APPLICATION OF DOUBLE SIDE APPROACH METHOD TO THE SOLUTION OF FULLY DEVELOPED LAMINAR FLOW IN DUCT PROBLEMS

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ABSTRACT

The double side approach method combines the method of weighted residuals (MWR) with mathematical programming to solve the differential equations. Once the differential equation is proved to satisfy the maximum principle, collocation method and mathematical programming are used to transfer the problem into a bilateral inequality. By utilizing Genetic Algorithms optimization method, the maximum and minimum solutions which satisfy the inequality can be found. Adopting this method, quite less computer memory and time are needed than those required for finite element method.

In this paper, the incompressible-Newtonian, fully-developed, steady-state laminar flow in equilateral triangular, rectangular, elliptical and super-elliptical ducts is studied. Based on the maximum principle of differential equations, the monotonicity of the Laplace operator can be proved and the double side approach method can be applied. Different kinds of trial functions are constructed to meet the no-slip boundary condition, and it was demonstrated that the results are in great agreement with the analytical solutions or the formerly presented works. The efficiency, accuracy, and simplicity of the double side approach method are fully illustrated in the present study to indicate that the method is powerful for solving boundary value problems.

Keywords: Double side approach method, Fully developed laminar flow, Genetic algorithms.

1. INTRODUCTION

Nowadays, it is still challenging to solve some differential equations by either numerical or theoretical methods. The analytic exact solutions for some problems are difficult or even impossible to find; only their approximate solutions can be obtained using methods such as the method of weighted residuals (MWR).

The method of weighted residuals which is based on the governing differential equation, is a mathematical procedure for obtaining approximate solutions. To improve the approximations, experience and intuition are sometimes needed for a good first guess of the trial function. The analytical form of the approximate solution is often more useful and requires less computation time than that generated by numerical integration [1]. MWR has advantages of program simplicity, a short computer running time, and low computational error.

Several methods based on MWR have been developed. Chen *et al.* proposed an error bounds estimate procedure to solve the boundary value problem for differential equations [2]. An approximate method that uses modern mathematical programming and the point collocation method to deal with the initial value problem for differential equations are presented by Xing *et al.* [3]. Appl and Hung proposed a principle that relates the convergence of the upper and lower bounds in certain continuous boundary value problems [4].

In this study, the proposed double side approach method uses the collocation method in the method of weighted residuals to transform the differential equation into a mathematical programming problem based on the monotonic relation of solutions. After forming the mathematical programming problem, adopting Genetic algorithms as the optimization method to solve the mathematical programming problem.

Genetic algorithms (GAs) are a class of probabilistic search algorithms for optimization problems, first proposed by Holland *et al.* [5-7]. GAs mimic the genetic processes that occur in nature. GAs start with a population of randomly generated candidates and evolve toward better solutions by applying genetic operators, such as reproduction, crossover, and mutation on each individuals, and the fitness which depends on the objective function of each individuals is calculated for evaluation. The individual with the highest fitness value is the optimum individual, that is, the optimum

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solution.

In the last decade, GAs have emerged as a practical and robust search method [8-11]. GAs have many advantages comparing to the traditional optimization methods. Particularly, information about function derivatives isn't required in GAs, they work directly on the function, and the optimum solution obtained by GAs is most likely a global optimum. Moreover, GAs are intellectual searching algorithms, they search under the guidance of "fitness", not all-round or randomly and they're easy to implement on problems with complex causality. High efficiency is also a superior characteristic of GAs because they work on a set of individuals rather than just a single one.

This method deals with the error bounds of solutions based on the monotonic relation of solutions and several techniques exist for the treatment of upper and lower bound solutions under this monotonic relation. Wang *et al.* [12] used residual correction method which applys the maximum principle to the finite difference method to analyze the error bounds of non-linear Burger's equation. This method was also employed by Peng, Chen and Li [13] to the laser heating process. Both works presented the efficiency and accuracy of this numerical method. In addition, Wang, Huang and Yang [14] proposed a new computing method for the cubic spline difference and applied it on the heat conduction problem. This method is validated to have simple computational procedures and numerical precision will be increased without increasing the computational load.

Ducts with noncircular cross sections are widely used in practical applications such as heat exchangers and chemical reactors. The present study proposes the double side approach method that combines MWR and GAs to solve laminar, fully developed flow through ducts with various cross section shapes.

2. FORMULATION

2.1 Problem Statement

Flow through a duct is a fundamental problem in fluid flow. The objective of the present study is to verify the validity of the double side approach method for analyzing the axial velocity field in ducts with various cross section shapes.

Consider a fully developed, incompressible, laminar, Newtonian fluid through a straight duct with an invariant cross section. The Navier-Stokes equations for this kind of flow can reduce to the equation of momentum, which is also called Poisson's equation:

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{\mu}\frac{dP}{dz} \tag{1}$$

where w is the component of axial velocity and dP/dz is the constant pressure gradient along the flow direction (a negative value).

After normalizing the axial velocities using $w_0 = -dP/\mu dz$, the momentum equation can then be transformed into the form:

$$\nabla^2 W = -1 \tag{2}$$

where $W = w/w_0$ is the normalized axial velocity.

On solid walls, the no-slip boundary condition is:

$$W = 0 \tag{3}$$

In this paper, four duct cross section shapes are considered, namely equilateral triangle, rectangle, ellipse, and super-ellipse, as illustrated in Fig. 1.

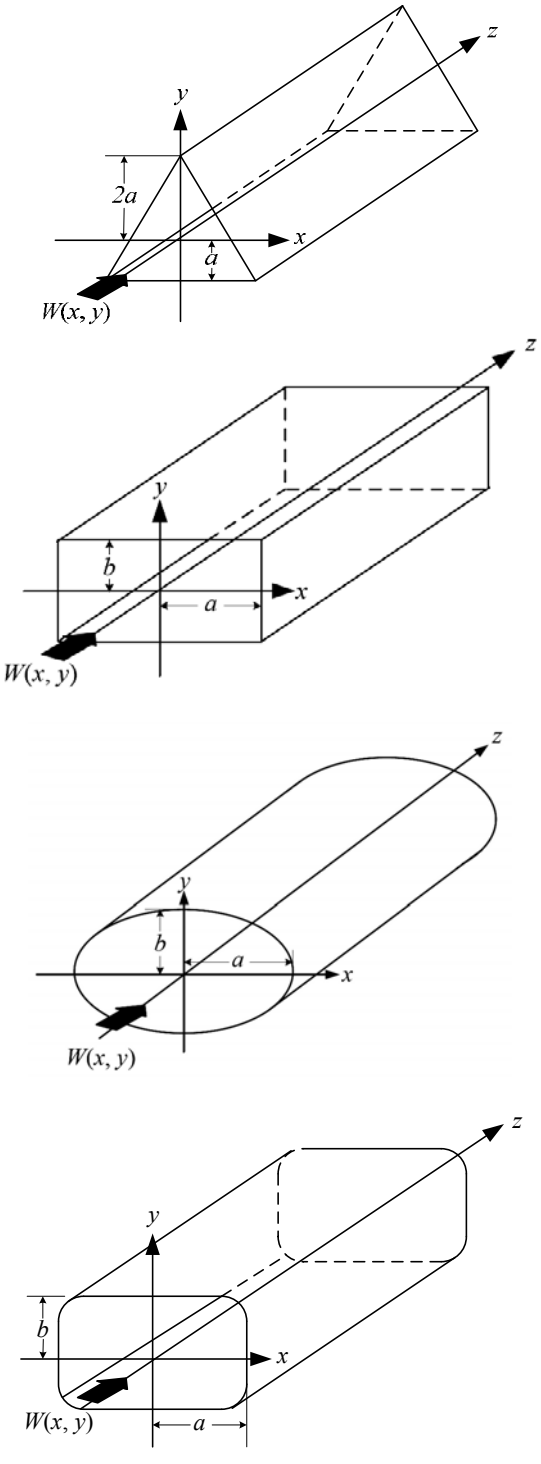


Fig. 1 Coordinate system and sketches of ducts

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2.2 Maximum Principle and Monotonicity of Elliptic Equation

In order to obtain the upper and lower approximate solutions of differential equations, the concept of maximum principle is utilized to establish the residual of differential equations and the monotonic relations of solutions are introduced. Poisson's equation is an elliptic equation, so theorems for elliptic equations are presented here.

2.2.1 One-Dimensional Elliptic Equations

At first, considering the function $u(x)$ which is well defined in domain V satisfies

$$(L+h)[u]=f(x) \qquad x\in V \tag{4}$$

and the boundary conditions:

$$\frac{\partial u(x)}{\partial \sigma}+A(x)u(x)=g_1(x) \ \text{ along } \ S_1 \ \text{ (partial boundary of } V) \tag{5}$$

$$u(x)=g_2(x) \ \text{ along } \ S_2 \ \text{ (partial boundary of } V) \tag{6}$$

Here, L is the n-dimensional elliptic operator and h is a function of x. The operator $(L+h)$ can be shown as:

$$(L+h)\equiv\sum_{i,j=1}^{n}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}b_i\frac{\partial}{\partial x_i}+h \tag{7}$$

For one-dimensional case, $n=1$, Eq. (4) can be written as:

$$a\frac{\partial^2 u(x)}{\partial x^2}+b\frac{\partial u(x)}{\partial x}+h(x)u(x)=f(x) \qquad x\in V \tag{8}$$

It should be pointed that

$$S_1\cup S_2=S \tag{9}$$

S is the complete boundary of domain V. $\partial u/\partial\sigma$ is the out normal derivative of arbitrary points on S_1.

Assume $\hat{u}(x)$ and $\check{u}(x)$ are two functions which are well defined in domain V and let they meet the following inequalities:

$$\begin{cases} R\hat{u}=-(L+h)[\hat{u}]+f(x)\ge 0 & x\in V \\ R_{B1}\hat{u}=\dfrac{\partial\hat{u}}{\partial\sigma}+A(x)\,\hat{u}-g_1(x)\ge 0 & \text{along boundary } S_1 \\ R_{B2}\hat{u}=\hat{u}-g_2(x)\ge 0 & \text{along boundary } S_2 \end{cases} \tag{10}$$

$$\begin{cases} R\check{u}=-(L+h)[\check{u}]+f(x)\le 0 & x\in V \\ R_{B1}\check{u}=\dfrac{\partial\check{u}}{\partial\sigma}+A(x)\,\check{u}-g_1(x)\le 0 & \text{along boundary } S_1 \\ R_{B2}\check{u}=\check{u}-g_2(x)\le 0 & \text{along boundary } S_2 \end{cases} \tag{11}$$

Here, R is the interior residual operator, R_{B1} and R_{B2} are the residual operator along boundary S_1 and S_2, respectively.

$u(x)$ is the solution for Eqs.(4) ~ (6), $g_1(x)$ and $g_2(x)$ are continuous functions along boundary S_1 and S_2, respectively. When Eqs. (10) and (11) hold, the monotonic relation of solutions exists as:

$$\hat{u}(x)\ge u(x)\ge\check{u}(x) \tag{12}$$

2.2.2 Two-Dimensional Elliptic Equations

In the two dimensional domain $V(x, y)$, the well-defined function $u(x, y)$ satisfies

$$F(x,\,y,\,u,\,u_x,\,u_y,\,u_{xx},\,u_{xy},\,u_{yy})=f(x,\,y) \qquad x,\,y\in V \tag{13}$$

$$u(x,\,y)=g(x,\,y) \qquad \text{along boundary } S \tag{14}$$

The function F is elliptic and has a continuous second derivative in domain V. Assume $\hat{u}(x, y)$ and $\check{u}(x, y)$ are two functions which are well defined in domain V and let they meet the following inequalities:

$$\begin{cases} R\hat{u}=-F(x,\,y,\,\hat{u},\,\hat{u}_x,\,\hat{u}_y,\,\hat{u}_{xx},\,\hat{u}_{xy},\,\hat{u}_{yy})+f(x,\,y)\ge 0 & x,\,y\in V \\ R_B\hat{u}=\hat{u}-g(x,\,y)\ge 0 & \text{along boundary } S \end{cases} \tag{15}$$

$$\begin{cases} R\check{u}=-F(x,\,y,\,\check{u},\,\check{u}_x,\,\check{u}_y,\,\check{u}_{xx},\,\check{u}_{xy},\,\check{u}_{yy})+f(x,\,y)\ge 0 & x,\,y\in V \\ R_B\check{u}=\check{u}-g(x,\,y)\ge 0 & \text{along boundary } S \end{cases} \tag{16}$$

If $\partial F/\partial u$ exists in domain V and the Eq. (17) holds as:

$$\frac{\partial F}{\partial u}\le 0 \qquad (x,\,y)\in V \tag{17}$$

Then the solution of Eqs. (13) ~ (14) has the monotonic relation as:

$$\hat{u}(x,\,y)\ge u(x,\,y)\ge\check{u}(x,\,y) \tag{18}$$

Now, check the monotonicity of solution of Poisson's equation. The residual of Eq. (2) can be written as:

$$RW(x,\,y)=-\nabla^2W(x,\,y)-1=0 \tag{19}$$

Compare Eq. (19) with Eq. (13), it can be seen that

$$F(x,\,y,\,W,\,W_x,\,W_y,\,W_{xx},\,W_{xy},\,W_{yy})=\nabla^2W \tag{20}$$

Further to know:

$$\frac{\partial F}{\partial W}=\frac{\partial(\nabla^2W)}{\partial W}=0 \tag{21}$$

Therefore, relation as Eq. (17) holds, that is, the monotonicity of solutions of Poisson's equation exists. Once the monotonicity is proved to exist, the double side approach method can be applied to solve the problems.

2.3 Formulation of Mathematical Programming

Based on the maximum principle of differential equations, the monotonicity of the Laplace operator had

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been approved in 2.2. Let $Z(x, y)$ be the appropriate trial function which satisfies the boundary conditions. The residuals of this problem can be formulated as:

$$R[Z] = -Z_{xx} - Z_{yy} - 1 \tag{22}$$

$$R_S[Z] = -Z \quad \text{on boundary } S \tag{23}$$

After the residuals have been constructed, the interior collocation method is applied. The problem can be further transformed into a bilateral inequality mathematical programming problem.

The upper bound approximate solution $Z_u(x, y, C)$ is to minimize $Z(x_i, y_i, C_j)$ under the conditions:

$$R\left[Z(x_i, y_i, C_j)\right] \geq 0 \tag{24}$$

$$R_S\left[Z(x_i, y_i, C_j)\right] \geq 0 \tag{25}$$

On the contrary, the lower bound approximate solution $Z_l(x, y, c)$ is to maximize $Z(x_i, y_i, C_j)$ under the conditions:

$$R\left[Z(x_i, y_i, C_j)\right] \leq 0 \tag{26}$$

$$R_S\left[Z(x_i, y_i, C_j)\right] \leq 0 \tag{27}$$

Here, $(x_i, y_i)_{i=1,2,\ldots,n}$ are the selected collocation points and $C_{j,j=1,2,\ldots,k}$ are the undetermined coefficients of the trial function.

After the solutions of the upper and lower bound have been derived, the approximate solution can be determined as:

$$\tilde{Z}(x, y) = \frac{Z_u + Z_l}{2} \tag{28}$$

2.4 Formulation of Genetic Algorithms

Finding the optimal solutions under the constructed inequalities is carried out using GAs. A simple GAs can be formulated as:

$$\text{GAs} = \left(P, \text{St}, \rho, \text{fit}, (\text{Se}, \text{Cr}, \text{Mu}), (P_s, P_c, P_m, G_m)\right) \tag{29}$$

where St is a string (chromosome) contained in a population P. ρ is a coding function maps the searching space of problem into a space of string. Selection (Se), crossover (Cr), and mutation (Mu) are genetic operators for generating new population. The fitness function (fit) is used to evaluate the search results. The probability of crossover P_c and the probability of mutation P_m are set to control the genetic operators. P_s is the population size and G_m is the setting maximum generation which is related to the temination criterion.

The basic procedures of GAs are as follows and the flow chart of the GAs is shown in Fig. 2.

1. Formulation — Formulate an optimization problem.
2. Initialization — Initialize an initial population P of several chromosomes St.

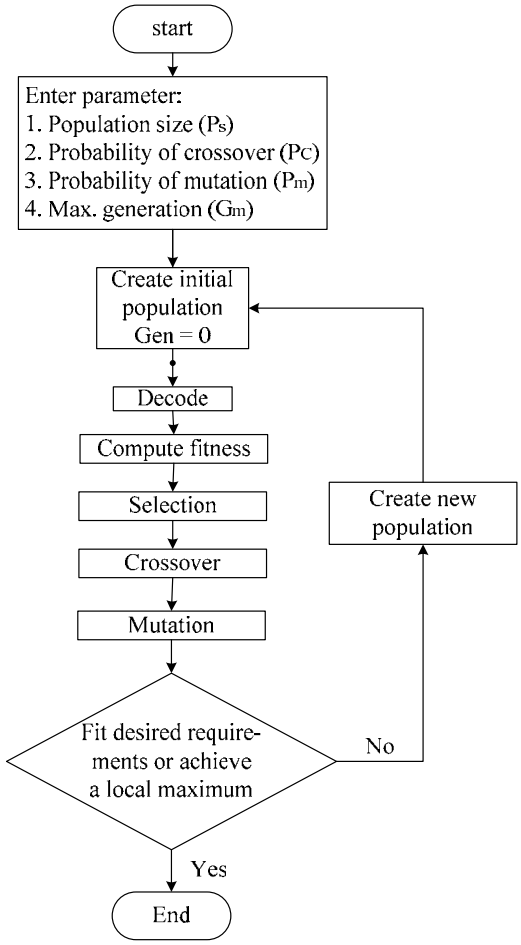


Fig. 2 Flow chart of GAs

3. Evaluation — Evaluate the fitness of each chromosome within the population. Using decoding structure maps the string into the real search point, and the fitness function maps the search point into a real number.
4. If termination criterion is not satisfied:
   {
5. Selection — Pairs of strings (parents) will be selected based on the fitness and undergo the subsequent genetic operations to produce new pairs of strings (children) that form a new population (generation). Use the concept of "survival of the fittest," the probability of a chromosome being selected is proportional to its fitness.
6. Crossover — An integer position k along the string St is selected at random between 1 and the l-1, l is the number of characters in the string. By swapping all characters between position $k + 1$ and l, two new strings are created.
7. Mutation — In selection operator, the gene is selected at random on a given chromosome. Mutation is used to prevent the genes from being lost from the population for purely stocahstic reasons.
8. Evaluation — Evaluate the new chromosomes within the new population.
   }
9. Return the best chromosome St.

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3. NUMERICAL RESULTS AND DISCUSSION

The double side approach method has three main parts. First, the trial functions of the axial velocity are established to satisfy the boundary conditions. Second, the objective in the mathematical programming is decided and then the inner collocation method is used to construct the inequalities. Finally, GAs are applied to solve the optimization problem. Table 1 shows the GAs parameters used.

For each trial function, only two undetermined parameters need to be found; therefore, few collocation points are required. Three collocation points are used here. It should be noted that due to the symmetry of the discussed cross section shapes, the points are only set in the first quadrant. Locations of collocation points are illustrated in Fig. 3 and the differences in the location of collocation points will be explained later in the text.

3.1 Equilateral Triangular Duct

The velocity profile of a fully developed, laminar flow for an equilateral triangular duct was determined by Dryden *et al.* [15] and reviewed by Shah and London in [16]:

$$W(x, y) = -\frac{1}{12a}\left[-y^3 + 3yx^2 + 3a(y^2 + x^2) - 4a^3\right] \quad (30)$$

The two types of trial function needed in the double side approach method are constructed as:

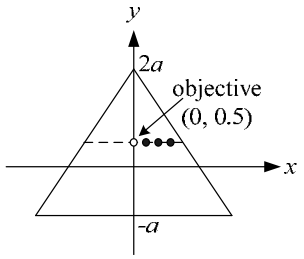
$$Z_1 = C_1(\sqrt{3}x + y - 2a)(-\sqrt{3}x + y - 2a)(y + a) + C_2(\sqrt{3}x + y - 2a)^2(-\sqrt{3}x + y - 2a)^2(y + a)^2 \quad (31)$$

$$Z_2 = \left[C_1 + C_2(x^2 + y^2)\right](\sqrt{3}x + y - 2a)(-\sqrt{3}x + y - 2a)(y + a) \quad (32)$$

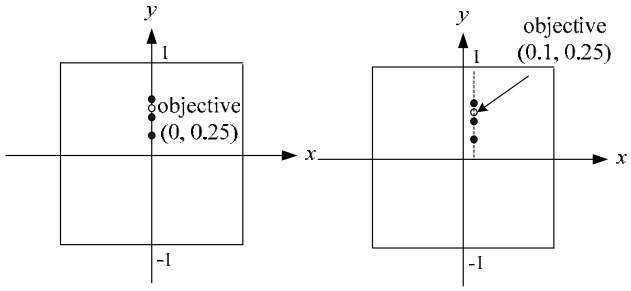
The objective function is arbitrarily chosen to be $Z(0, 0.5)$. The dimension of the triangular changes as the edge, a, varies. Therefore, it should be noted that if the fixed number of collocation points are expected, points should be shifted following the variation of a to avoid failure caused by the narrow range of their locations. An insufficient area of the collocation points cannot represent the characteristics of the whole duct section area. Hence, three collocation points $(x_i, y_i)_{i=1,2,3}$ are selected on the straight line $y = 0.5$ according to the rule:

$$\begin{cases} x_i = 2(i-1)\, a/5 \\ y_i = 0.5 \end{cases} \quad (33)$$

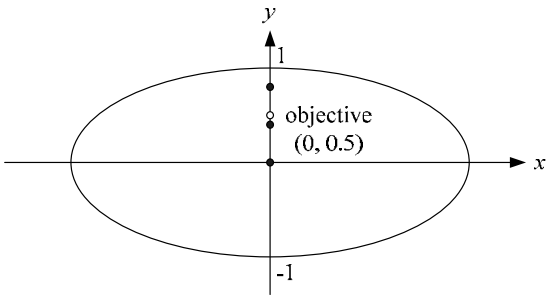
The collocation points selected by this rule, as shown in Fig. 3, are either located within the triangle or are wide enough to represent the whole domain. Results are tabulated values of a. Table 2 shows the optimal coefficients of the upper and lower bounds and Table 3 shows the upper, lower, and approximate solutions of the velocity, both obtained using Z_1. Tables 4 and 5 are the results using from Z_2. For both different trial functions, the results are satisfactory compared to the analytical solutions.



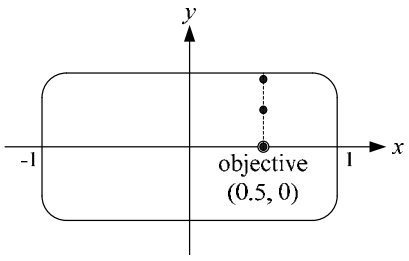
(a) Equilateral triangular duct



(b) Rectangular duct



(c) Elliptical duct



(d) Super-elliptical duct

Fig. 3 Locations of collocation points

Table 1 GAs parameters

Parameter	Value
Population size	100
Number of generations	1000
Probability of crossover	0.6
Probability of mutation	0.01
Selection of reference point	0

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Table 2 Optimal coefficients of Z_1

a/b	C_1	C_2
1	$C_{1u}=0.08333$	$C_{2u}=0$
	$C_{1l}=0.08330$	$C_{2l}=0$
2	$C_{1u}=0.04167$	$C_{2u}=0$
	$C_{1l}=0.04166$	$C_{2l}=0$
4	$C_{1u}=0.02083$	$C_{2u}=0$
	$C_{1l}=0.02083$	$C_{2l}=0$
5	$C_{1u}=0.01667$	$C_{2u}=0$
	$C_{1l}=0.01666$	$C_{2l}=0$
10	$C_{1u}=0.00833$	$C_{2u}=0$
	$C_{1l}=0.00833$	$C_{2l}=0$
20	$C_{1u}=0.00417$	$C_{2u}=0$
	$C_{1l}=0.00416$	$C_{2l}=0$

Table 3 Normalized velocity of Z_1 at (0, 0.5)

a/b	Upper	Lower	Approximate	Analytical	Difference
1	0.28124	0.28114	0.28119	0.28125	0.021%
2	1.27614	1.27584	1.27599	1.27604	0.004%
4	5.27259	5.27259	5.27259	5.27344	0.016%
5	8.27457	8.26961	8.27209	8.27292	0.010%
10	33.25861	33.25861	33.25861	33.27188	0.040%
20	133.37787	133.05802	133.21795	133.27135	0.040%

Table 4 Optimal coefficients of Z_2

a/b	C_1	C_2
1	$C_{1u}=0.08334$	$C_{2u}=0.00002$
	$C_{1l}=0.08331$	$C_{2l}=0.00001$
2	$C_{1u}=0.04172$	$C_{2u}=-0.00001$
	$C_{1l}=0.04167$	$C_{2l}=0$
4	$C_{1u}=0.02083$	$C_{2u}=0$
	$C_{1l}=0.02083$	$C_{2l}=0$
5	$C_{1u}=0.01667$	$C_{2u}=0$
	$C_{1l}=0.01667$	$C_{2l}=0$
10	$C_{1u}=0.00833$	$C_{2u}=0$
	$C_{1l}=0.00833$	$C_{2l}=0$
20	$C_{1u}=0.00417$	$C_{2u}=0$
	$C_{1l}=0.00417$	$C_{2l}=0$

Table 5 Normalized velocity of Z_2 at (0, 0.5)

a/b	Upper	Lower	Approximate	Analytical	Difference
1	0.28129	0.28118	0.28124	0.28125	0.004%
2	1.27760	1.27614	1.27687	1.27604	0.065%
4	5.27259	5.27259	5.27259	5.27344	0.016%
5	8.27457	8.27457	8.27457	8.27292	0.020%
10	33.25861	33.25861	33.25861	33.27188	0.040%
20	133.37787	133.37787	133.37787	133.27135	0.080%

The number of collocation points plays an important role in the double side approach method. It can be seen from Table 2 to Table 5 that the results have already agreed well with the analytical solution when three collocation points are selected. To further investigate the effect of the number of collocation points on calculation results in this problem, five collocation points are further examined as:

$$\begin{cases} x_i = (i-1)\ a/5 \\ y_i = 0.5 \end{cases} \tag{34}$$

Tables 6 and 7 present the comparisons between three and five collocation points of Z_1 and Z_2 respectively. It can be observed that using three collocation points has been sufficient enough to get high accuracy in this problem. In consideration of the computing efficiency, three collocation points are adopted in the following calculations.

3.2 Rectangular Duct

In 1994, Spiga and Morini proposed analytical solutions for the velocity distribution in fully developed, laminar flow through rectangular ducts using the finite Fourier transform [17]:

$$W(x,\ y)=\frac{16\beta^2}{\pi^4}\sum_{n\ odd}^{\infty}\sum_{m\ odd}^{\infty}\frac{\sin(n\pi x)\sin(m\pi y)}{nm(\beta^2n^2+m^2)} \tag{35}$$

where β is the aspect ratio b/a.

In the double side approach method, two kinds of trial function which satisfy the boundary conditions of the rectangle are supposed as:

$$Z_1=C_1(a^2-x^2)(b^2-y^2)+C_2(a^2-x^2)^2(b^2-y^2)^2 \tag{36}$$

$$Z_2=\left[C_1+C_2(x^2+y^2)\right](a^2-x^2)(b^2-y^2) \tag{37}$$

In order to verify the validity of this method for rectangular ducts, the normalized velocity values at points (0, 0.25) and (0.1, 0.25) are checked with the aspect ratio a/b set to unity, and the results are compared with those presented in Spiga and Morini's previous work [17].

Three collocation points are selected in each case. For objective function $Z(0, 0.25)$, the selected points are (0, 0.1), (0, 0.2), (0, 0.3), and for objective function $Z(0.1, 0.25)$, they are (0.1, 0.1), (0.1, 0.2), (0.1, 0.3), as shown in Fig. 3.

Table 8 shows the optimal coefficients of the upper and lower bounds for various trial functions and Table 9 shows the upper, lower, and approximate solutions of the velocity at point (0, 0.25). Tables 10 and 11 show the results at (0.1, 0.25). It can be seen from Table 8 to Table 11 that the results obtained using the double side method have only slight differences with those presented in [17].

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Table 6 Comparison of normalized velocity of Z_1 at (0, 0.5)

a/b	Approximate		Analytical	Difference	
	Three points	Five points		Three points	Five points
1	0.28119	0.28120	0.28125	0.021%	0.017%
2	1.27599	1.27609	1.27604	0.004%	0.004%
4	5.27259	5.27305	5.27344	0.016%	0.007%
5	8.27209	8.27226	8.27292	0.010%	0.008%
10	33.25861	33.26157	33.27188	0.040%	0.031%
20	133.21795	133.25418	133.27135	0.040%	0.013%

Table 7 Comparison of normalized velocity of Z_2 at (0, 0.5)

a/b	Approximate		Analytical	Difference	
	Three points	Five points		Three points	Five points
1	0.28124	0.28124	0.28125	0.004%	0.004%
2	1.27687	1.27639	1.27604	0.065%	0.027%
4	5.27259	5.27398	5.27344	0.016%	0.010%
5	8.27457	8.27403	8.27292	0.020%	0.013%
10	33.25861	33.26415	33.27188	0.040%	0.023%
20	133.37787	133.22416	133.27135	0.080%	0.035%

Table 8 Optimal coefficients of Z_1 and Z_2 for objective (0, 0.25)

	C_1	C_2
Z_1	$C_{1u} = 1.20685$ $C_{1l} = 1.27670$	$C_{2u} = 0.50370$ $C_{2l} = -2.27878$
Z_2	$C_{1u} = 1.21594$ $C_{1l} = 1.14921$	$C_{2u} = 0.03115$ $C_{2l} = 0.61470$

Table 9 Normalized velocity of Z_1 and Z_2 at (0, 0.25) under $a = b = 1$

	Upper	Lower	Approximate	Difference
Z_1	0.05768	0.05484	0.05626	2.1%
Z_2	0.05709	0.05567	0.05638	1.9%
[17]	–	–	0.05750	–

Table 10 Optimal coefficients of Z_1 and Z_2 for objective (0.1, 0.25)

	C_1	C_2
Z_1	$C_{1u} = 1.26200$ $C_{1l} = 1.30833$	$C_{2u} = -0.53624$ $C_{2l} = -2.60349$
Z_2	$C_{1u} = 1.20879$ $C_{1l} = 1.15742$	$C_{2u} = 0.29428$ $C_{2l} = 0.66140$

Table 11 Normalized velocity of Z_1 and Z_2 at (0.1, 0.25) under $a = b = 1$

	Upper	Lower	Approximate	Difference
Z_1	0.05570	0.05360	0.05465	1.8%
Z_2	0.05536	0.05424	0.05480	1.6%
[17]	–	–	0.05567	–

3.3 Elliptical Duct

For elliptical ducts with $2a$ and $2b$ as the major and minor axes, respectively, the fully developed, laminar velocity profile can be represented as [16,17]:

$$W(x, y) = -\frac{1}{2(1+\beta^2)}(\beta^2 x^2 + y^2 - b^2) \tag{38}$$

where β is the aspect ratio b/a.

The two kinds of trial function which contain the shape equation of an ellipse are set to:

$$Z_1 = C_1\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)+C_2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^2 \tag{39}$$

$$Z_2 = \left[C_1 + C_2(x^2+y^2)\right]\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) \tag{40}$$

Choose $Z(0, 0.5)$ as the objective function. For convenience, b is set to 1. Because b is set to an invariant value and the collocation points are arranged on the y-axis, the collocation points do not need to move as the value of a changes. Three collocation points, (0, 0), (0, 0.4), and (0, 0.8), as shown in Fig. 3, are selected. Results are tabulated for $b = 1$ and various values of a. Tables 12 and 13 show the results for Z_1 and Tables 14 and 15 show the results for Z_2. The solutions obtained using the double side method are in good agreement with the analytical solutions (0.02% relative difference).

3.4 Super-Elliptical Duct

The shape equation of a super ellipse in the Cartesian coordinate can be shown as:

$$\frac{x^{2n}}{a^{2n}}+\frac{y^{2n}}{b^{2n}}=1 \tag{41}$$

where n is the power of the super ellipse. Ellipse ($n = 1$) and rectangle ($n = \infty$) are two limiting cases of the super ellipse.

No data has been reported for the super-elliptical duct before Wang [18]. Wang analyzed the flow through a super-elliptical duct and discussed the effect of corner rounding by using the Ritz method, which is mostly used for elastic eigenvalue problems. The velocity distribution under the conditions $a = 1$, $b = 0.5$ and $n = 5$ was derived [18]. To verify the double side approach method for super-elliptical ducts, the same conditions are considered. The trial function is in the form:

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Table 12 Optimal coefficients of Z_1

a/b	C_1	C_2
1	$C_{1u}=0.25002$	$C_{2u}=0$
	$C_{1l}=0.24998$	$C_{2l}=0.00001$
2	$C_{1u}=0.40009$	$C_{2u}=-0.00004625$
	$C_{1l}=0.39994$	$C_{2l}=0.00003$
4	$C_{1u}=0.47058$	$C_{2u}=0$
	$C_{1l}=0.47055$	$C_{2l}=0.00002$
5	$C_{1u}=0.48076$	$C_{2u}=0$
	$C_{1l}=0.48071$	$C_{2l}=0.00003$
10	$C_{1u}=0.49504$	$C_{2u}=0$
	$C_{1l}=0.49493$	$C_{2l}=0.00006$
20	$C_{1u}=0.49883$	$C_{2u}=-0.00004$
	$C_{1l}=0.49870$	$C_{2l}=0.00003$

Table 13 Normalized velocity of Z_1 at (0, 0.5)

a/b	Upper	Lower	Approximate	Analytical	Difference
1	0.18751	0.18749	0.18750	0.18750	0%
2	0.30000	0.29997	0.29999	0.30000	0.003%
4	0.35294	0.35292	0.35293	0.35294	0.003%
5	0.36057	0.36055	0.36056	0.36058	0.006%
10	0.37128	0.37123	0.37126	0.37129	0.008%
20	0.37410	0.37404	0.37407	0.37406	0.003%

Table 14 Optimal coefficients of Z_2

a/b	C_1	C_2
1	$C_{1u}=0.25008$	$C_{2u}=-0.00005$
	$C_{1l}=0.25000$	$C_{2l}=0$
2	$C_{1u}=0.40000$	$C_{2u}=0$
	$C_{1l}=0.39987$	$C_{2l}=0.00005$
4	$C_{1u}=0.47062$	$C_{2u}=-0.00002$
	$C_{1l}=0.47053$	$C_{2l}=0.00003$
5	$C_{1u}=0.48076$	$C_{2u}=0$
	$C_{1l}=0.48064$	$C_{2l}=0.00006$
10	$C_{1u}=0.49504$	$C_{2u}=0$
	$C_{1l}=0.49500$	$C_{2l}=0.00001$
20	$C_{1u}=0.49875$	$C_{2u}=0$
	$C_{1l}=0.49871$	$C_{2l}=0.00001$

Table 15 Normalized velocity of Z_2 at (0, 0.5)

a/b	Upper	Lower	Approxi-mate	Analytical	Difference
1	0.18756	0.18750	0.18753	0.18750	0.016%
2	0.30000	0.29992	0.29996	0.30000	0.013%
4	0.35296	0.35290	0.35293	0.35294	0.003%
5	0.36057	0.36049	0.36053	0.36058	0.014%
10	0.37128	0.37125	0.37127	0.37129	0.005%
20	0.37406	0.37403	0.37405	0.37406	0.003%

$$Z=\left[C_1+C_2(x^2+y^2)\right]\left(1-\frac{x^{2n}}{a^{2n}}-\frac{y^{2n}}{b^{2n}}\right) \tag{42}$$

Take point (0.5, 0) as the objective. Because the velocity distribution at $x=0.5$ is desired, three collocation points located as wide as possible, as shown in Fig. 3, are selected as (0.5, 0), (0.5, 0.25), (0.5, 0.499). Table 16 shows the optimal coefficients of the upper and lower bounds and Table 17 shows the upper, lower, and approximate solutions. Figure 4 shows the velocity profile at $x=0.5$. It is clear from Fig. 4 that good agreement is obtained compared to the results reported in [18].

Table 16 Optimal coefficients of Z for objective (0.5, 0)

	C_1	C_2
Upper	$C_{1u}=0.15193$	$C_{2u}=-0.24466$
Lower	$C_{1l}=0.13950$	$C_{2l}=-0.22426$

Table 17 Normalized velocity of Z at (0.5, 0)

	Upper	Lower	Approximate
Z	0.09068	0.08335	0.08702

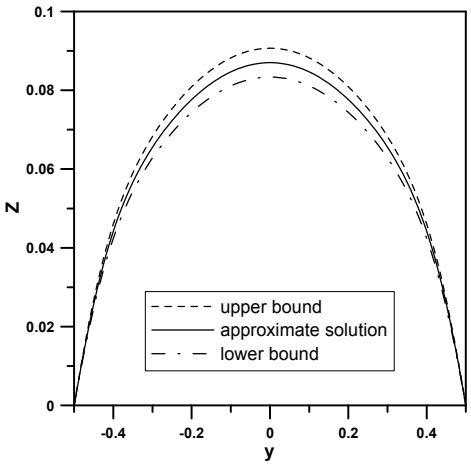


Fig. 4 Velocity distribution at $x=0.5$

4 CONCLUSIONS

The double side approach method, which combines the method of weighted residuals with mathematical programming was proposed for solving incompressible Newtonian, fully-developed, steady-state laminar flow in ducts of various shapes. It was demonstrated that all results generated by the method are in good agreement with the analytical solutions or previously reported results. The proposed method is efficient at establishing the upper and lower bounds for solutions to boundary value problems.

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