An octet of circles

MICHAEL SEWELL

1. *Sources*

This article was triggered by a single sentence in a book by Pickover [1]. He quotes, without providing a proof, or a reference to the original source, what he calls 'Johnson's Theorem'. He states it in the form that 'if three equal circles are drawn through a point, the circle through their other three intersections is equal to each of them'. Such a simple assertion has the quality of surprise, prompting curiosity as to its background. Johnson published an eight-line proof [2] in 1916. I am indebted to Mr T. J. Chapman (of Reading University Library) for finding the original publication for me.

In response to a referee's comment, it may be thought helpful to quote Johnson's proof verbatim. It is short, as follows, and has no diagram.

> Denote the centers of the circles by C_1, C_2, C_3, the intersections of C_2 and C_3 by O and P_1, those of C_3 and C_1 by O and P_2, those of C_1 and C_2 by O and P_3. Then $OC_2P_1C_3$ is a rhombus, and so is $OC_3P_2C_1$. Hence, C_2P_1 and C_1P_2 are equal, $C_1C_2P_1P_2$ is a parallelogram, and P_1P_2 is equal to C_1C_2. Thus the triangles $C_1C_2C_3$ and $P_1P_2P_3$ are congruent, and have equal circumcircles. But the circumcircle of the former has its center at O, and is equal to each of the given circles. Hence, the circle through $P_1P_2P_3$ is equal to each of the given circles.
>
> *Corollary*: Each of the four points O, P_1, P_2, P_3, is the orthocenter of the other three, and the set of points has all the well-known properties of an orthocentric system.

Immediately following Johnson's article are some 'Remarks on the foregoing circle theorem' by A. Emch about a 'connection with the theory of circular inversion', preceded by 'another proof of the theorem, equally notable for its extreme simplicity'.

More brief information is provided by A. Bogomolny [3], who remarks in particular that, although Johnson 'believed that the problem was new, it appeared a hundred years earlier' in 1819 [4]. Further associated information is cited by D. N. Mackenzie [5] in a 14-page article which does not overlap with what I shall say here. Wikipedia [6] also provides associated facts which complement what I shall say below. I am grateful to Professor P. M. Sewell (of Cambridge University Computing Laboratory) for references [4, 5 and 6]. What follows in this paper is substantially different from the information in references [3] to [6].

In particular the octet is not mentioned or hinted at in any of the foregoing references.

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2. *Configurations of three circles*

In this investigation we first need to identify all the possibly relevant configurations of three intersecting circles which have the same diameter, and pass through a common point, but are not necessarily symmetrically disposed.

We begin by drawing two intersecting circles. In so doing we have to choose whether their two centres lie within both circles, or have each centre on the circumference of the other, or have each centre outside the companion circle. In the last case the circles could be either touching or not touching.

For brevity we develop the consequences for this last and non-touching case, so that there are two distinct intersection points. The reader may wish to explore the corresponding results for the other cases.

In this chosen case there can then be six possible locations for the centre of the third circle which can be regarded as distinct: (a) within both of the first two; (b) within one, and on the circumference of the other; (c) within one and outside the other; (d) outside one and on the other; (e) outside both; (f) on an intersection of both.

The locations (b), (d) and (f), in which the third centre is on the circumference of one or both of the first two circles, can be regarded as non-generic. Therefore we leave those aside and regard the three locations (a), (c) and (e) as generic cases.

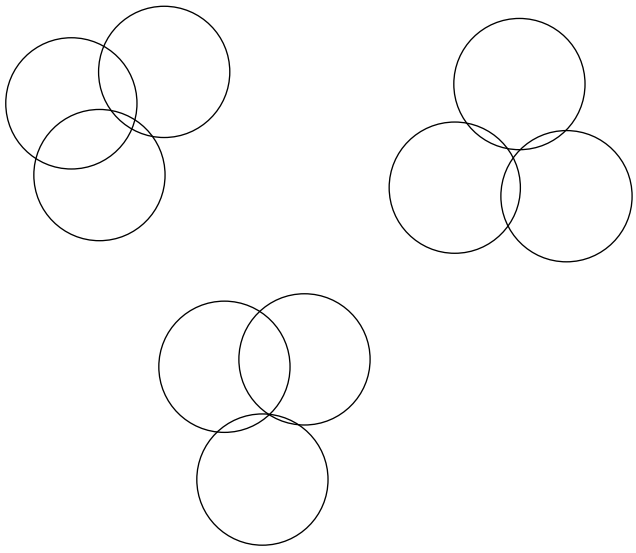


FIGURE 1: Three intersection cases (g) and (i) [above] and (h) [below]

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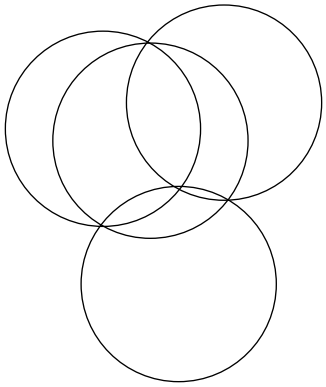


FIGURE 2: Johnson's circle

For the sake of brevity we pursue the investigation further in just one of those cases, and we choose the case (e).

Even that has two possibilities for the location of the third circle. Its centre might be on the ('produced') chord joining the two intersections of the first two circles (this would be the symmetric case), or off it (the non-symmetric case).

3. *The most generic case*

Having thoroughly investigated possible alternative configurations (which seems not to be done to this extent in any of the cited references), it may now be considered that the non-symmetric case in the last paragraph above is the 'most generic', and we confine subsequent attention to that. (Enumeration of cases can seem unrewarding, but on the other hand one needs to be aware of all options. So we try to strike a balance here, by first listing the cases, and then making this unsymmetric choice.)

Even for this one choice there are three ways in which the third circle may intersect the first two. They are as follows. The third circle may intersect the first two (and each twice) so that either (g) each of one intersection with one circle is within the other circle; or (h) each is at an intersection of the first two, so that the three circles have one common point; or (i) both intersections are outside the zone common to the intersection of both. We show these last three cases in the first trio of diagrams, i.e. in Figure 1.

Johnson's 1916 Theorem states a further property of case (h), namely that if three identical (but not necessarily symmetrically disposed) circles pass through a common point, their other three intersections must lie on another circle which has the same size as the first three. (Google gives more information about this case, albeit in quite technical language which we do not need here). This fourth circle is also shown in Figure 2, for case (h).

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4. *Further symmetries*

When we draw Figure 2 we can see that there are further symmetries, which I have not seen mentioned elsewhere, and one purpose of this paper is to describe the consequences of them. First we notice that any three of the four circles could be regarded as the starting trio of circles having a common intersection point, with the fourth circle passing through the three pairwise intersections of those first three. Thus there are four examples of Johnson's Theorem in the same diagram. In other words, if the starting three are labelled circles A, B, C and the fourth one is labelled D, we could equally well have A, B, D as the starting trio and C as the fourth one; and then A, C, D with B; and then B, C, D with A.

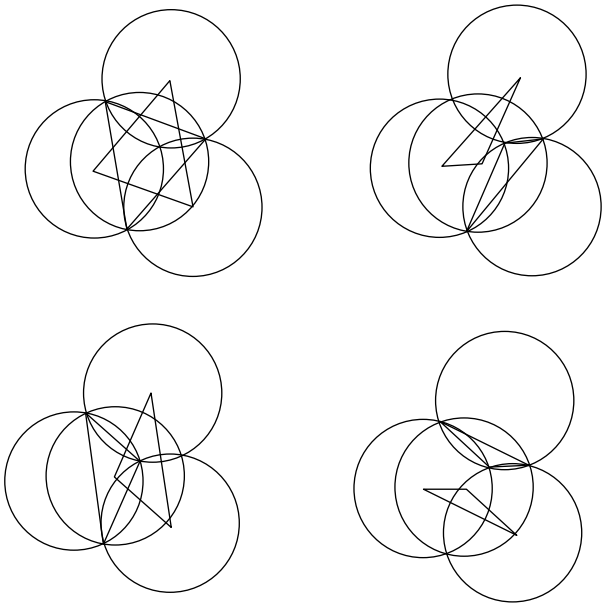


FIGURE 3: Four pairs of congruent triangles

5. *Four pairs of congruent triangles*

Next, we can add to Figure 2 the triangle whose vertices are at the centres of those first three circles which have the common intersection required of them. That triangle is neither equilateral nor isosceles in the general case. Even so, we can see from the construction that this triangle is congruent to a second triangle which can be drawn, whose vertices are at the three other and pairwise intersections of the first three circles. Moreover, the construction shows that corresponding pairs of sides of those two triangles are parallel. Also, the joins of centres of each pair of circles are equal in length to the chord of the third circle where it meets this pair. Further, this parallel congruent and reversed property is true for each of the four different pairs of triangles created by this construction. Figure 3 shows them.

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The spirit of this article can be seen to be that the results are established here by geometrical construction. Anyone who gets out ruler and compasses will be convinced by this, as I am, and this has been the approach which I have followed in this investigation. The details cited above from Johnson's paper are given to show a different approach which some readers may prefer. More information in this direction can be found in reference [6], which also contains the first pair of triangles in Figure 3, but not the other three pairs there.

6. *Octet of circles*

The foregoing observation about Figure 3 shows that, by symmetry, the circle which circumscribes the second triangle, in each of the four cases there, is the same size as each of the first trio of circles, and as the circle which passes through their centres.

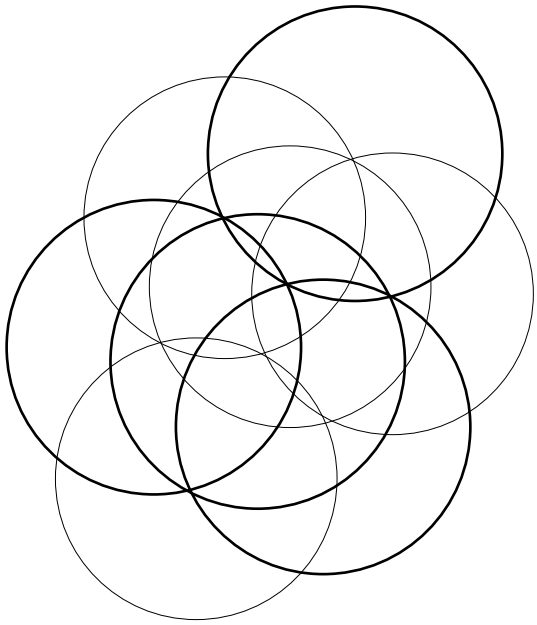


FIGURE 4: Octet of circles

This further means that, by symmetry, a second quartet of circles can be constructed in each case, of the same size as the first quartet, but reversed in orientation. One of them circumscribes the first triangle, and the other three have their centres at the vertices of the new triangle. An additional circle of the same size circumscribes the second triangle.

Thus we finally have not merely four identical circles, mutually intersecting as described in Johnson's theorem, but eight circles having such identical properties. An example is shown in the last diagram, i.e. in Figure 4.

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The construction further shows that, although there are four different pairs of oppositely disposed congruent triangles, the octet of equal circles constructed in each of the four cases is the same.

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