4.1 and Theorem 4.3, we know that $(h^u_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n))_n$ is a sequence decreasing to $\mathcal{H}_\Phi(w_0)$ and $(h^l_{\Phi_{\varepsilon_n}}(w_0, \alpha\varepsilon_n))_n$ is a sequence increasing to $\mathcal{H}_\Phi(w_0)$. Therefore, upper and lower approximations for $h^u_{\Phi_{\varepsilon_n}}$ and $h^l_{\Phi_{\varepsilon_n}}$, respectively, bound and converge to $\mathcal{H}_\Phi(w_0)$. Thus, we can compute $\mathcal{H}_\Phi(w_0)$ to any desired precision. □

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Remark 4.7. Briefly, we assume that the assumptions of Lemma 3.2 and Corollary 3.8 hold. We observe that when X is recursively precompact, the distance to the boundary function is computable. Moreover, if the oracles for $h^l_{\Phi_{\varepsilon_n}}$ and $h^u_{\Phi_{\varepsilon_n}}$ are given, uniformly, by Turing machines, then $\mathcal{H}_\Phi(w_0)$ is computable. More generally, Theorem 4.6 implies that if the functions $(n, s, w) \mapsto h^l_{\Phi_{\varepsilon_n}}(w, 2^{-s})$ and $(n, s, w) \mapsto h^u_{\Phi_{\varepsilon_n}}(w, 2^{-s})$ are given by oracles, then there is a Turing machine whose inputs include the oracles for $h^l_{\Phi_{\varepsilon_n}}$ and $h^u_{\Phi_{\varepsilon_n}}$ which computes $\mathcal{H}_\Phi$. In particular, if the functions $(n, s, w) \mapsto h^l_{\Phi_{\varepsilon_n}}(w, 2^{-s})$ and $(n, s, w) \mapsto h^u_{\Phi_{\varepsilon_n}}(w, 2^{-s})$ are computable, then $\mathcal{H}_\Phi$ is computable.

We now study the computability of the local maximal and minimal entropy. The main idea is to apply the thermodynamic formalism with the goal to identify the localized measures of maximal entropy within a family of equilibrium states. Recall the following from §2.2: let $\Phi \in C(X, \mathbb{R}^m)$ and let $w \in \operatorname{int}\operatorname{Rot}(\Phi)$. Since $\mu \mapsto h_\mu(f)$ is upper semi-continuous, there exists at least one $\mu \in \mathcal{M}_\Phi(w)$ with $h_\mu(f) = \mathcal{H}(w)$, that is, μ is a localized measure of maximal entropy at w. For $v \in \mathbb{R}^m$, we consider the one-dimensional potential $v \cdot \Phi = v_1\Phi_1 + \cdots + v_m\Phi_m$. Recall that $ES(v \cdot \Phi)$ denotes the set of equilibrium states of the one-dimensional potential $v \cdot \Phi$; see equation (2). The analogous upper semi-continuity argument shows that $ES(v \cdot \Phi)$ is non-empty. It is a result of Jenkinson [**33**] that there exist $v \in \mathbb{R}^m$ and $\mu_{v\cdot\Phi} \in ES(v \cdot \Phi)$ such that $\mu_{v\cdot\Phi}$ is a localized measure of maximal entropy at w. Moreover, the variational principle implies that every localized measure of maximal entropy at w belongs to $ES(v \cdot \Phi)$; see equation (2). The following result provides an estimate for the norm of v.

PROPOSITION 4.8. *Let* $\Phi \in C(X, \mathbb{R}^m)$. *Let* $v \in \mathbb{R}^m \setminus \{0\}$ *and let* $\mu_{v\cdot\Phi} \in ES(v \cdot \Phi)$. *Let* $r = \operatorname{dist}(\operatorname{rv}(\mu_{v\cdot\Phi}), \partial\operatorname{Rot}(\Phi))$. *Then* $\|v\| \le (2/r)h_{\text{top}}(f)$.

Proof. If $r = 0$, then $\operatorname{rv}(\mu_{v\cdot\Phi}) \in \partial\operatorname{Rot}(\Phi)$ and the inequality is trivial. Assume now that $r > 0$, in which case $\operatorname{rv}(\mu_{v\cdot\Phi}) \in \operatorname{int}\operatorname{Rot}(\Phi)$. Suppose, for contradiction, that $\|v\| > (2/r)h_{\text{top}}(f)$. Let $H_v(\Phi)$ be the unique supporting hyperplane of $\operatorname{Rot}(\Phi)$ for which v is the outward-pointing normal vector. By the compactness of $\operatorname{Rot}(\Phi)$, $F_v(\Phi) = \operatorname{Rot}(\Phi) \cap H_v(\Phi)$ is a (non-empty) face of $\operatorname{Rot}(\Phi)$.

Let $\nu \in \mathcal{M}$ be an invariant measure with $\operatorname{rv}(\nu) \in F_v(\Phi)$. Since $h_{\text{top}}(f) \ge h_{\mu_{v\cdot\Phi}}(f)$ and $h_\nu(f) \ge 0$, we have that $h_\nu(f) \ge h_{\mu_{v\cdot\Phi}}(f) - h_{\text{top}}(f)$. Using equation (2), we have that

$$\begin{aligned} P_{\text{top}}(v \cdot \Phi) &\ge h_\nu(f) + \int v \cdot \Phi \, d\nu \\ &\ge h_{\mu_{v\cdot\Phi}}(f) - h_{\text{top}}(f) + v \cdot \operatorname{rv}(\nu) \\ &= h_{\mu_{v\cdot\Phi}}(f) - h_{\text{top}}(f) + v \cdot \operatorname{rv}(\mu_{v\cdot\Phi}) + v \cdot (\operatorname{rv}(\nu) - \operatorname{rv}(\mu_{v\cdot\Phi})). \end{aligned}$$

We observe that $v \cdot (\operatorname{rv}(\nu) - \operatorname{rv}(\mu_{v\cdot\Phi}))$ is at least $\|v\|$ times the distance $\operatorname{dist}(\operatorname{rv}(\mu_{v\cdot\Phi}), H_v(\Phi))$. Since $H_v(\Phi)$ does not intersect the interior of $\operatorname{Rot}(\Phi)$, $\operatorname{dist}(\operatorname{rv}(\mu_{v\cdot\Phi}), H_v(\Phi)) \ge r$. Therefore,

$$P_{\text{top}}(v \cdot \Phi) \ge h_{\mu_{v\cdot\Phi}}(f) - h_{\text{top}}(f) + v \cdot \operatorname{rv}(\mu_{v\cdot\Phi}) + r\|v\|.$$

Using the assumption on $\|v\|$, we find that

$$P_{\text{top}}(v \cdot \Phi) > h_{\mu_{v\cdot\Phi}}(f) + v \cdot \operatorname{rv}(\mu_{v\cdot\Phi}) + h_{\text{top}}(f).$$

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This implies that

$$P_{\text{top}}(v \cdot \Phi) - \left(h_{\mu_{v\cdot\Phi}}(f) + \int v \cdot \Phi \, d\mu \right) > h_{\text{top}}(f).$$

Hence, $\mu_{v\cdot\Phi}$ is not an equilibrium state of $v \cdot \Phi$. This contradiction completes the proof. □

The following result is the main result of this section.

THEOREM 4.9. *Let $f : X \to X$ be a continuous map on a computable compact metric space X such that $\mu \mapsto h_\mu(f)$ is upper semi-continuous. Let $\Phi : X \to \mathbb{R}^m$ and* $\text{Rot}(\Phi)$ *be computable with* $\text{int Rot}(\Phi) \neq \varnothing$. *Suppose that there exists an approximating sequence $(\Phi_{\varepsilon_n})_n$ of Φ such that for all $n \in \mathbb{N}$ and all $v \in \mathbb{R}^m$, the potential $v \cdot \Phi_{\varepsilon_n}$ has a unique equilibrium state $\mu_{v\cdot\Phi_{\varepsilon_n}}$. Moreover, assume that there are oracles approximating the functions $n \mapsto \varepsilon_n$, $(v, n) \mapsto h_{\mu_{v\cdot\Phi_{\varepsilon_n}}}(f)$, and $(v, n) \mapsto \text{rv}(\mu_{v\cdot\Phi_{\varepsilon_n}})$ to arbitrary precision. Then there is a Turing machine whose inputs include these oracles which computes $\mathcal{H}_\Phi$ on* $\text{int Rot}(\Phi)$.

Proof. By Theorem 4.6 and Remark 4.7, it is enough to show that $(n, s, w) \mapsto h^l_{\Phi_{\varepsilon_n}}(w, 2^{-s})$ and $(n, s, w) \mapsto h^u_{\Phi_{\varepsilon_n}}(w, 2^{-s})$ can be approximated by oracles. In this proof, we focus on $h^u_{\Phi_{\varepsilon_n}}(w, 2^{-s})$; the case for h^l is similar. We fix $w_0 \in \text{int Rot}(\Phi)$ and assume that there is an oracle that approximates w_0. By Corollary 3.8, we observe that the distance to the boundary function r is computable.

By passing to a subsequence of the ε_n (and using sufficiently good approximations for r, $\sqrt{m}$, and ε_n), we may assume that the decreasing ε_n satisfy $4\sqrt{m}\varepsilon_n < r$ for all n. We let $w_1, \ldots, w_{2^m} \in \overline{B}(w_0, \frac{1}{2}r) \subset \text{int Rot}(\Phi)$ as in Lemma 4.2. For each w_i, let $\mu_i \in \mathcal{M}$ be an invariant measure such that $\text{rv}_\Phi(\mu_i) = w_i$. We observe that since $\|\Phi - \Phi_{\varepsilon_n}\|_\infty < \varepsilon_n$, $\|\text{rv}_\Phi(\mu_i) - \text{rv}_{\Phi_{\varepsilon_n}}(\mu_i)\| < \varepsilon_n < (1/4\sqrt{m})r$. Therefore, by Lemma 4.2, $B(w_0, (1/4\sqrt{m})r) \subset \text{conv}(\text{rv}_{\Phi_{\varepsilon_n}}(\mu_i))_i \subset \text{Rot}(\Phi_{\varepsilon_n})$, uniformly, for all n. Fix s to be a sufficiently large integer so that $2^{-s} < (1/4\sqrt{m})r$.

We observe that the variational principle for the topological entropy, see equation (2) with $\Phi \equiv 0$, implies that $h_{\text{top}}(f) = h_{\mu_{0\cdot\Phi_{\varepsilon_n}}}(f)$. By assumption, the map $(v, n) \mapsto h_{\mu_{v\cdot\Phi_{\varepsilon_n}}}(f)$ is approximated by an oracle, so we can approximate $h_{\text{top}}(f)$ to any precision. In addition, by computing an approximation to w_0 of high enough precision, by Proposition 4.8, we can compute an upper bound R for $\|v\|$ that applies to all n and to all w within a fixed neighborhood of w_0. Throughout the remainder of this proof, we restrict our attention to the closed ball $\overline{B}(0, R)$ in $\mathbb{R}^m$.

Since the equilibrium states achieve the localized measure of maximal entropy at any $w \in \text{int Rot}(\Phi)$, it follows that

$$h^u_{\Phi_n}(w_0, 2^{-s}) = \max\{h_{\mu_{v\cdot\Phi_{\varepsilon_n}}}(f) : v \in \overline{B}(0, R), \text{rv}(\mu_{v\cdot\Phi_{\varepsilon_n}}) \in B(w_0, 2^{-s})\}.$$

We observe that $\overline{B}(0, R)$ and $\overline{B}(w_0, 2^{-s})$ are compact subsets of $\mathbb{R}^m$, and we can compute coverings of them using arbitrarily small balls. Moreover, the maximum, entropy, and rotation vector functions can all be approximated, so we can approximate this maximum to arbitrary precision.

A similar argument holds for $h^l_{\Phi_{\varepsilon_n}}$. Thus, we have established the conditions of Theorem 4.6 and, hence, there is a Turing machine that computes $\mathcal{H}$ on $\text{int Rot}(\Phi)$. □

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Remark 4.10. We note that, in Theorem 4.9, we can replace the assumption on the uniqueness of the equilibrium states of the potentials $v \cdot \Phi_{\varepsilon_n}$ by a slightly more general condition: namely, it is sufficient to require that for all $v \in \mathbb{R}^m$ and all $n \in \mathbb{N}$, the equilibrium states of the potentials $v \cdot \Phi_{\varepsilon_n}$ have the same rotation vector. When the rotation vectors agree, the equilibrium states also have the same entropy. This more general condition holds if and only if $v \mapsto P_{\text{top}}(v \cdot \Phi_{\varepsilon_n})$ is differentiable on $\mathbb{R}^m$ for all $n \in \mathbb{N}$; see [**33**].

In addition, we note that if the functions $n \mapsto \varepsilon_n$, $(v, n) \mapsto h_{\mu_{v\cdot\Phi_{\varepsilon_n}}}(f)$, and $(v, n) \mapsto \text{rv}(\mu_{v\cdot\Phi_{\varepsilon_n}})$ are computable, then $\mathcal{H}_\Phi$ is computable on $\text{int Rot}(\Phi)$.

5. *Computability of rotation sets for shift maps*

In this and the following section, we apply the theoretical results of the previous sections to shift maps.

5.1. *Shift maps.* We first collect some basic facts on shift maps. Let $d \in \mathbb{N}$ and let $\mathcal{A} = \{0, \ldots, d-1\}$ be a finite alphabet of d symbols. The (one-sided) *shift space* Σ_d on the alphabet $\mathcal{A}$ is the set of all sequences $x = (x_k)_{k=1}^{\infty}$, where $x_k \in \mathcal{A}$ for all $k \in \mathbb{N}$. We endow Σ_d with the *Tychonov product topology* which makes Σ_d a compact metrizable space. For example, given $0 < \theta < 1$, the metric given by

$$d(x, y) = d_\theta(x, y) \stackrel{\text{def}}{=} \theta^{\min\{k\in\mathbb{N}:\, x_k \neq y_k\}} \quad \text{and} \quad d(x, x) = 0$$

induces the Tychonov product topology on Σ_d. The *shift map* $f : \Sigma_d \to \Sigma_d$, defined by $f(x)_k = x_{k+1}$, is a continuous d-to-1 map on Σ_d.

If $X \subset \Sigma_d$ is an f-invariant set, we say that $f|_X$ is a *subshift* with shift space X. In the following, we use the symbol X for any shift space including the full shift $X = \Sigma_d$. A particular class of subshifts are subshifts of finite type. Namely, suppose that A is a $d \times d$ matrix with values in $\{0, 1\}$; then consider the set of sequences given by $X = X_A = \{x \in \Sigma_d : A_{x_k x_{k+1}} = 1\}$. The set X_A is a closed (and, therefore, compact) f-invariant set, and we say that $f|_{X_A}$ a *subshift of finite type*. By reducing the alphabet, if necessary, we always assume that $\mathcal{A}$ does not contain symbols that do not occur in any of the sequences in X_A.

A continuous map $f : Y \to Y$ on a compact metric space Y is called *(topologically) transitive* if, for any pair of non-empty open sets $U, V \subset Y$, there exists $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$. We note that if f is onto, then transitivity is equivalent to having a dense orbit. Moreover, we say that $f : Y \to Y$ is *topologically mixing* if, for any pair of non-empty open sets $U, V \subset Y$, there exists $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$ for all $n \geq N$. A SFT $f|_{X_A}$ is transitive if and only if A is *irreducible*, that is, for each i and j, there exists an $n \in \mathbb{N}$ such that $A^n_{ij} > 0$. Moreover, $f|_{X_A}$ is topologically mixing if and only if A is *aperiodic*, that is, there exists $n \in \mathbb{N}$ such that $A^n_{ij} > 0$ for all i and j.

In the context of symbolic dynamics, we always consider the case of one-sided shift maps. However, all our results carry over to the case of two-sided shift maps. For details on how to make the connection between one-sided shift maps and two-sided shift maps, we refer the interested reader to [**33**].

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Let $f : X \to X$ be a one-sided subshift. Given $x \in X$, we write $\pi_k(x) = (x_1, \ldots, x_k) \in \mathcal{A}^k$. Let $\tau = (\tau_1, \ldots, \tau_k) \in \mathcal{A}^k$. We denote the *cylinder of length k generated* by τ by $\mathcal{C}(\tau) = \{x \in X : x_1 = \tau_1, \ldots, x_k = \tau_k\}$. We note that $\mathcal{C}(\tau)$ may be empty. If $\mathcal{O}(\tau) \stackrel{\text{def}}{=} (\tau_1, \ldots, \tau_k, \tau_1, \ldots, \tau_k, \ldots) \in X$, we call $\mathcal{O}(\tau)$ the *periodic point in X generated by* τ *of period k*. We say that $x \in X$ is a *preperiodic* point if $f^n(x)$ is periodic for some $n \in \mathbb{N}$. If f is a SFT and $\mathcal{C}(\tau)$ is not empty, then the preperiodic points in $\mathcal{C}(\tau)$ form a dense (and, in particular, non-empty) subset of $\mathcal{C}(\tau)$.

Given $x \in X$ and $k \in \mathbb{N}$, we call $\mathcal{C}_k(x) = \mathcal{C}(\pi_k(x))$ the *cylinder of length k generated by* x. We denote the *cardinality of the set of cylinders of length k in X* by $m_c(k)$. We note that $m_c(k) \leq d^k$, with equality for the full shift.

We recall the definitions from §1.2 for $\mathrm{Per}_n(f)$ and $\mathrm{Per}(f)$, i.e., the set of periodic points with prime period n and the set of periodic points of f, respectively. Let $x \in \mathrm{Per}_n(f)$. We call $\tau_x = (x_1, \ldots, x_n)$ the *generating segment* of x, i.e., $x = \mathcal{O}(\tau_x)$. Next, we define certain periodic points that play a crucial role when dealing with locally constant potentials.

Definition 5.1. Let $n, k \in \mathbb{N}$. We say that $x \in \mathrm{Per}_n(f)$ is a *k-elementary periodic point with period n* if the cylinders $\mathcal{C}_k(x), \ldots, \mathcal{C}_k(f^{n-1}(x))$ are pairwise disjoint. Fixed points are the k-elementary points with period 1. In the case $k = 1$, we simply say that x is an *elementary periodic point*. We denote by $\mathrm{EPer}^k(f)$ the set of all k-elementary periodic points.

We observe that the period n of a k-elementary periodic point is at most $m_c(k) \leq d^k$. In particular, $\mathrm{EPer}^k(f)$ is finite.

Let $\Phi \in C(X, \mathbb{R}^m)$. Given $k \in \mathbb{N}$, we define the *k-variation* of Φ as the maximum difference of Φ applied to elements of the same cylinder of length k, i.e., $\mathrm{var}_k(\Phi) = \sup\{\|\Phi(x) - \Phi(y)\| : x_1 = y_1, \ldots, x_k = y_k\}$. We say that Φ is constant on cylinders of length k if $\mathrm{var}_k(\Phi) = 0$. It is easy to see, from the compactness of X, that Φ is locally constant if and only if Φ is constant on cylinders of length k for some $k \in \mathbb{N}$.

5.2. *Computability of shift spaces and potentials.* Throughout this section, we assume that X is a shift space on the alphabet $\mathcal{A} = \{0, \ldots, d-1\}$. Since, in general, there are uncountably many elements of X, we develop a computability theory for shift spaces. We begin by explicitly applying the definition of computability from Definition 2.1 to this case.

Definition 5.2. An *oracle* for a sequence $x = \{x_n\}_{n \in \mathbb{N}} \in X$ is a function ϕ such that on input n, $\phi(n) = x_n$. Moreover, x is called *computable* if there is a Turing machine ϕ which is an oracle for x.

Since, in general, X is uncountable and there are only countably many Turing machines, most sequences in X are not computable. Preperiodic sequences, however, are computable. Additionally, the definition of a computable function is identical to Definition 2.1. We observe that the distance function d_θ generating the Tychonov product topology, see §5.1, is computable if and only if $\theta \in (0, 1)$ is computable. Therefore, throughout the remainder

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of this paper, we assume that θ is a computable real number in $(0, 1)$. This makes X into a computable metric space with $\mathcal{S}_X$ consisting of the preperiodic points.

Briefly, we outline why the preperiodic (and periodic) points are computable. Since the allowable transitions are given by the non-zero entries in the matrix A, for any finite sequence $\tau = (\tau_1, \ldots, \tau_k)$, it is straight-forward to check that all transitions in τ are allowable. Moreover, if $\tau\tau_1$, i.e., τ concatenated with its first symbol, is an allowable sequence, then τ can be repeated forever. Since there are only finitely many symbols, given any finite-length prefix, τ_{prefix}, one can extend τ_{prefix} by all sequences of length at most $d + 1$. Either none of these sequences will be allowable or there will be a sequence that ends with a repeated symbol, which can be extended to a preperiodic sequence.

We note that if θ in d_θ is computable, then any subshift of finite type is recursively precompact. In particular, for any $n > 0$, we can compute a k so that $\theta^k < 2^{-n}$. Let $\{\tau_{k_i}\}_{i=1}^{d^{k-1}}$ be the set of all sequences of length k on d symbols. By following the argument above, we can either verify that $\mathcal{C}(\tau_{k_i})$ is empty or compute a (computable) preperiodic point $x_{k_i} \in \mathcal{C}(\tau_{k_i})$. In this case, $\overline{B}(x_{k_i}, \theta^k) = \mathcal{C}(\tau_{k_i})$.

We observe that when X is a shift space for a SFT, a function Φ which is locally constant is computable if and only if its range is a set of computable numbers. More precisely, we recall that every non-empty cylinder $C(\tau)$ contains a computable point and the value of Φ on a computable point is computable. Additionally, the definition of computable sets carries over directly to the case of shift spaces.

We recall the definition of the total parameter space $T \subset C(X, \mathbb{R}^m) \times \mathbb{R}^m$ given in Definition 4.4. We make the definition of an oracle for a point in this space explicit as follows.

Definition 5.3. Let (Φ, w) be a point in T, i.e., $w \in \text{Rot}(\Phi)$. An oracle for (Φ, w) is a pair of oracles (ϕ, ψ) with the following properties:

(1) ψ is a function such that for any n, $\psi(n)$ is a point in $\mathbb{Q}^m$ which is within 2^{-n} of w; and

(2) $\phi : X \to \mathbb{R}^m$ is a function such that for any n and $x \in X$, $\phi(x)$ and $\Phi(x)$ differ by at most 2^{-n}.

Moreover, (Φ, w) is called *computable* if both Φ and w are computable. In this case, ψ is a Turing machine approximating w and ϕ and is also a Turing machine for Φ.

For a distance on T, we use the sum of the supremum norm on $C(X, \mathbb{R}^m)$ and the Euclidean distance on $\mathbb{R}^m$. We also explicitly define computable functions on subsets of T.

Definition 5.4. Let $U \subset T$ and $F : U \to \mathbb{R}$. Then F is *computable* if there is a Turing machine χ such that for any $(\Phi, w) \in U$, any oracle (ψ, η) of (Φ, w), and natural number n, $\chi(\psi, \eta, n)$ is a rational number of distance at most 2^{-n} to $F(\Phi, w)$.

5.3. *Locally constant potentials for SFTs.* Let $f : X \to X$ be a SFT on the alphabet $\mathcal{A} = \{0, \ldots, d - 1\}$ with transition matrix A. For $k, m \in \mathbb{N}$, we denote the set of potentials $\Phi : X \to \mathbb{R}^m$ that are constant on cylinders of length k by $LC_k(X, \mathbb{R}^m)$. Based on work of Ziemian [**60**] and Jenkinson [**33**] (see also [**46**]), we provide the necessary tools for the

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study of the computability of rotation sets and their entropies. We start with the following key result.

PROPOSITION 5.5. *Let $k, m \in \mathbb{N}$, $\Phi \in LC_k(X, \mathbb{R}^m)$, and $d' = m_c(k)$. Then there exists a subshift $g : Y \to Y$ of finite type with alphabet $\mathcal{A}' = \{0, \ldots, d' - 1\}$ and transition matrix A' with the following properties:*

(1) *there exists a homeomorphism $h : X \to Y$ that conjugates† f and g (i.e., $h \circ f = g \circ h$);*
(2) *the transition matrix A' has at most d non-zero entries in each row; and*
(3) *the potential $\Phi' = \Phi \circ h^{-1}$ is constant on cylinders of length one.*

Proof. First, we define the map h. Let $\{\mathcal{C}_k(0), \ldots, \mathcal{C}_k(m_c(k) - 1)\}$ denote the set of cylinders of length k in X, which we identify with $\mathcal{A}' = \{0, \ldots, d' - 1\}$. The transition matrix A' is defined by $a'_{ij} = 1$ if and only if there exists an $x \in X$ with $\mathcal{C}_k(x) = i$ and $\mathcal{C}_k(f(x)) = j$.

Let $Y = Y_{\mathcal{A}'}$ be the shift space in $\mathcal{A}'^{\mathbb{N}}$ given by the transition matrix A'. Furthermore, let $g : Y \to Y$ be the corresponding map for the SFT. For $x \in X$, we define $h(x) = y = (y_n)_{n=1}^{\infty}$ by $y_n = \mathcal{C}_k(f^{n-1}(x))$. It follows from the definition that $h : X \to Y$ is a bijection. Next, we show that h is a homeomorphism. Let $(\xi^n)_{n=1}^{\infty}$ be a sequence in X with $\xi = \lim \xi^n$. This means that for any $\ell \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\xi_i^n = \xi_i$ for all $i = 1, \ldots, \ell$. We conclude that $h(\xi^n)_i = h(\xi)_i$ for $i = 1, \ldots, \ell + 1 - k$ for all $n \geq N$, which establishes the continuity of h. Finally, since h is a continuous bijection with compact domain, h is a homeomorphism. Let $x = (x_n)_{n=1}^{\infty} \in X$ and $y = (y_n)_{n=1}^{\infty} = h(f(x))$. By definition, $y_n = \mathcal{C}_k(f^{n-1}(f(x))) = \mathcal{C}_k(f^n(x))$. On the other hand, $g(h(x))_n = h(x)_{n+1} = \mathcal{C}_k(f^{(n+1)-1}(x)) = \mathcal{C}_k(f^n(x)) = y_n$. This shows that $h \circ f = g \circ h$.

The assertion that A' has at most d non-zero entries in each row follows from the fact that for each cylinder $\mathcal{C}_k(i) \subset X$, the set $f(\mathcal{C}_k(i))$ can be written as the disjoint union of at most d cylinders of length k. More precisely, for any $x \in f(\mathcal{C}_k(i))$, the first $k - 1$ letters of x are determined by $\mathcal{C}_k(i)$ and there are at most d possible letters for the kth position. Let $y, \widetilde{y} \in Y$ with $y_1 = \widetilde{y}_1$. Then $h^{-1}(y), h^{-1}(\widetilde{y}) \in \mathcal{C}_k(i)$ for some $i \in \{1, \ldots, m_c(k) - 1\}$. Since Φ is constant on cylinders of length k, we conclude that $\Phi(h^{-1}(y)) = \Phi(h^{-1}(\widetilde{y}))$. This shows that Φ' is constant on cylinders of length one. □

Ziemian [**60**] proved that the rotation set of a potential Φ that is constant on cylinders of length two is a polyhedron. This result extends to potentials that are constant on cylinders of length $k \geq 2$; see Jenkinson [**33**] (also see Reitsam [**46**] for further details). For completeness we provide a short proof.

THEOREM 5.6. *Let $f : X \to X$ be a transitive SFT and let $\Phi \in LC_k(X, \mathbb{R}^m)$. Then* $\mathrm{Rot}(\Phi)$ *is a polyhedron; in particular,* $\mathrm{Rot}(\Phi)$ *is the convex hull of* $\mathrm{rv}(\{\mu_x : x \in \mathrm{EPer}^k(f)\})$.

† We exclusively use h_μ and h_{top} for entropies and h for the conjugate map. Both uses are fairly common in the respective literature.

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Proof. Let (Y, g, Φ') be as in Proposition 5.5 and h the conjugate map. Since being transitive is a topological property that is preserved by topological conjugation, it follows that g is also transitive. It follows from the definition that $h(\mathrm{EPer}^k(f)) = \mathrm{EPer}^1(g)$. Therefore, by Proposition 5.5, it suffices to prove the statement for g and Φ'. Let $y \in Y$ be a periodic point of g. The generating segment of y can be obtained as a finite concatenation of generating segments of elementary periodic points. Therefore, $\mathrm{rv}_{\Phi'}(\mu_y)$ is a convex combination of $\mathrm{rv}_{\Phi'}(\mu)$ for $\mu \in \mathrm{EPer}^1(g)$; in other words, $\mathrm{rv}_{\Phi'}(\mu_y) \in \mathrm{conv}(\mathrm{rv}(\mathrm{EPer}^1(g)))$. The result now follows from the fact that the periodic point measures are dense in $\mathcal{M}$ and the finiteness of $\mathrm{EPer}^1(g)$. □

Remark 5.7. From the proof of Theorem 5.6, one deduces that the conclusions of Theorem 5.6 hold if f is a subshift with the following properties:

(1) $\{\mu_x : x \in \mathrm{Per}(f)\}$ is dense in $\mathcal{M}$;
(2) there exists a finite set $P \subset \mathrm{Per}(f)$ such that the rotation vector of every periodic point measure can be written as a convex sum of rotation vectors of periodic point measures in P.

For the discussion of Property (1), we refer the interested reader to [**22**] and the references therein. To the best of our knowledge, Property (2) has not been studied in the literature beyond SFTs and k-elementary periodic points. In principle, however, one can check Property (2) for particular classes of shift maps.

Next, we prove the following useful lemma.

LEMMA 5.8. *Let $\Phi \in C(X, \mathbb{R}^m)$. For all $\varepsilon > 0$, there exists an ε-approximation Φ_ε of Φ such that Φ_ε is constant on cylinders of length $k(\varepsilon)$ for some $k(\varepsilon) \in \mathbb{N}$. If Φ is not locally constant, then $k(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Moreover, if Φ and ε are computable, then Φ_ε and $k(\varepsilon)$ can be chosen to be computable.*

Proof. Since X is compact, Φ is continuous, and cylinders form a basis for the topology on X, for any ε, there exists a k such that in every cylinder $\mathcal{C}(\tau)$ of length k, if $x, y \in C(\tau)$, then $\|\Phi(x) - \Phi(y)\| < \varepsilon$. From this, Φ_ε can be constructed by choosing $x_\tau \in \mathcal{C}(\tau)$ for each non-empty cylinder of length k in X and setting $\Phi_\varepsilon(x) = \Phi(x_\tau)$ whenever $x \in \mathcal{C}(\tau)$.

Suppose that Φ is not locally constant and fix $k > 0$. Since Φ is not locally constant, for some cylinder $\mathcal{C}(\tau)$ of length k, there exist $x, y \in \mathcal{C}(\tau)$ such that $\Phi(x) \neq \Phi(y)$. So, we choose $0 < \varepsilon < \frac{1}{2}\|\Phi(x) - \Phi(y)\|$ to get $k(\varepsilon) > k$.

Finally, in the case when ε and Φ are computable, we have the following: since X is recursively precompact, Φ has a global modulus of continuity. Since θ is computable, we can use the global modulus of continuity to find a k as above. Since every non-empty cylinder of X has some point of $\mathcal{S}_X$ in the cylinder, we can use this point to define Φ_ε. □

Finally, we can prove the main result of this section.

THEOREM 5.9. *Let $f : X \to X$ be a transitive SFT with computable distance d_θ. Let $\Phi \in C(X, \mathbb{R}^m)$ be computable; then* $\mathrm{Rot}(\Phi)$ *is computable.*

Proof. We use the previous results of this section to verify the conditions in Observation 3.4. Let Φ_ε be as constructed in Lemma 5.8 with corresponding $k = k(\varepsilon)$;

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then we observe that for any measure ν, $\|\mathrm{rv}_\Phi(\nu) - \mathrm{rv}_{\Phi_\varepsilon}(\nu)\| < \varepsilon$. Moreover, the points in $\mathrm{EPer}^k(f)$ are enumerable as they correspond to allowable (via the transition matrix A) sequences from a finite alphabet without repeats. Finally, since $\mathrm{PM}(X)$ is recursively precompact, the map $\mu \mapsto \mathrm{rv}_{\Phi_\varepsilon}(\mu)$ is computable; see Lemma 3.2. Therefore, we can use the global modulus of continuity of this map along with Theorem 5.6 to find a finite set of rational convex combinations of μ_x for $x \in \mathrm{EPer}^k(f)$ such that for all $\nu \in \mathcal{M}$, there is some μ in the finite set of rational convex combinations of the μ_x so that $\|\mathrm{rv}_{\Phi_\varepsilon}(\nu) - \mathrm{rv}_{\Phi_\varepsilon}(\mu)\|$ is arbitrarily small. Since

$$\begin{aligned}\|\mathrm{rv}_\Phi(\nu) - \mathrm{rv}_\Phi(\mu)\| &\\ \leq \|\mathrm{rv}_\Phi(\nu) - \mathrm{rv}_{\Phi_\varepsilon}(\nu)\| &+ \|\mathrm{rv}_{\Phi_\varepsilon}(\nu) - \mathrm{rv}_{\Phi_\varepsilon}(\mu)\| + \|\mathrm{rv}_{\Phi_\varepsilon}(\mu) - \mathrm{rv}_\Phi(\mu)\|\end{aligned}$$

can be made arbitrarily small, the conditions in Observation 3.4 hold. □

6. *Computability of localized entropy for shift maps*

In this section, we build upon the results of §§4 and 5 to prove that the localized entropy function is computable on the interior of the rotation set for transitive SFTs. Our main goal is to verify the hypotheses of Theorem 4.9 in this case. By applying Lemma 5.8, it is enough to consider locally constant potentials. For these potentials, we establish the computability of the entropies and rotation vectors of their equilibrium states. Throughout this section, we assume that $f : X \to X$ is a transitive SFT over an alphabet with d symbols and transition matrix $A = (A_{ij})$. We recall that transitivity is equivalent to A being irreducible.

We begin by introducing some notation. For an $n \times m$ matrix $B = (B_{ij})$, we write $B \geq 0$ or $B > 0$ if all the entries of B are non-negative or positive, respectively. Moreover, for two $n \times m$ matrices B and C, we write $B \geq C$ or $B > C$ if $B - C \geq 0$ or $B - C > 0$, respectively.

Next, we review some basic facts about Markov measures; for details, see [**37**]. We say that a $d \times d$ matrix $B \geq 0$ is *compatible* with A provided $A_{ij} = 0$ implies that $B_{ij} = 0$. Moreover, B is *faithfully compatible* with A provided $A_{ij} > 0$ if and only if $B_{ij} > 0$. For example, for any function $\Phi : \{1, \ldots, d\}^2 \to \mathbb{R}$, the matrix B with $B_{ij} = e^{\Phi(i,j)} A_{ij}$ is faithfully compatible with A. In fact, any matrix B faithfully compatible with A can be written as $B_{ij} = e^{\Phi(i,j)} A_{ij}$ for some $\Phi : \{1, \ldots, d\}^2 \to \mathbb{R}$. Such a B is irreducible since A is irreducible. A $d \times d$ matrix $B \geq 0$ is *row stochastic* if the sum of the entries in each row is 1. The set of all row stochastic matrices compatible with A is denoted by $M_{\mathrm{stoch}}(A)$.

By the Perron–Frobenius theorem, any matrix $B \geq 0$ which is faithfully compatible to an irreducible matrix A has positive left and right eigenvectors $l = (l_1, \ldots, l_d) > 0$ and $r = (r_1, \ldots, r_d) > 0$ associated to its real Perron–Frobenius eigenvalue $\lambda > 0$. Let the matrix $P = (P_{ij})$ be given by

$$P_{ij} := B_{ij} \frac{r_j}{\lambda r_i}. \tag{14}$$

We observe that P is row stochastic, has Perron–Frobenius eigenvalue 1, and is faithfully compatible with A. It follows that $p = (r_1 l_1, \ldots, r_d l_d) > 0$ is a left eigenvector of the Perron–Frobenius eigenvalue 1 of P. Moreover, if r and l are normalized so that their inner product is 1, i.e., $r \cdot l = 1$, then p is a probability vector. In this situation, the pair

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(p, P) defines a probability measure $\mu = \mu_{(p,P)}$ characterized by its value on cylinders. Namely,

$$\mu(\mathcal{C}(j_0, j_1, \ldots, j_r)) = p_{j_0} P_{j_0 j_1} \ldots P_{j_{r-1} j_r}. \tag{15}$$

We say that $\mu = \mu_{(p,P)}$ is the *(one-step) Markov measure* associated with P. A direct computation, see, e.g., [**37**], shows that μ is f-invariant and that its measure-theoretic entropy is given by

$$h_\mu(f) = -\sum_{i,j} p_i P_{ij} \log P_{ij}. \tag{16}$$

If P is defined as in equation (14) for some matrix B faithfully compatible to an irreducible $A \geq 0$, we can write the entropy of μ in terms of B as follows:

$$\begin{aligned} h_\mu(f) &= -\sum_{i,j} p_i P_{ij} \log P_{ij} = -\sum_{i,j} l_i B_{ij} \frac{r_j}{\lambda} (\log B_{ij} + \log r_j - \log r_i - \log \lambda) \\ &= \log \lambda - \sum_j l_j r_j \log r_j + \sum_i l_i r_i \log r_i - \sum_{i,j} l_i \frac{r_j}{\lambda} B_{ij} \log B_{ij} \\ &= \log \lambda - \sum_{i,j} l_i \frac{r_j}{\lambda} B_{ij} \log B_{ij}. \end{aligned} \tag{17}$$

We observe that since $A_{ij} \in \{0, 1\}$ for all $i, j = 1, \ldots, d$, if $B = A$, then

$$h_\mu(f) = \log \lambda = h_{\text{top}}(f).$$

In the case where $B_{ij} = e^{\Phi(i,j)} A_{ij}$, by substituting this equality into equation (17), it follows that

$$\begin{aligned} h_\mu(f) &= \log \lambda - \sum_{i,j=1}^{d} l_i r_i \frac{r_j}{\lambda r_i} B_{ij} \Phi(i, j) \\ &= \log \lambda - \sum_{i,j=1}^{d} p_i P_{ij} \Phi(i, j) = \log \lambda - \int_X \Phi \, d\mu. \end{aligned} \tag{18}$$

We note that we can interpret Φ in the integral in equation (18) as a real-valued potential defined by $\Phi(\mathcal{C}(i, j)) = \Phi(i, j)$. In particular, $\Phi \in LC_2(X, \mathbb{R})$. We obtain the following useful result.

PROPOSITION 6.1. *Let $f : X \to X$ be a transitive SFT with transition matrix A and let $\Phi \in LC_2(X, \mathbb{R})$. Let $B \in \mathbb{R}^{d \times d}$ be defined by $B_{ij} = e^{\Phi(\mathcal{C}(i,j))} A_{ij}$. Let λ denote the Perron–Frobenius eigenvalue of B. Let P be defined as in equation (14) and let $\mu = \mu_{(p,P)}$ be the Markov measure associated with P defined in equation (15). Then μ is the unique equilibrium measure of the potential Φ, that is, the unique invariant measure satisfying*

$$P_{\text{top}}(\Phi) = h_\mu(f) + \int_X \Phi \, d\mu = \log \lambda.$$

Proof. The result follows from the Ruelle–Perron–Frobenius theorem; see, e.g., [**7**, **50**], the variational principle, i.e., equation (2), and equation (18). □

Next, we consider the computability of the Perron–Frobenius eigenvalue and eigenvectors.

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PROPOSITION 6.2. *Let M_d be the set of non-negative and irreducible $d \times d$ matrices. The maps assigning $B \in M_d$ to the Perron–Frobenius eigenvalue, or the left and right eigenvectors with first entry* 1*, are computable.*

Proof. We follow the approach in [**20**, **52**, **58**]. Let S be the closure of the intersection of the $(d-1)$-dimensional sphere S^{d-1} with the first orthant in $\mathbb{R}^d$; we observe that this set is computable in $\mathbb{R}^d$. Since B is irreducible, $(I+B)^{d-1} > 0$. Moreover, by [**20**, **58**], the Perron–Frobenius eigenvalue can be computed using the following formula:

$$\max_{x \in S} \min_{i} \frac{(B(I+B)^{d-1}x)_i}{((I+B)^{d-1}x)_i}.$$

The computation of the left and right eigenvectors is nearly identical, so we only show that one of them is computable. By Perron–Frobenius theory, the Perron–Frobenius eigenvalue is an eigenvalue of (algebraic) multiplicity one and all of its entries are non-zero. Therefore, there is a unique eigenvector r associated to the Perron–Frobenius eigenvalue λ whose first entry is 1. Therefore, after substituting 1 for the first entry of r, $(B - \lambda I)r = 0$ is a square system with $d-1$ variables with a unique solution. This can be computed with Cramer's rule. □

Next, we establish the main assumptions of Theorem 4.9 for transitive SFTs.

THEOREM 6.3. *Let $f : X \to X$ be a transitive SFT with transition matrix A and computable distance d_θ. Let $\Phi : X \to \mathbb{R}^m$ be a computable potential. Suppose that $(\varepsilon_n)_n$ is a uniformly computable, convergent sequence of positive numbers converging to* 0*. Then there exists an approximating sequence $(\Phi_{\varepsilon_n})_n$ of Φ such that for all $v \in \mathbb{R}^m$ and $n \in \mathbb{N}$, the potential $v \cdot \Phi_{\varepsilon_n}$ has a unique equilibrium state $\mu_{v \cdot \Phi_{\varepsilon_n}}$. Moreover, the maps $(v, n) \mapsto h_{\mu_{v \cdot \Phi_{\varepsilon_n}}}(f)$ and $(v, n) \mapsto \mathrm{rv}(\mu_{v \cdot \Phi_{\varepsilon_n}})$ are computable.*

Proof. By Lemma 5.8, since ε_n is computable, there is a computable ε_n-approximation Φ_{ε_n} to Φ which is locally constant. Since locally constant potentials are Lipschitz continuous, we may conclude from Property 4 of the topological pressure in §2.2 that for all $v \in \mathbb{R}^m$ and $n \in \mathbb{N}$, the potential $v \cdot \Phi_{\varepsilon_n}$ has a unique equilibrium state $\mu_{v \cdot \Phi_{\varepsilon_n}}$. Suppose that an oracle for v is given; then there is a Turing machine that produces Φ_v, a locally constant approximation to $v \cdot \Phi_{\varepsilon_n}$.

By applying Proposition 5.5 (and computing a larger alphabet of size d'), we consider the conjugate SFT g, with transition matrix A', and potential Φ'_v. We let h be the conjugating map between f and g. We recall that Φ'_v is constant on cylinders of length one. Moreover, the pressure and equilibrium states (after taking the pushforward) for Φ_v and Φ'_v are preserved under this conjugation; see, e.g., [**56**]. We observe that since Φ'_v is constant on cylinders of length one, it is, in particular, constant on cylinders of length two.

For each cylinder $\mathcal{C}(i, j)$ of length two, we define $B'_{ij} = e^{\Phi'_v(\mathcal{C}(i,j))} A'_{ij}$ and let $B' = (B'_{ij})$. Since g is transitive, A' is irreducible, which implies that B' is also irreducible. By Lemma 6.2, we can compute the Perron–Frobenius eigenvalue λ and eigenvectors r' and l', so we can compute the matrix P', where $P'_{ij} = B'_{ij}(r_j/\lambda r_i)$. Moreover, we can also compute the probability vector $p' = (r'_1 l'_1, \ldots, r'_{d'} l'_{d'})$. Let $\mu'_v = \mu_{(p', P')}$ be defined

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as in equation (15). It follows from Proposition 6.1 and Corollary 3.5 that $\mu'_v = h_*\mu_v$. Furthermore, we can compute the rotation vector of μ_v with respect to Φ_{ε_n} by

$$\mathrm{rv}(\mu_v) = \left(\int (\Phi'_v)_k \, d\mu'_v\right)_k = \left(\sum_{i,j} (\Phi'_v(\mathcal{C}(i,\, j)))_k \, p_i P_{ij}\right)_k.$$

Similarly, the measure-theoretic entropy can be calculated using Formula (16) for $h_{\mu'_v}(f) = h_{\mu_v}(f)$. □

Finally, we are able to prove the main result of this section.

THEOREM 6.4. *Let $f : X \to X$ be a transitive SFT with computable distance d_θ. Let $\Phi \in C(X, \mathbb{R}^m)$ be computable. Then $\mathcal{H}$ is computable on* $\operatorname{int}\mathrm{Rot}(\Phi)$.

Proof. If $\operatorname{int}\mathrm{Rot}(\Phi) = \varnothing$, there is nothing to prove. Otherwise, we apply Theorem 4.9. The existence of r follows from Corollary 3.8 and Theorem 5.9. The remaining assumptions are established in Theorem 6.3. □

7. *Entropy at boundary points*

In this section, we construct a class of examples that show that, in general, the localized entropy is not computable at the boundary of rotation sets. More precisely, we show that the global entropy function may be discontinuous at the boundary of the total parameter space of rotation sets; cf. Theorem 4.5. Recall that at an interior point w_0 of the rotation set, we have

$$\lim_{n\to\infty} h^l_{\Phi_{\varepsilon_n}}(w_0, \varepsilon_n) = \mathcal{H}_\Phi(w_0) = \lim_{n\to\infty} h^u_{\Phi_{\varepsilon_n}}(w_0, \varepsilon_n). \tag{19}$$

We now construct a family of examples for which there is an exposed boundary point where the two limits in equation (19) do not coincide.

Example 7.1. Let $f : X \to X$ be the one-sided full shift with alphabet $\{0, 1, 2, 3\}$. We construct a potential function Φ as follows: fix a real number $a > 0$ and consider a function $\ell_1 : [0, a] \to \mathbb{R}$ which is continuous, non-negative, increasing, and strictly concave with $\ell_1(0) = 0$. Let $\ell_2 = -\ell_1$; therefore, ℓ_2 is continuous, non-positive, decreasing, and strictly convex with $\ell_2(0) = 0$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence with $x_k \in (0, a)$ for all k that is exponentially and strictly decreasing to 0; see Figure 1. Let $x = \mathcal{O}(1) \in X$ and $y = \mathcal{O}(3) \in X$ be the fixed points of repeating 1's and 3's, respectively.

Next, we define several subsets of X. Let $S_1 = \{0, 1\}$ and $S_2 = \{2, 3\}$. Fix a natural number $\lambda \in \mathbb{N}$ with $\lambda \geq 3$. For $k \geq \lambda$, define

$$\begin{aligned}
\widetilde{Y}_i(k) &= \{\xi \in X : \xi_1, \ldots, \xi_k \in S_i\},\\
Y_i(\infty) &= \{\xi \in X : \xi_j \in S_i \text{ for all } j\} = \bigcap_k \widetilde{Y}_i(k),\\
Y_i(k) &= \{\xi \in \widetilde{Y}_i(k-1) : \xi \notin \widetilde{Y}_i(k)\} = \widetilde{Y}_i(k-1) \setminus \widetilde{Y}_i(k).
\end{aligned}$$

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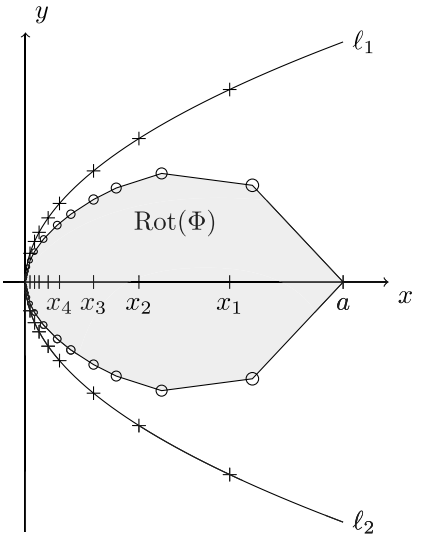


FIGURE 1. The rotation set defined in Example 7.1. The rotation set is an infinite polygon and the origin is an exposed point.

For $k > \lambda$, define

$$\begin{aligned} X_1(k) &= Y_1(k) \setminus \mathcal{C}_{k-1}(x), \\ X_2(k) &= Y_2(k) \setminus \mathcal{C}_{k-1}(y), \\ X(k) &= X_1(k) \cup X_2(k), \\ Y_0(\lambda) &= X \setminus (\widetilde{Y}_1(\lambda) \cup \widetilde{Y}_2(\lambda)). \end{aligned}$$

We define a potential $\Phi : X \to \mathbb{R}^2$ by

$$\Phi(\xi) = \begin{cases} (a, 0) & \text{if } \xi \in Y_0(\lambda), \\ (x_{k-\lambda}, 0) & \text{if } \xi \in X(k),\ k > \lambda, \\ (x_{k-\lambda}, \ell_1(x_{k-\lambda})) & \text{if } \xi \in \mathcal{C}_{k-1}(x) \cap Y_1(k),\ k > \lambda, \\ (x_{k-\lambda}, \ell_2(x_{k-\lambda})) & \text{if } \xi \in \mathcal{C}_{k-1}(y) \cap Y_2(k),\ k > \lambda, \\ (0, 0) & \text{if } \xi \in Y_1(\infty) \cup Y_2(\infty). \end{cases} \tag{20}$$

Throughout the rest of this section, we study the potential Φ defined in the example above.

LEMMA 7.2. *The potential Φ defined in equation (20) is continuous and $(0, 0)$ is an exposed point of* $\mathrm{Rot}(\Phi)$.

Proof. Let (ξ^n) be a convergent sequence in X with $\xi = \lim \xi^n$. If $\Phi(\xi) = (a, 0)$, $(x_{k-\lambda}, 0)$, or $(x_{k-\lambda}, \ell_i(x_{k-\lambda}))$, then the behavior of ξ is determined by the first k terms of ξ. Since ξ^n converges to ξ, for n sufficiently large, ξ^n agrees with ξ on the first k terms. Therefore, $\Phi(\xi^n) = \Phi(\xi)$.

If $\Phi(\xi) = (0, 0)$, then $\xi \in Y_i(\infty)$ for $i = 1, 2$. For any k, for all n sufficiently large, ξ^n and ξ share the first k terms, so $\xi^n \in \widetilde{Y}_i(k)$. Therefore, for $k > \lambda$, $\Phi(\xi^n)$ is one of

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$(x_{k-\lambda}, 0)$, $(x_{k-\lambda}, \ell_i(x_{k-\lambda}))$, or $(0, 0)$. As k grows, the value for all of these expressions approaches $(0, 0)$.

To show that $(0, 0)$ is an exposed point of $\mathrm{Rot}(\Phi)$, we observe that $\mathrm{Rot}(\Phi) \subset \mathrm{conv}(\Phi(X))$. Since all points in the image of Φ other than $(0, 0)$ have positive x-coordinate, $(0, 0)$ is extremal. Finally, the compactness of $\Phi(X)$ implies that $(0, 0)$ is an exposed point with the y-axis as supporting line. □

We now restrict x_k and ℓ_1 to control the shape of $\mathrm{Rot}(\Phi)$. Let $w(0) = (a, 0)$ and $w(\infty) = (0, 0)$. Suppose that $\sum_{k=1}^{\infty} x_k < a$ and $\ell_1(x_1) < (\lambda + 1)\ell_1(x_2)$. For $i = 1, 2$, define

$$w_i(\lambda) = \frac{1}{3\lambda}[3(\lambda - 1)(a, 0) + 2(x_1, \ell_i(x_1)) + (x_1, \ell_{3-i}(x_1))].$$

Moreover, for $j > \lambda$ and $i \in \{1, 2\}$, we define

$$w_i(j) = \frac{1}{j}\left[\lambda(a, 0) + \sum_{k=1}^{j-\lambda}(x_k, \ell_i(x_k))\right].$$

Using techniques similar to those in the proofs in [**40**, Example 2], one can show that

$$\mathrm{Rot}(\Phi) = \mathrm{Conv}\{w(0), w(\infty), w_i(j) : j \geq \lambda,\ i = 1, 2\}.$$

In particular, $\partial\mathrm{Rot}(\Phi)$ is an infinite polygon. Furthermore, by requiring additional properties on ℓ_1, it can be arranged that $w(\infty)$ is a smooth boundary point. We refer the interested reader to [**40**] for details.

Next, we define approximations of the potential Φ. Fix $n \in \mathbb{N}$ and let $\varepsilon_n = 2^{-n}$. Since ℓ_1 is continuous and $\ell_1(0) = 0$, there exists $K = K(n) \geq 2\lambda$ such that $\|(x_{k-\lambda}, \ell_1(x_{k-\lambda}))\| < \varepsilon_n$ and $\|(x_{k-\lambda}, \ell_1(x_{k-\lambda})) - (x_{l-\lambda}, \ell_1(x_{l-\lambda}))\| < \varepsilon_n$ for all $k, l \geq K$. We note that since $\ell_2 = -\ell_1$, the corresponding inequalities also hold for points on the curve ℓ_2. We define

$$X_{\varepsilon_n} = \bigcup_{\lambda < k \leq K} (X(k) \cup (\mathcal{C}_{k-1}(x) \cap Y_1(k)) \cup (\mathcal{C}_{k-1}(y) \cap Y_2(k))).$$

Finally, we define the potentials $\Phi_{\varepsilon_n} : X \to \mathbb{R}^2$ by changing the behavior of the potential function near $Y_1(\infty) \cup Y_2(\infty)$. Let

$$\Phi_{\varepsilon_n}(\xi) = \begin{cases} \Phi(\xi) & \text{if } \xi \in X_{\varepsilon_n} \cup Y_0(\lambda), \\ (x_{K+1-\lambda}, 0) & \text{if } \xi \in (\widetilde{Y}_1(K) \cup \widetilde{Y}_2(K)) \setminus (\mathcal{C}_K(x) \cup \mathcal{C}_K(y)), \\ (x_{K+1-\lambda}, \ell_1(x_{K+1-\lambda})) & \text{if } \xi \in \mathcal{C}_K(x), \\ (x_{K+1-\lambda}, \ell_2(x_{K+1-\lambda})) & \text{if } \xi \in \mathcal{C}_K(y). \end{cases} \tag{21}$$

It follows, from the construction, that Φ_{ε_n} is constant on cylinders of length K and that $(\Phi_{\varepsilon_n})_n$ is a converging sequence of ε_n-approximations of Φ.

THEOREM 7.3. *Let Φ and $(\Phi_{\varepsilon_n})_n$ be defined as in equations (20) and (21); then*

$$0 = \lim_{n\to\infty} h^l_{\Phi_{\varepsilon_n}}(w(\infty), \varepsilon_n) < \lim_{n\to\infty} h^u_{\Phi_{\varepsilon_n}}(w(\infty), \varepsilon_n) = \log 2. \tag{22}$$

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Proof. First, we prove the right-hand-side equality of inequality (22). In Lemma 7.2, we proved that $w(\infty) = (0, 0)$ is an exposed point of $\mathrm{Rot}(\Phi)$. Moreover, $w(\infty)$ is an extreme point of $\mathrm{conv}(\Phi(X))$, which implies that each invariant measure μ with $\mathrm{rv}_\Phi(\mu) = w(\infty)$ must be supported on $Y_1(\infty) \cup Y_2(\infty)$. Since each $Y_i(\infty)$ is a full shift on two symbols, it follows that $h_{\mathrm{top}}(f|_{Y_1(\infty)\cup Y_2(\infty)}) = \log 2$. From the variational principle, it follows that $\mathcal{H}_\Phi(w(\infty)) = \log 2$. Then Proposition 4.1 implies that $\lim_{n\to\infty} h^u_{\Phi_{\varepsilon_n}}(w(\infty), \varepsilon_n) = \log 2$.

Next, we show the left-hand-side equality of inequality (22). By construction, $(x_{K(n)+1-\lambda}, \ell_i(x_{K(n)+1-\lambda})) \in \overline{B}(w(\infty), \varepsilon_n)$ for all $n \in \mathbb{N}$ and $i = 1, 2$. We claim that $\mathcal{H}_{\Phi_{\varepsilon_n}}(x_{K+1-\lambda}, \ell_i(x_{K+1-\lambda})) = 0$ as follows: first, we observe that $(x_{K(n)+1-\lambda}, \ell_i(x_{K(n)+1-\lambda}))$ is an extreme point of $\mathrm{Rot}(\Phi_{\varepsilon_n})$ since the point is in $\mathrm{Rot}(\Phi_{\varepsilon_n})$ and it is an extreme point of $\mathrm{conv}(\Phi_{\varepsilon_n}(X))$. More precisely, for $i = 1$, the point is in the rotation set since $\mathrm{rv}_{\Phi_{\varepsilon_n}}(\mu_x) = (x_{K(n)+1-\lambda}, \ell_1(x_{K(n)+1-\lambda}))$. The case $i = 2$ is analogous. Moreover, it is an extreme point of the convex hull since all other images of Φ have either a larger x-coordinate or the same x-coordinate and a smaller y-coordinate.

Now let $\mu \in \mathcal{M}$ with $\mathrm{rv}_{\Phi_{\varepsilon_n}}(\mu) = (x_{K(n)+1-\lambda}, \ell_1(x_{K(n)+1-\lambda}))$. Therefore, the support of μ is a subset of $\Phi^{-1}_{\varepsilon_n}(x_{K(n)+1-\lambda}, \ell_1(x_{K(n)+1-\lambda}))$. Our goal is to show that $\mu = \mu_x$. We observe that we can write

$$\Phi^{-1}_{\varepsilon_n}(x_{K(n)+1-\lambda}, \ell_1(x_{K(n)+1-\lambda})) = \mathcal{C}_K(x) = \{x\} \cup \bigcup_{k>K} A_k,$$

where $A_k = \mathcal{C}_{k-1}(x) \setminus \mathcal{C}_k(x)$. By construction, $f^{-1}(A_k)$ is the disjoint union of A_{k+1} and cylinders of the form $\mathcal{C}(\xi_1\pi_{k-1}(x)\xi_{k+1})$, where $\xi_1, \xi_{k+1} \in \{0, 2, 3\}$. Let $\xi \in \mathcal{C}(\xi_1\pi_{k-1}(x)\xi_{k+1})$; then $\Phi_{\varepsilon_n}(\xi) \neq (x_{K(n)+1-\lambda}, \ell_1(x_{K(n)+1-\lambda}))$. We conclude that $\mu(\mathcal{C}(\xi_1\pi_{k-1}(x)\xi_{k+1})) = 0$, since otherwise $\mathrm{rv}_{\Phi_{\varepsilon_n}}(\mu)$ would not be an extreme point. By the f-invariance of μ, it must be that $\mu(A_k) = \mu(A_{k+1})$. Therefore, $\mu(A_k) = \mu(A_l)$ for all $k, l > K$. Since the A_k are pairwise disjoint, $\mu(A_k) = 0$ for all $k > K$. Hence, the support of μ is x and $\mu = \mu_x$. It follows that $h^l_{\Phi_{\varepsilon_n}}(w(\infty), \varepsilon_n) = 0$ for all $n \in \mathbb{N}$ and we obtain $\lim_{n\to\infty} h^l_{\Phi_{\varepsilon_n}}(w(\infty), \varepsilon_n) = 0$. □

Remark 7.4. By choosing a, x_k to be computable real numbers, and ℓ_1 to be a computable function, the result of Theorem 7.3 also applies in the computable case.

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